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Time-varying interaction leads to amplitude death in coupled nonlinear oscillators

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Abstract. A new form of time-varying interaction in coupled oscillators is introduced. In this interaction, each individual oscillator has always time-independent self-feedback while its interaction with other oscillators are modulated with time-varying function. This interaction gives rise to a phenomenon called amplitude death even in diffusively coupled identical oscillators. The nonlinear variation of the locus of bifurcation point is shown. Results are illustrated with Landau–Stuart (LS) and Rössler oscillators.

Keywords. Time-varying interaction; amplitude death; synchronization.

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Natural systems are rarely isolated, and hence the interactions between such systems have been extensively studied from both theoretical and experimental points of view in a variety of contexts in physical, biological, and social sciences [1,2]. Several new interesting phomonena, namely, synchronization, hysteresis, phase locking, phase shifting, phase-flip, riddling etc. [1–6] arise in such interacting systems. Amplitude death (AD) is an important phenomenon that can occur in coupled nonlinear oscillators when their interaction causes the fixed points to become stable and attracting [7–23]. Although AD was first observed in chemical systems [8], it is now becoming evident, through a number of recent theoretical and experimental studies, that this phenomenon is quite widespread [7]. Initially, it was believed that the system parameters of the instantaneously coupled oscillators should be different [14,15]. However, either by introducing a delay in the coupling [16] or by using conjugate coupling [11], amplitude death can occur without mismatch in oscillators. Several theoretical and experimental studies of AD in coupled systems have contributed further to the understanding of this effect [18], for example the work on distributed delay [20], nonlinear interaction [12], linear augmentation [23], dynamical coupling [13], environmental coupling [7,9] etc.

One aspect of interest is in identifying the different types of interactions that facilitate the phenomenon of AD. This is of particular relevance in natural systems where the parameters of individual systems may not be accessible, and AD can be achieved mainly by using an appropriate form of interaction. In all the previously reported studies, that lead to AD, explicitly time-independent interactions were considered [9,11–16,18–21,23]. In this paper we consider a new form of time-varying diffusive interaction. This can lead to AD without increasing the dimensionality. Recall that AD has been achieved by using time delay [16], time-varying delay [22] through the environment [9], by using additional dynamical interaction [13], linear augmentation [23], all of which increase the dimensionality of the system. However, the present interaction proposed is simple and also useful for controlling AD regime.

Let us consider a configuration of N coupled nonlinear oscillators that is specified by the equation

$$\dot{\mathbf{X}}_j = \mathbf{F}_j(\mathbf{X}_j) + \epsilon \sum_{k=1,k\neq j}^{N} \mathbf{G}(\mathbf{X}_j, \mathbf{X}_k, t), \qquad j = 1, \ldots, N, \tag{1}$$

where $\mathbf{X}_j$ is the m_j-dimensional vector of dynamical variables for the jth oscillator and $\mathbf{F}_j$s specify their evolution equations. ϵ is the coupling strength which we consider identical for all connections. In order to present the results, we consider a linear diffusive interaction as

$$\mathbf{G}(\mathbf{X}_k, \mathbf{X}_j, t) = H(t)\mathbf{X}_k - \mathbf{X}_j, \tag{2}$$

where $H(t)$ is a time-varying function. Here the function $H(t)$ modulates the interaction due to other oscillators ($\mathbf{X}_k$) while the self-feedback ($\mathbf{X}_j$) remains unaffected. Note that for $H(t) = 1$ the phenomenon of AD is not possible in coupled identical oscillators even if we consider ϵ as either constant [14,15] or time-varying [24–26].

In this paper we consider $H(t)$ as periodic step function of period T defined as

$$H(t) = a, \qquad 0 < t \leq \tau \tag{3}$$

$$= b, \qquad \tau < t \leq T, \tag{4}$$

where a and b are constants. This type of interaction is quite common in those systems in which the individual can commute from one place to another, e.g. mine workers (or navy personnel) interact with their families for a period of time (say, τ) while for another period of time (say, $T - \tau$) they remain away for their work in mines (or sea) which are almost isolated. In the extreme cases, e.g. for $a = b = 0$ the systems are completely independent of each other while for $a = b = 1$ AD is impossible [14,15]. However, for other combinations of a, b and τ, the possibility of AD exists. The details are discussed below.

First we consider two coupled Landau–Stuart (LS) limit cycle oscillators, modelled as

$$\dot{Z}_j = [1 + i\omega_j - |Z_j|^2]Z_j + \epsilon[H(t)Z_k - Z_j], \tag{5}$$

where $j, k = 1, 2$ and $j \neq k$. The variable $Z_j(= x_j + iy_j)$ is the complex amplitude of the jth oscillator with frequency ω_j. In this paper we consider identical oscillators, i.e. $\omega_1 = \omega_2 = \omega = 10$. Usually, a linear stability analysis is carried out at a fixed point of the system which is time-independent. As the interaction in eq. (2) is time-varying, we consider an average eigenvalue $\lambda = [\tau\lambda_a + (T - \tau)\lambda_b]/T$, where λ_a and λ_b are the eigenvalues of the stability matrix at the fixed point over the period τ and $T - \tau$

respectively. Therefore, the linear stability analysis at fixed point (zero in eq. (5)) gives nontrivial characteristic equations

$$1 - i\omega - \epsilon - \lambda = \pm\epsilon\left[a\frac{\tau}{T} + b\frac{(T-\tau)}{T}\right] \tag{6}$$

$$= \pm\epsilon[a\tau' + b(1-\tau')], \tag{7}$$

where the factors τ/T and $(T-\tau)/T$ are considered to be due to an average over a period T. Here, we define $\tau' = \tau/T \in [0, 1]$. Letting $\lambda = \alpha + i\beta$, where α and β are real and imaginary parts of the eigenvalues, eq. (7) leads to

$$\alpha = 1 - \epsilon + \epsilon[(a-b)\tau' + b] \tag{8}$$

and

$$\beta = \omega. \tag{9}$$

For selecting the largest value of α we consider the negative sign in eq. (7). The various possible dynamics in parameter space $\epsilon - \tau'$ for the given a and b values are explored below.

Case I: $a = 1, b = 0$ *and* $0 < \tau' < 1$

In this case both the oscillators mutually interact with each other during τ while remaining independent for $T - \tau$. The phase diagram in parameter space $\epsilon - \tau'$ for $T = 100$ [27] is shown in figure 1a. The dashed line (with circles) is the contour curve taken from the largest Lyapunov exponent [28] at $\Lambda_1 = -0.01$ where all the Lyapunov exponents are negative (which confirms the occurrence of AD) [29]. The details of the dynamics across this curve are shown in figure 1b where the two largest Lyapunov exponents are plotted as a function of coupling strength ϵ at $\tau' = 0.5$. The asymptotic periodic (P) and the transient AD dynamics are shown in the inset figures. The solid line in figure 1a corresponds to the locus ($\alpha = 0$ in eq. (8))

$$\tau' = \frac{1}{a}\left(1 - \frac{1}{\epsilon}\right), \tag{10}$$

which matches well with the numerically calculated dashed line. Here the dynamics goes from periodic to a fixed point via Hopf bifurcation [30] as real part of the eigenvalue α becomes negative while β remains the same as coupling strength is increased. Here the stabilized fixed point (origin) is the same as that of the uncoupled systems.

The details of the dynamics due to switching interaction is shown in figure 1c where trajectories x_1 and x_2 are plotted as a function of time. These show that, at every τ and T, there are discontinuous changes in the slope of the trajectory. However, both the oscillators remain in complete synchronous state (in-phase) as shown in the inset figure in relative phase space $x_1 - x_2$.

It is also interesting to see that the locus of bifurcation point, eq. (10), varies nonlinearly (concave type) even though τ and $T - \tau$ are linearly related. If $\tau' \to 1$ then $\epsilon \to \infty$, which indicates that AD is not possible at any higher value of the coupling strength in such instantaneously coupled oscillators [14,15]. However, for $\tau' < 1$ we do observe AD

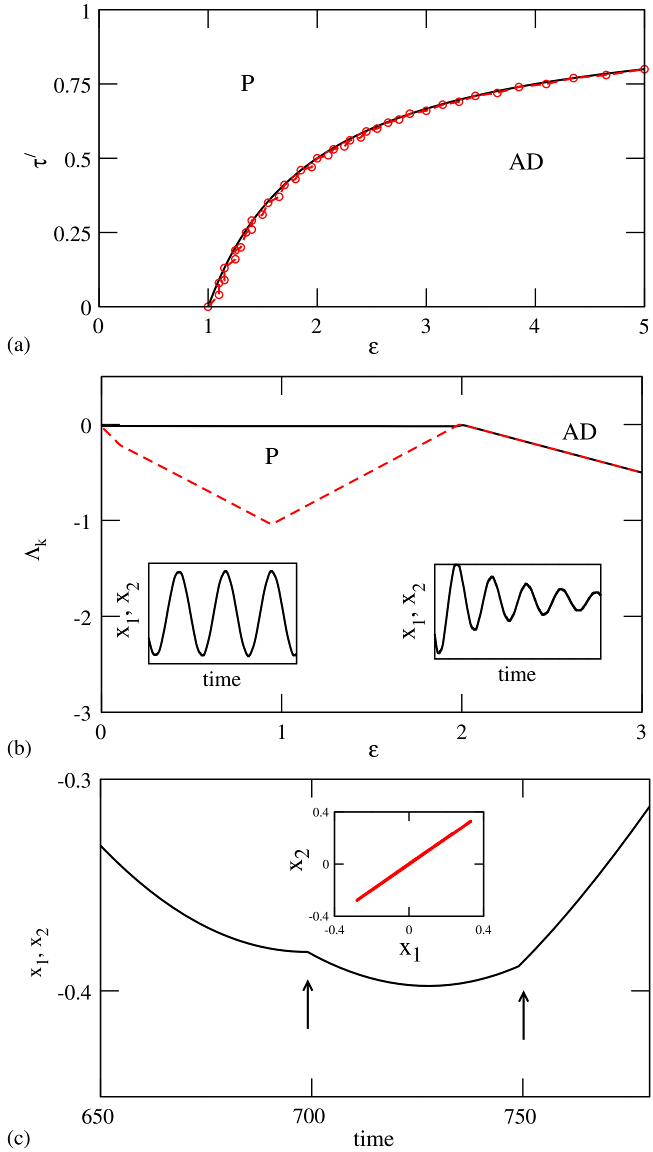

Figure 1. (**a**) The phase diagram in the parameter space $\epsilon - \tau'$ for $a = 1$ and $b = 0$. Solid and dashed (with circles) lines represent eq. (10) and contour at $\Lambda_1 = -0.01$ respectively. (**b**) The two largest Lyapunov exponents as a function of coupling strength at fixed $\tau' = 0.5$. Inset figures in (**b**) show the overlapped asymptotic periodic and transient AD trajectories at $\epsilon = 1.5$ and 3 respectively. (**c**) The variables x_1 and x_2 as a function of time at $\epsilon = 3$. The inset figure in figure 1c shows the complete synchronized (in-phase) motion in relative phase-space $x_1 - x_2$ while arrows indicate the times of switching.

with this type of interaction at appropriate coupling strengths even when the oscillators are identical.

Case II: $a = 1, b < 1$ *and* $0 < \tau' < 1$

In this case the locus of bifurcation point (from periodic to AD) turns out to be

$$\tau' = \frac{1}{a-b}\left[1 - b - \frac{1}{\epsilon}\right]. \tag{11}$$

Figure 2a shows the analytical result (solid line) given by eq. (11) along with the corresponding numerical data (dashed line with circles). These curves show similar behaviour as in Case I. However, the derivatives of eqs (10) and (11) with respect to ϵ suggest that the former has a slower variation. It is also evident from the curves in figures 1a and 2a that the locus reaches faster to $\tau' = 1$ in the latter case.

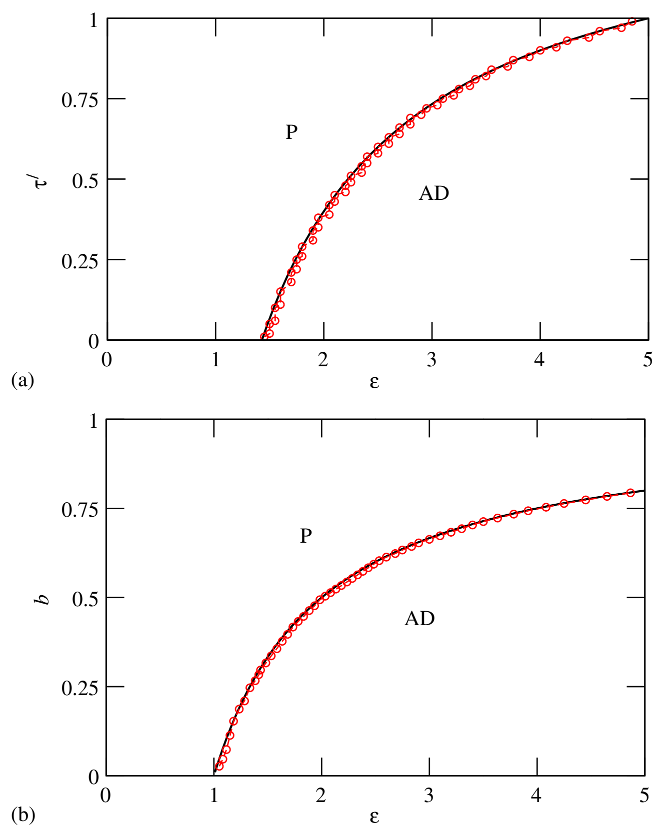

Figure 2. The phase diagrams in the parameter space (**a**) $\epsilon - \tau'$ at $b = 0.5$ (Case II) and (**b**) $b - \epsilon$ at $\tau' = 0$ (Case III). The locus of bifurcation point (solid lines) in (**a**) and (**b**) are the plots of eqs (11) and (12) respectively. The dashed lines (with circles) correspond to the contour at $\Lambda_1 = -0.01$.

Case III: $0 < a = b < 1$ *and* $\tau' = 0$

In this case the oscillators always interact with each other, but with reduced constant strength. This situation can also be considered as a mean-field interaction as discussed recently in ref. [31]. Here the characteristic equation, eq. (8), gives a critical value of the coupling strength

$$\epsilon_c = \frac{1}{1-b}, \tag{12}$$

which depends only on b. Here AD can be found whenever $\epsilon > \epsilon_c$ (see figure 2b). It is important to note that as $b \to 1$, the critical value of the coupling strength $\epsilon_c \to \infty$ and hence AD is not possible [14,15]. However, if $b \leq 1$ then AD can be possible even in instantaneously coupled identical oscillators. This result also suggests that AD can be easily targeted by tuning only the parameter b.

In order to see the effect of such types of time-varying interaction in high-dimensional systems, we consider two coupled chaotic Rössler [32] oscillators,

$$\begin{aligned} \dot{x}_j &= -y_j - z_j + \epsilon[H(t)x_k - x_j], \\ \dot{y}_j &= x_j + 0.2y_j, \\ \dot{z}_j &= 0.2 + z_j(x_j - 10), \end{aligned} \tag{13}$$

where $H(t)$ is the same as in eq. (3). As it is not possible to obtain the compact form for eigenvalues for this system [33], we present the numerical results only. Shown in figure 3a is the schematic phase diagram in the parameters space, $\epsilon - \tau'$ for $a = 1, b = 0$ (the results for other cases are similar to LS oscillators, eq. (5)). The curves, dashed and solid lines, are the contour plots of the largest Lyapunov exponent at $\Lambda_1 = 0.01$ and -0.01 respectively to indicate the transition from chaotic (C) to periodic and from periodic to AD regimes. In order to see further details of these transitions, we plot in figure 3b the spectrum of Lyapunov exponents as a function of coupling strength at fixed $\tau' = 0.1$. This indicates that, as coupling strength is increased from zero, the chaotic dynamics gets suppressed and becomes periodic (figure 3c). Note that the periodic orbit presented here is different from those reported in refs [1,11] in the sense that here the trajectories are always piecewise continuous due to the step-type of interaction. Therefore, it shows a small spread in the attractor (see inset in figure 3c). With further increase of coupling strength the dynamics goes from periodic to AD regime as shown in figure 3d. Although, in the AD regime, the amplitude of oscillation is quenched near a stable fixed point, there is always a small amplitude oscillation (see the inset of figure 3d) due to the switching of interactions [34].

The network of oscillators with active–passive interaction also show the occurrence of AD. We consider a representative example of globally $N = 10$ identical coupled LS and Rössler oscillators with interactions $\sum_{k=1,k\neq j}^{N}(aZ_k + bZ_k - Z_j)/(N-1)$ and $\sum_{k=1,k\neq j}^{N}(ax_k + bx_k - x_j)/(N-1)$ in eqs (5) and (13) respectively. Figures 4a and 4b show the schematic phase diagrams for $a = 1$ and $b = 0$ for the respective oscillators. These figures clearly show that AD occurs in a manner similar to that of the two coupled oscillators.

In summary, we observe that the introduction of a time-varying interaction in coupled oscillators can lead to amplitude death. Here, AD occurs even in identical oscillators

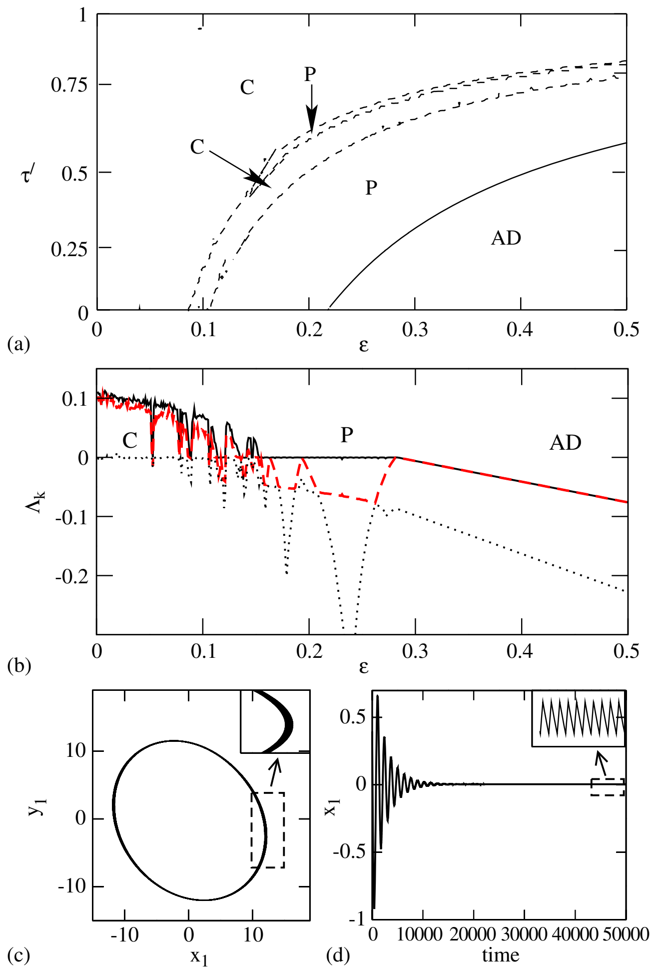

Figure 3. (**a**) The schematic phase diagram in the parameter space $\epsilon - \tau'$ for $a = 1$ and $b = 0$ of Rössler oscillators (eq. (13)). (**b**) The largest three Lyapunov exponents as a function for coupling strength at $\tau' = 0.3$. (**c**) The periodic attractor at $\epsilon = 0.25$ in phase-space $x_1 - y_1$ and (**d**) the transient AD trajectory at $\epsilon = 0.5$ as a function of time. The inset figures in (**c**) and (**d**) are the expansion of dashed boxes in the respective figures.

which are instantaneously coupled. In the coupling parameter space $\epsilon - \tau'$, a concave nonlinear locus of the bifurcation point (from periodic to AD) is observed. These results are found to be universal in the sense that these can occur in different types as well as in networks of oscillators. These results also suggest that the interaction, such as Case III, can be used to control the AD regime in parameter space.

In this paper, the study is restricted to well-known models of dynamical systems. However, as the results are quite generic [35], these results can be applied to other systems as

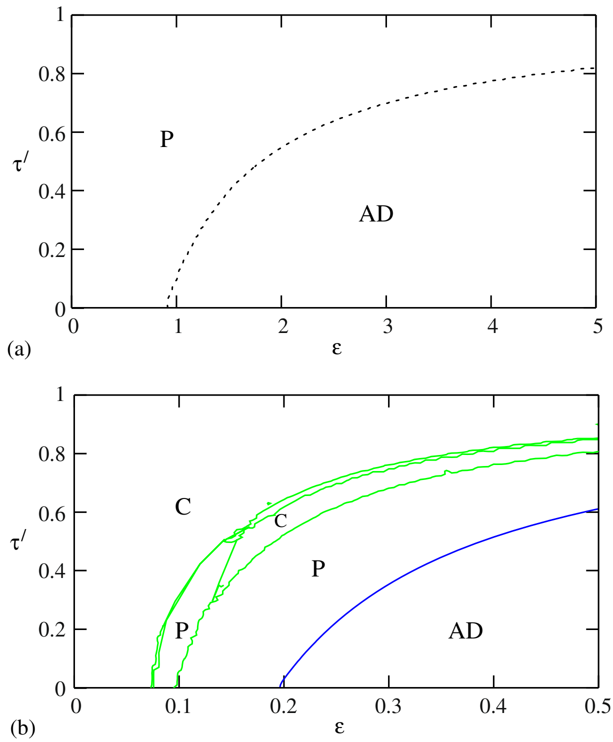

Figure 4. Schematic phase diagrams in the parameter space $\epsilon - \tau'$ for network of $N = 10$ (**a**) Landau–Stuart and (**b**) Rössler oscillators for $a = 1$ and $b = 0$.

well. AD can be of considerable importance in controlling oscillatory dynamics. Therefore, the present results are of potential utility in devising appropriate design strategies wherever the oscillation suppression is required, for example in lasers [36,37].

In this paper we considered the interaction as a step-type of function (eq. (3)) only. However, the use of some other time-varying modulating functions can be quite interesting too. For instance, $H(t) = \sin(t)$ also leads the system into a regime of AD (results are not reported here). The simplicity of the modulating function used in the paper makes it a good example to realize in an experiment. It would also be interesting to explore the behaviour of such a time-varying modulation in other possible coupling schemes.

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