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Projective synchronization of chaotic systems with bidirectional nonlinear coupling

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Abstract. This paper presents a new scheme for constructing bidirectional nonlinear coupled chaotic systems which synchronize projectively. Conditions necessary for projective synchronization (PS) of two bidirectionally coupled chaotic systems are derived using Lyapunov stability theory. The proposed PS scheme is discussed by taking as examples the so-called unified chaotic model, the Lorenz–Stenflo system and the nonautonomous chaotic Van der Pol oscillator. Numerical simulation results are presented to show the efficiency of the proposed synchronization scheme.

Keywords. Projective synchronization; unified chaotic system; Lyapunov stability theory.

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1. Introduction

The study of chaos in the last four decades had a tremendous impact on the foundations of science and engineering. One of the most exciting recent developments in nonlinear dynamics is the discovery of synchronized chaos. The possibility of synchronization in a coupled chaotic system composed of identical chaotic oscillators was first reported by Fujisaka and Yamada [1,2] and later by Pecora and Carroll [3]. Synchronization is a universal phenomenon in a variety of natural and engineering systems [4]. Over the past two decades, chaos synchronization has received much attention because of its fundamental importance in nonlinear dynamics and potential applications in laser dynamics, electronic circuits, chemical and biological systems and secure communications [5]. Different regimes of synchronization, e.g. complete synchronization [3], generalized synchronization [5–8], phase synchronization [9], lag synchronization [10], antisynchronization [11], projective synchronization [12,17] etc. of chaotic oscillators, have been theoretically investigated and experimentally observed. In 2004, Yu and Zhang [13] have presented

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synchronization of two coupled chaotic systems with bidirectional linear error feedback. Again, Yu [14] has observed synchronization of linearly bidirectional coupled chaotic system with time delay. Tarai *et al* [15] have studied synchronization of bidirectionally coupled chaotic Chen systems with delay. Very recently, Khan [16] has looked into the generalized synchronization of bidirectionally coupled chaotic systems via linear transformation. Projective synchronization (PS) was introduced by Mainieri and Rehacek in 1999 [17]. Li and Xu [18] investigated stability criterion for PS in three-dimensional chaotic systems. Wen and Xu [19] have analysed in detail a PS scheme for two coupled chaotic dynamical systems

$$\dot{X} = F(X) \qquad \text{(drive system)}$$

and

$$\dot{Y} = G(X, Y) \qquad \text{(response system)}.$$

In PS, the drive and response vectors synchronize up to a scaling factor, i.e. the vectors become proportional if $\lim_{t\to\infty} |X - \alpha Y| = 0$, where α is a nonsingular constant diagonal matrix. Jia [20] reported PS of coupled new hyperchaotic Lorenz systems. Du *et al* [21] have investigated PS of different chaotic systems with uncertain parameters. Very recently, Tarai and Khan [22] discussed the PS scheme for chaotic systems using backstepping design.

Most of the physical and biological systems are bidirectionally and nonlinearly coupled. But very few cases of PS of bidirectionally coupled chaotic systems were reported, which motivates us to study the PS of bidirectionally coupled chaotic systems. In this paper, we introduce a new scheme for constructing bidirectionally coupled nonlinear chaotic systems which synchronize projectively. Sufficient conditions for PS of two bidirectionally coupled chaotic systems are derived. We discuss the proposed theory by considering two bidirectionally coupled unified chaotic systems, Lorenz–Stenflo (LS) systems and the chaotic Van der Pol–Duffing oscillators. Finally, simulation results are presented and discussed.

2. Synchronization of bidirectionally coupled nonlinear chaotic systems

Any chaotic system can be written in the following form:

$$\dot{X} = AX + \Psi(X), \tag{1}$$

where $X \in R^n$ is the state vector, A is a $n \times n$ matrix and Ψ is a continuous nonlinear function from R^n to R^n. Now we introduce the following bidirectional coupling scheme:

$$\dot{X} = AX + \Lambda\Psi(X) + \Lambda\Phi(Y) + D_1(\Lambda Y - X), \tag{2}$$

$$\dot{Y} = AY + \Psi(X) + \Phi(Y) + D_2(\Lambda^{-1}X - Y), \tag{3}$$

where $Y \in R^n$ is the state vector of the response system and Φ is a continuous nonlinear function from R^n to R^n. Λ is a $n \times n$ nonsingular diagonal matrix and D_1, D_2 are $n \times n$ diagonal matrices.

Let $E(t) = X - \Lambda Y$. Therefore,

$$\dot{E}(t) = \dot{X} - \Lambda\dot{Y} = (A - D_1 - D_2)E(t) \tag{4}$$

when Λ commutes with the matrices A, D_1 and D_2. In order to realize the chaos synchronization between two bidirectionally coupled nonlinear systems, we have to choose suitable matrices D_1 and D_2 such that $\lim_{t\to\infty} E(t) = 0$.

Theorem. *If there exists a positive definite matrix M and a constant $\epsilon > 0$ such that*

$$(A - D_1 - D_2)^T M + M(A - D_1 - D_2) \leq -\epsilon I \tag{5}$$

then dynamical system (4) is globally and asymptotically stable at the origin.

Proof. We choose the Lyapunov function as

$$L(t) = E^T M E,$$

where M is a positive definite matrix. Therefore, $L(t)$ has a positive definite quadratic form. Now,

$$\begin{aligned}\dot{L}(t) &= \dot{E}^T M E + E^T M \dot{E}\\ &= E^T[(A - D_1 - D_2)^T M + M(A - D_1 - D_2)]E\\ &\leq -\epsilon E^T E\\ &< 0.\end{aligned}$$

Hence, by Lyapunov stability theory, the error dynamical system (4) is globally and asymptotically stable at the origin. Hence, the coupled systems (2) and (3) will synchronize projectively. Different types of coupling can be constructed for suitable choice of A, D_1 and D_2.

3. Examples

In this section, we shall discuss our methods for different chaotic systems (autonomous and nonautonomous).

3.1 *PS of coupled unified chaotic systems*

A unified chaotic system with continuous periodic switch between Lorenz, Chen and Lu systems is presented by Lu and Wu [23]. The unified chaotic system can be described by the following system of differential equations:

$$\begin{aligned}\dot{x} &= (25a + 10)(y - x),\\ \dot{y} &= (28 - 35a)x - y - xz,\\ \dot{z} &= xy - \frac{8 + a}{3}z,\end{aligned} \tag{6}$$

where $a \in [0, 1]$. When $a = 0, 0.8, 1$, this system represents the Lorenz chaotic system, Lu chaotic system and Chen chaotic system respectively. Practically, unified chaotic system is chaotic for any $a \in [0, 1]$ (figure 1). The system (6) can be rewritten as

$$\dot{X} = AX + \Psi(X), \tag{7}$$

where

$$A = \begin{pmatrix} -(25a+10) & (25a+10) & 0 \\ 28-35a & -1 & 0 \\ 0 & 0 & -(8+a)/3 \end{pmatrix}$$

and

$$\Psi(X) = \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix}.$$

Following eqs (2) and (3) the master–slave systems are constructed for system (6) as

$$\begin{aligned} \dot{x_1} &= (25a+10)(x_2 - x_1) + d_{11}(\lambda_{11}y_1 - x_1) \\ \dot{x_2} &= (28-35a)x_1 - x_2 - \lambda_{22}(x_1x_3 + y_1y_3) + d_{12}(\lambda_{22}y_2 - x_2) \\ \dot{x_3} &= -\frac{8+a}{3}x_3 + \lambda_{33}(x_1x_2 + y_1y_2) + d_{13}(\lambda_{33}y_3 - x_3) \end{aligned} \tag{8}$$

and

$$\begin{aligned} \dot{y_1} &= (25a+10)(y_2 - y_1) + d_{21}(\mu_{11}x_1 - y_1) \\ \dot{y_2} &= (28-35a)y_1 - y_2 - x_1x_3 - y_1y_3 + d_{22}(\mu_{22}x_2 - y_2) \\ \dot{y_3} &= -\frac{8+a}{3}y_3 + x_1x_2 + y_1y_2 + d_{23}(\mu_{33}x_3 - y_3) \end{aligned} \tag{9}$$

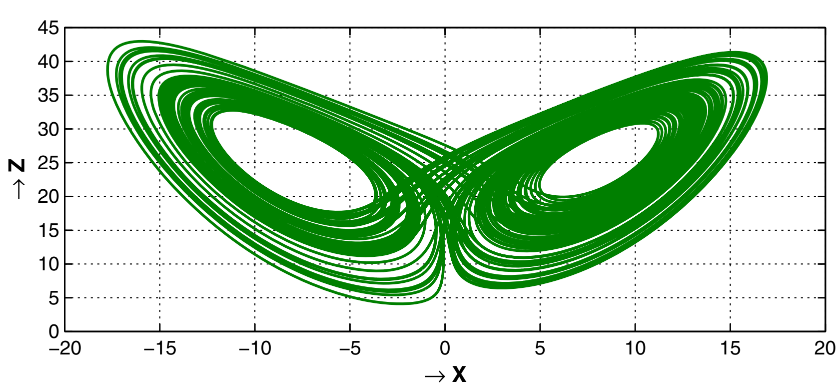

Figure 1. Chaotic attractor of the unified chaotic system for $a = 0$ (Lorenz system).

where

$$D_1 = \mathrm{diag}(d_{11}, d_{12}, d_{13}),$$

$$D_2 = \mathrm{diag}(d_{21}, d_{22}, d_{23}),$$

$$\Lambda = \begin{pmatrix} \lambda_{11} & 0 & 0 \\ 0 & \lambda_{22} & 0 \\ 0 & 0 & \lambda_{33} \end{pmatrix}$$

and

$$\Lambda^{-1} = \begin{pmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{pmatrix}.$$

Here, the PS error will follow the following dynamical system:

$$\dot{E}(t) = (A - D_1 - D_2)E(t) \quad \text{with } E = (e_1, e_2, e_3)^T.$$

Choose a positive definite symmetric constant matrix $M = \mathrm{diag}(m_1, m_2, m_3)$, with $m_i > 0\,(i = 1, 2, 3)$ and any constant $\epsilon > 0$, then

$$\begin{aligned} H &= (A - D_1 - D_2)^T M + M(A - D_1 - D_2) + \epsilon I \\ &= \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{12} & h_{22} & 0 \\ 0 & 0 & h_{33} \end{pmatrix}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} h_{11} &= -2m_1(25a + 10 + d_{11} + d_{21}) + \epsilon \\ h_{12} &= m_1(25a + 10) + m_2(28 - 35a) \\ h_{22} &= -2m_2(1 + d_{12} + d_{22}) + \epsilon \\ h_{33} &= -2m_3\left(\frac{8 + a}{3} + d_{13} + d_{23}\right) + \epsilon. \end{aligned} \tag{11}$$

Now, the matrix (10) is negative definite if and only if its subdeterminants satisfy

$$\begin{aligned} &\Delta_1 = -2m_1(25a+10+d_{11}+d_{21})+\epsilon < 0 \\ &\Delta_2 = \mathrm{Det}\begin{pmatrix} -2m_1(25a+10+d_{11}+d_{21})+\epsilon & m_1(25a+10)+m_2(28-35a) \\ m_1(25a+10)+m_2(28-35a) & -2m_2(1+d_{12}+d_{22})+\epsilon \end{pmatrix} > 0 \\ &\Delta_3 = \mathrm{Det}(H) < 0. \end{aligned} \tag{12}$$

Under these conditions, the coupled unified chaotic systems synchronize projectively. The matrix Λ should be such a diagonal matrix which commutes with the matrix A. Here one such choice for Λ is $\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & l \end{pmatrix}$ with $k, l \in R$, $k \neq 0$ and $l \neq 0$.

3.2 *PS of coupled Lorenz–Stenflo systems*

Now we consider the following four-coupled nonlinear autonomous first-order differential equations

$$\begin{aligned}\dot{x} &= \alpha(y-x)+\gamma w\\ \dot{y} &= x(r-z)-y\\ \dot{z} &= xy-bz\\ \dot{w} &= -x-\alpha w\end{aligned} \tag{13}$$

which are formulated by Stenflo [24]. Here $r(>0)$, $\alpha(>0)$, $\gamma(>0)$ and $b(>0)$ are parameters. With the following parameter values $\alpha = 1.0$, $b = 0.7$, $\gamma = 1.5$ and $r = 26.0$, the LS system exhibits chaotic motion. The system (13) can be written as

$$\dot{X} = AX + \Psi(X), \tag{14}$$

where

$$A = \begin{pmatrix} -\alpha & \alpha & 0 & \gamma \\ r & -1 & 0 & 0 \\ 0 & 0 & -b & 0 \\ -1 & 0 & 0 & -\alpha \end{pmatrix}$$

and

$$\Psi(X) = \begin{pmatrix} 0 \\ -xz \\ xy \\ 0 \end{pmatrix}.$$

According to eqs (2) and (3) the master–slave systems with nonlinear bidirectional coupling are

$$\begin{aligned}\dot{x_1} &= \alpha(x_2-x_1)+\gamma x_4+d_{11}(\lambda_{11}y_1-x_1)\\ \dot{x_2} &= rx_1-x_2-\lambda_{22}(x_1x_3+y_1y_3)+d_{12}(\lambda_{22}y_2-x_2)\\ \dot{x_3} &= -bx_3+\lambda_{33}(x_1x_2+y_1y_2)+d_{13}(\lambda_{33}y_3-x_3)\\ \dot{x_4} &= -x_1-\alpha x_4+d_{14}(\lambda_{44}y_4-x_4)\end{aligned} \tag{15}$$

and

$$\begin{aligned}\dot{y_1} &= \alpha(y_2-y_1)+\gamma y_4+d_{21}(\mu_{11}x_1-y_1)\\ \dot{y_2} &= ry_1-y_2-(x_1x_3+y_1y_3)+d_{22}(\mu_{22}x_2-y_2)\\ \dot{y_3} &= -by_3+(x_1x_2+y_1y_2)+d_{23}(\mu_{33}x_3-y_3)\\ \dot{y_4} &= -y_1-\alpha y_4+d_{24}(\mu_{44}x_4-y_4)\end{aligned} \tag{16}$$

with $D_1 = \mathrm{diag}(d_{11}, d_{12}, d_{13}, d_{14})$, $D_2 = \mathrm{diag}(d_{21}, d_{22}, d_{23}, d_{24})$ and

$$\Lambda = \begin{pmatrix} \lambda_{11} & 0 & 0 & 0 \\ 0 & \lambda_{22} & 0 & 0 \\ 0 & 0 & \lambda_{33} & 0 \\ 0 & 0 & 0 & \lambda_{44} \end{pmatrix}$$

and

$$\Lambda^{-1} = \begin{pmatrix} \mu_{11} & 0 & 0 & 0 \\ 0 & \mu_{22} & 0 & 0 \\ 0 & 0 & \mu_{33} & 0 \\ 0 & 0 & 0 & \mu_{44} \end{pmatrix}.$$

From (15) and (16), the synchronization error system becomes

$$\dot{E}(t) = (A - D_1 - D_2)E(t) \quad \text{with } E = (e_1, e_2, e_3, e_4)^T.$$

Choose a positive definite symmetric constant matrix $M = \mathrm{diag}(m_1, m_2, m_3, m_4)$, with $m_i > 0$ $(i = 1, 2, 3, 4)$ and any constant $\epsilon > 0$ then

$$\begin{aligned} H &= (A - D_1 - D_2)^T M + M(A - D_1 - D_2) + \epsilon I \\ &= \begin{pmatrix} h_{11} & h_{12} & 0 & h_{14} \\ h_{21} & h_{22} & 0 & 0 \\ 0 & 0 & h_{33} & 0 \\ 0 & 0 & 0 & h_{44} \end{pmatrix}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} h_{11} &= 2m_1(\alpha - d_{11} - d_{21}) + \epsilon, \quad h_{12} = m_2 r + m_1 \alpha \\ h_{14} &= m_1 \gamma, \quad h_{21} = m_2 r + m_1 \alpha, \quad h_{22} = -2m_2(1 + d_{12} + d_{22}) + \epsilon \\ h_{33} &= -2m_3(b + d_{13} + d_{23}) + \epsilon, \quad h_{44} = -2m_4(\alpha + d_{14} + d_{24}) + \epsilon. \end{aligned}$$

According to the basic matrix theory, the matrix H is negative definite if and only if its subdeterminants satisfy

$$\begin{aligned} \Delta_1 &= h_{11} < 0 \\ \Delta_2 &= \mathrm{Det}\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} > 0 \\ \Delta_3 &= \mathrm{Det}\begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & h_{33} \end{pmatrix} < 0 \\ \Delta_4 &= \mathrm{Det}(H) > 0. \end{aligned} \tag{18}$$

Under these conditions the coupled chaotic Lorenz–Stenflo systems synchronize projectively. One choice for Λ is $\begin{pmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & l & 0 \\ 0 & 0 & 0 & k \end{pmatrix}$, where $k, l \in R$, $k \neq 0$ and $l \neq 0$, which commutes with the matrix A.

3.3 *PS of coupled 2D nonautonomous chaotic systems*

It is well known that Van der Pol–Duffing oscillator has chaotic dynamics under suitable external time periodic forcing. Here, we consider the forced Van der Pol–Duffing oscillator as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu(1 - x^2)y - \alpha x - \beta x^3 + f\cos(\omega t). \end{aligned} \tag{19}$$

Here, f is the amplitude of forcing and ω is the frequency of forcing. This system exhibits chaotic motion for $\mu = 0.1, \alpha = -0.5, \beta = 0.5, f = 0.14$ and $\omega = 0.53$. Now, following the rule for the construction of master–slave system, we obtain the master system as

$$\begin{aligned}\dot{x_1} &= x_2 + \lambda_{12}(-\mu x_1^2 x_2 - \mu y_1^2 y_2 - \beta x_1^3 - \beta y_1^3 + 2f\cos(\omega t)) \\ &\quad + d_{11}(\lambda_{11} y_1 - x_1) \\ \dot{x_2} &= -\alpha x_1 + \mu x_2 + \lambda_{22}(-\mu x_1^2 x_2 - \mu y_1^2 y_2 - \beta x_1^3 - \beta y_1^3 + 2f\cos(\omega t)) \\ &\quad + d_{12}(\lambda_{22} y_2 - x_2)\end{aligned} \tag{20}$$

and the slave system as

$$\begin{aligned}\dot{y_1} &= y_2 + d_{21}(\mu_{11} x_1 - y_1) \\ \dot{y_2} &= -\alpha y_1 + \mu y_2 - \mu x_1^2 x_2 - \mu y_1^2 y_2 - \beta x_1^3 - \beta y_1^3 \\ &\quad + 2f\cos(\omega t) + d_{22}(\mu_{22} x_2 - y_2)\end{aligned} \tag{21}$$

with

$$D_1 = \mathrm{diag}(d_{11}, d_{12}), \quad D_2 = \mathrm{diag}(d_{21}, d_{22}),$$
$$\Lambda = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix}, \quad \Lambda^{-1} = \begin{pmatrix} \mu_{11} & 0 \\ 0 & \mu_{22} \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & 1 \\ -\alpha & \mu \end{pmatrix}.$$

Now, by Lyapunov stability theorem, the error dynamical system becomes

$$\dot{E}(t) = (A - D_1 - D_2)E(t), \quad \text{where } E = (e_1, e_2)^T.$$

Choose a positive definite symmetric constant matrix $M = \mathrm{diag}(m_1, m_2)$, with $m_i > 0$ $(i = 1, 2)$ and any constant $\epsilon > 0$. Then

$$\begin{aligned}H &= (A - D_1 - D_2)^T M + M(A - D_1 - D_2) + \epsilon I \\ &= \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},\end{aligned} \tag{22}$$

where

$$\begin{aligned}h_{11} &= -2m_1(d_{11} + d_{21}) + \epsilon, \qquad h_{12} = m_1 - m_2\alpha, \\ h_{21} &= m_1 - m_2\alpha, \qquad h_{22} = -2m_2(d_{12} + d_{22} - \mu) + \epsilon.\end{aligned}$$

Now the matrix H will be negative definite if

$$\begin{aligned}\Delta_1 &= h_{11} < 0 \\ \Delta_2 &= \mathrm{Det}(H) > 0.\end{aligned} \tag{23}$$

Under the above conditions the coupled nonautonomous chaotic Van der Pol–Duffing oscillators synchronize projectively. One choice for Λ is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, with $k \in R$ and $k \neq 0$, which commutes with the matrix A.

4. Results

The fourth-order Runge–Kutta method is used to solve coupled system of differential equations. For all simulations we choose matrix M as the identity matrix of proper size and $\epsilon = 0.1$. Firstly, in figure 1 we have drawn the chaotic attractor of the unified chaotic system for $a = 0$, i.e. for Lorenz system. In figure 2 we have drawn the time evolution of the synchronization errors of the unified chaotic systems for $a = 0$, i.e. for coupled Lorenz systems taking $D_1 = \text{diag}(12, 10, 0)$ and $D_2 = \text{diag}(1, 10, 0)$. Figure 2 shows that error goes to zero asymptotically and therefore the two chaotic systems synchronize projectively. We draw the time evolution of the synchronization errors of unified chaotic system for $a = 0.8$, i.e. for Lu system in figure 3 taking $D_1 = \text{diag}(4, 10, 0)$ and $D_2 = \text{diag}(1, 1, 0)$. Clearly, the figure establishes the occurrence of projective synchronization. In figure 4, we plot the time evolution of synchronization errors of the unified chaotic system for $a = 1.0$, i.e. for Chen system taking $D_1 = \text{diag}(3, 8, 0)$ and $D_2 = \text{diag}(1, 3, 0)$. The initial values of the errors are taken as $e_1(0) = 12.0, e_2(0) = -3.0, e_3(0) = -14.0$ in all the above three cases. Figures show that errors go to zero asymptotically, i.e. the two chaotic systems synchronize

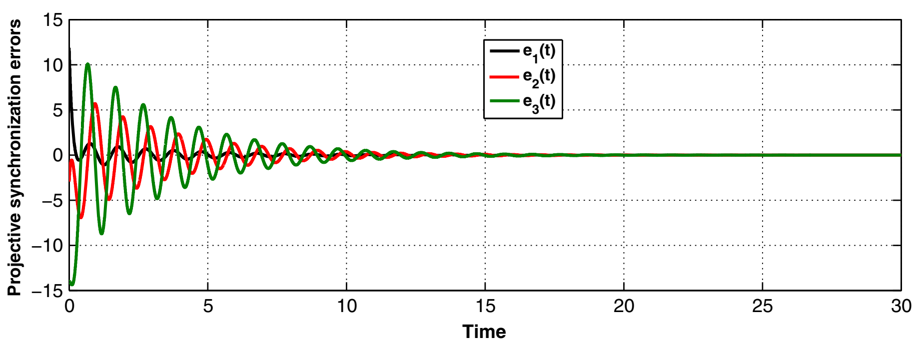

Figure 2. Time evolution of the projective synchronization errors for coupled Lorenz systems ($a = 0.0$).

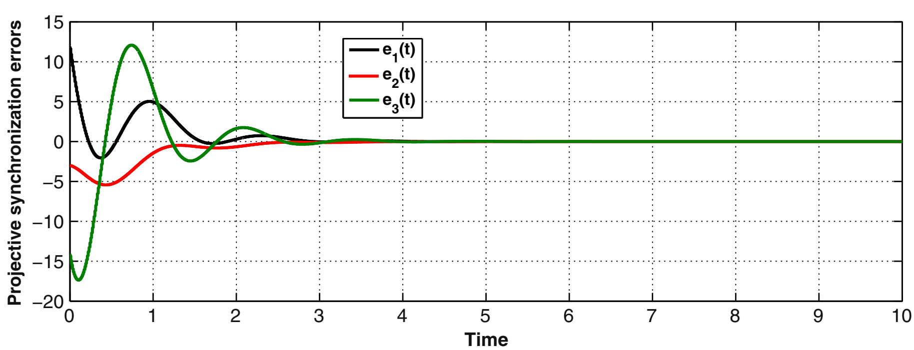

Figure 3. Time evolution of the projective synchronization errors for coupled Lu systems ($a = 0.8$).

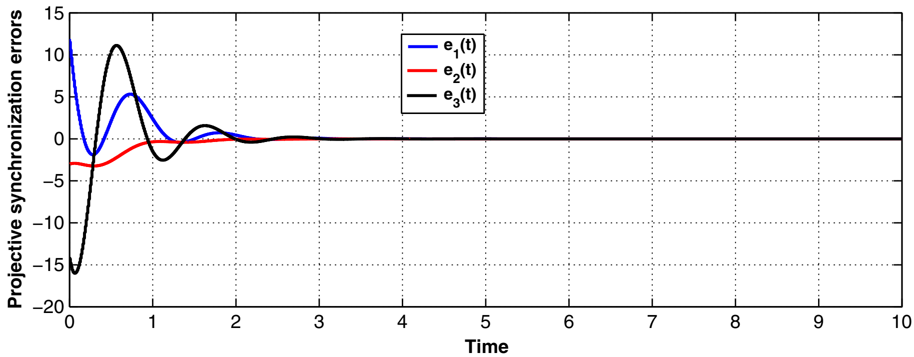

Figure 4. Time evolution of the projective synchronization errors for coupled Chen systems ($a = 1.0$).

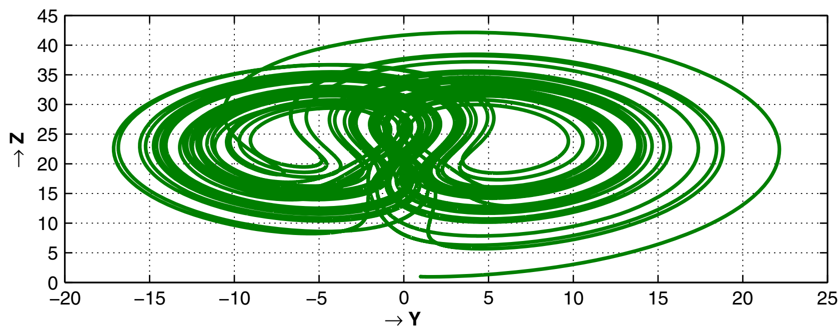

Figure 5. Phase diagram of the chaotic Lorenz–Stenflo system for $\alpha = 1.0$, $\beta = 0.7$, $\gamma = 1.5$ and $r = 26.0$ in the yz plane.

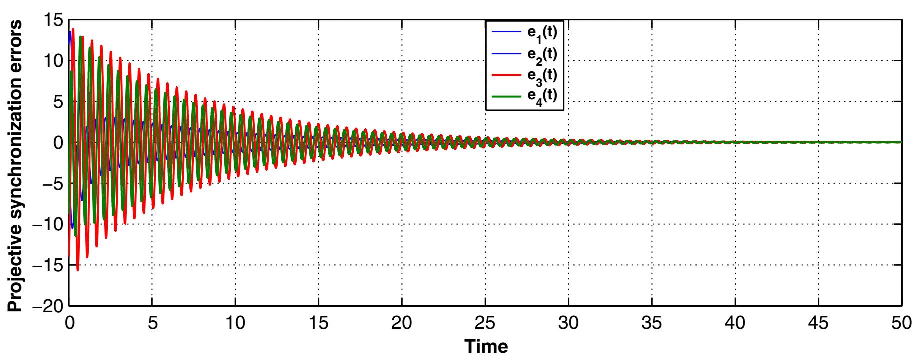

Figure 6. Time evolution of the projective synchronization errors for the coupled Lorenz–Stenflo systems.

projectively. Chaotic phase portrait of Lorenz–Stenflo system is drawn in figure 5 for $\alpha = 1.0, b = 0.7, \gamma = 1.5$ and $r = 26.0$. We draw the time evolution of the synchronization error for four-dimension and autonomous Lorenz–Stenflo system in figure 6

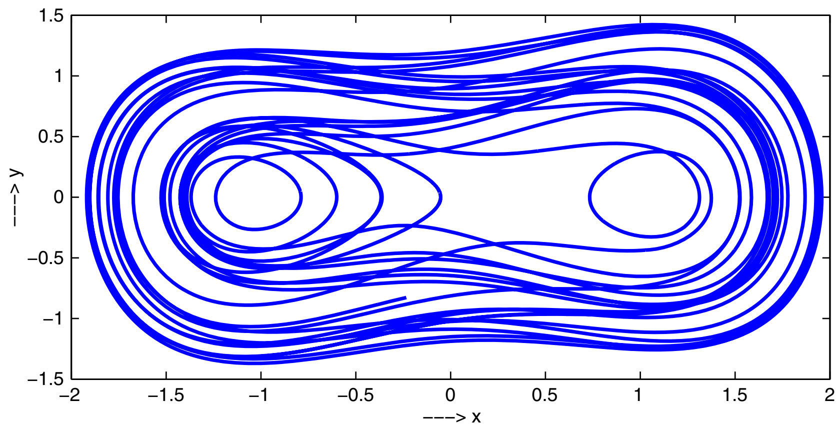

Figure 7. Phase diagram of the chaotic Van der Pol–Duffing oscillator for $\mu = 0.1$, $\alpha = -0.5$, $\beta = 0.5$, $f = 0.14$ and $\omega = 0.53$.

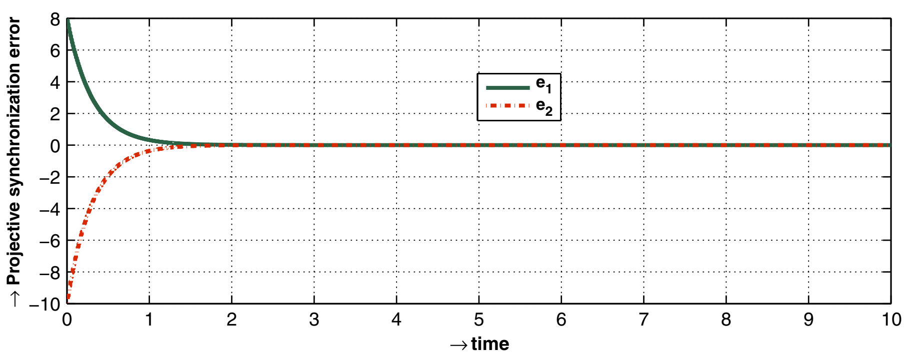

Figure 8. Time evolution of the projective synchronization errors for the coupled Van der Pol–Duffing oscillators.

taking $D_1 = \mathrm{diag}(3.6, 1.6, 3.3, 2.0)$ and $D_2 = \mathrm{diag}(3.0, 5.0, 4.0, 4.0)$. The initial values of the errors are taken as $e_1(0) = 0.0, e_2(0) = 1.0, e_3(0) = 1.0, e_4(0) = 2.0$. Figure 6 confirms that master–slave systems synchronize projectively. The phase diagram of the periodically forced Van del Pol–Duffing oscillator (chaotic) is plotted in figure 7 for $\mu = 0.1, \alpha = -0.5, \beta = 0.5, f = 0.14$ and $\omega = 0.53$. We draw the projective synchronization error for nonautonomous coupled chaotic Van der Pol–Duffing oscillators in figure 8 with $D_1 = \mathrm{diag}(1, 1.5)$ and $D_2 = \mathrm{diag}(1, 1)$ taking initial values of the errors as $e_1(0) = 8.0, e_2(0) = -10$. Synchronization errors going to zero asymptotically implies that two chaotic systems synchronize projectively.

5. Conclusions

A novel nonlinear and bidirectional coupling scheme was introduced to achieve projective synchronization between the coupled systems. Based on Lyapunov stability theory, a set of necessary conditions were derived for projective synchronization of coupled chaotic systems. The proposed scheme can be used for projective synchronization of autonomous as well as nonautonomous chaotic systems. We discussed the proposed projective synchronization scheme by taking as examples the unified chaotic system, Lorenz–Stenflo system and periodically forced Van der Pol–Duffing oscillator system. Numerical simulation results were presented to show the efficiency of the proposed scheme.

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