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Spin–momenta entanglement in moving frames: Properties of von Neumann entropy

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Abstract. The fact that spin–momentum of massive particles become entangled (disentangled) as seen by moving observers, is used to investigate the properties of von Neumann entropy, as a measure of spin–momentum entanglement. To do so, we partition the total Hilbert space into momentum and spin subspaces so that the entanglement occurs between total spin states and total momenta of two spin-$\frac{1}{2}$ particles. Assuming that the occurrence of spin–momentum states is determined by Gaussian probability distributions, we show that the degree of entanglement ascends for small rapidities, reaches a maximum and diminishes at high rapidity. We further report how the characteristics of this behaviour vary as the widths of distributions change. In particular, a separable state, resulting from equal distribution widths, indeed becomes entangled in moving frames.

Keywords. Relativistic entanglement; reduced von Neumann entropy; Gaussian distributions.

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1. Introduction

It is well known that any information processing scheme involves a part emitting the information and a part measuring the received information. The emitted information, as a quantum state, is described by a density operator ρ, which is a positive one, while the detection is described by other positive operators d_i. The probability that the ith detector registers a signal is then given by $\mathrm{Tr}(d_i\rho)$ [1–5]. Moreover, if the detectors are sensitive to one set of degrees of freedom, the detection probability is obtained by using reduced density operator (traced over all the disregarded degrees of freedom) [6–8]. It is also well established that the von Neumann entropy and its reduced one, measure the availability, as well as locality, of information [9–11]. Expressed differently, the von Neumann entropy measures the stored qubits of information per states of the combined system while the reduced one quantifies the qubits per states that may be stored in one of the subsystems [1]. It is then obvious that a deeper understanding of von Neumann entropy, as compared to, for instance, concurrence, is vital for theories of information [12]. It is also commonly accepted that, generally, the von Neumann entropy (the measure of information availability) is relativistically covariant, while the reduced one (the measure of information locality) is not [13,14]. The non-locality of information, a purely quantum mechanical trend, reflects itself into entanglement whose degree is then determined by the reduced von Neumann entropy [15,16]. Thus the main aim of the present article is to elucidate the properties of the reduced von Neumann entropy, and consequently, the entanglement of spin–momentum states, for a system of two spin-$\frac{1}{2}$ particles, as seen by moving observers.

The entanglement, or disentanglement, between spin and spatial degrees of freedom for massive particles depends upon the frame in which it is measured because under a Lorentz transformation the momenta are boosted, while the spin states are Wigner-rotated [17,18]. Since the Wigner rotations of spin states depend on the momentum states, they rotate differently and entanglement between spin and spatial degrees of freedom does occur [19,20]. This point reflects itself into the partitioning of the system into a part containing the spin states and a second one with the momentum states alone. The entanglement is thus measured by the reduced von Neumann entropy. The reduced density matrix, however, forms a mixed ensemble, so that if the entanglement between spin states is desired, one has to resort to measures such as concurrence, negativity, etc. [13,14,21–23]. Since the main interest in the present work is the behaviour of entanglement between the spin and momentum states, we do not enter into the subtleties surrounding the spin measurement [24,25]. We thus employ the von Neumann entropy as the measure of spin–momentum entanglement. Even though the entanglement of spin-spatial degrees of freedom has been investigated, for a system of two massive spin-$\frac{1}{2}$ particles in which the spin–momentum states occur with specific probability distributions, such a study has not been reported. In what follows, therefore, we present a thorough examination of the spin–momentum entanglement of such a system under the assumption that spin–momentum states occur with Gaussian probability distributions of different widths. Since the width (second moment) of each probability distribution is directly proportional to the number of momentum states available to that particle, the variation in the widths ratio is expected to drastically change the spin–momentum entanglement. We shall elaborate more on this point in the concluding section. It may be worth mentioning that in refs [13,14] the authors used Gaussian probability distributions of equal widths, while in refs [19,26–28] sharp distributions (single momentum states) are employed to elaborate on spin–spin entanglement. With the presupposition that the system is pure in the fixed frame, we proceed by Lorentz transforming the stateket and calculate the degree of entanglement as seen by a moving observer. In so doing, we obtain relatively simple expressions for the eigenvalues of the reduced density matrix which allows us to investigate the effect of rapidity as well as the distributions' widths on the entanglement. We further show that if the state in the fixed frame is a separable one, it becomes entangled in the moving frame. Moreover, from these simple expressions we conclude that as the ratio of widths increases the degree of entanglement, as expected, also increases. This conclusion stems from the fact that as the ratio of widths increases, more momentum states participate in the entanglement. The result of the present report clearly shows how the amount of information that may be stored in the momentum and spin subspaces varies under a Lorentz transformation.

This work is organized as follows. After the Introduction, §2 is devoted to the calculation of the reduced von Neumann entropy and the corresponding eigenvalues, as seen by a moving observer. We then adopt the results of §2 to the case of Gaussian probability distributions in §3. In this section we also discuss the effect of rapidity along with the distributions' widths on the spin–momentum entanglement. Finally, concluding remarks are made in §4.

2. Von Neumann entropy in moving frames

Our investigation of von Neumann entropy, consequently the entanglement, begins by considering two spin-1/2 particles of mass, m, with spin states $|s\rangle$ and $|\sigma\rangle$. Such a bipartite state, as viewed in a fixed (laboratory) frame may be expressed as

$$|\Psi\rangle = \sum_{s,\sigma} \int\int \mathrm{d}^3p\, \mathrm{d}^3q a_{s,\sigma}(\mathbf{p},\mathbf{q})|\mathbf{p},\mathbf{q},s,\sigma\rangle\,, \tag{1}$$

where $\mathbf{p}$ and $\mathbf{q}$ represent the 3-momenta of the two particles and momentum representation has been assumed. Physically, eq. (1) describes a state in which the spin states occur with momentum-dependent probabilities, $a_{s,\sigma}(\mathbf{p},\mathbf{q})$, such that $\sum_{s,\sigma}\int\int \mathrm{d}^3p\,\mathrm{d}^3q|a_{s,\sigma}(\mathbf{p},\mathbf{q})|^2 = 1$. In this normalization condition we have deliberately omitted factors of the form $\sqrt{p^2c^2+m^2c^4}$ which we include as we proceed [15]. In a frame, Λ, moving relative to the fixed one, with a velocity $\vec{v} = v\hat{n}$ (rapidity ζ), the state of eq. (1) experiences a boost and Wigner rotation [14,15] according to,

$$\begin{aligned}|\Psi\rangle^\Lambda =& \sqrt{\frac{p_0^\Lambda}{p_0}}\sqrt{\frac{q_0^\Lambda}{q_0}}\sum_{s,\sigma}\int\int \mathrm{d}^3p\,\mathrm{d}^3q a_{s,\sigma}(\mathbf{p}^\Lambda,\mathbf{q}^\Lambda)\\ &\times W(\mathbf{p},\mathbf{q},s,\sigma)|\mathbf{p}^\Lambda,\mathbf{q}^\Lambda,s,\sigma\rangle,\end{aligned} \tag{2}$$

where the boosted momenta are

$$\mathbf{P}_0^\Lambda = \mathbf{P}_0\cosh(\zeta) + (\mathbf{P}\cdot\mathbf{n})\sinh(\zeta) \tag{3}$$

and

$$\mathbf{P}^\Lambda = \mathbf{P} + [(\mathbf{P}\cdot\mathbf{n})\cosh(\zeta) + \mathbf{P}_0\sinh(\zeta) - (\mathbf{P}\cdot\mathbf{n})]\mathbf{n} \tag{4}$$

for $\mathbf{P} = \mathbf{p},\mathbf{q}$. The Wigner rotation $W(\mathbf{p},\mathbf{q},s,\sigma)$ in eq. (2), which acts upon the spin states, is given by

$$W(\mathbf{p},\mathbf{q},s,\sigma) = W(\mathbf{p},s)\otimes W(\mathbf{q},\sigma), \tag{5}$$

where

$$\begin{aligned}W(\mathbf{P},\eta) =& \frac{1}{\sqrt{(P_0^\Lambda+mc)(P_0+mc)}}[(P_0+mc)\cosh(\zeta/2)\\ &+[\mathbf{P}\cdot\mathbf{n} - i\eta(\mathbf{P}\times\mathbf{n})]\sinh(\zeta/2)]\end{aligned} \tag{6}$$

for $\eta = s,\sigma$. Forming the transformed density operator $\rho^\Lambda = |\Psi\rangle^{\Lambda\,\Lambda}\langle\Psi|(\rho = |\Psi\rangle\langle\Psi|$ in the fixed frame) and tracing over momentum states gives,

$$\begin{aligned}\rho_{\text{spin}}^\Lambda =& \left(\frac{p_0^\Lambda q_0^\Lambda}{p_0 q_0}\right)\sum_{s,\acute{s}}\sum_{\sigma,\sigma'}\int\int \mathrm{d}^3p\,\mathrm{d}^3q a_{s,\sigma}(\mathbf{p}^\Lambda,\mathbf{q}^\Lambda)a^*_{s',\sigma'}(\mathbf{p}^\Lambda,\mathbf{q}^\Lambda)\\ &\times W(\mathbf{p},\mathbf{q},s,\sigma)|s,\sigma\rangle\langle s',\sigma'|W^\dagger(\mathbf{p},\mathbf{q},s',\sigma').\end{aligned} \tag{7}$$

The von Neumann entropy, as a measure of entanglement [9,10], is then obtained from

$$S = -\mathrm{Tr}[\rho_{\mathrm{spin}}^{\Lambda} \log(\rho_{\mathrm{spin}}^{\Lambda})] = -\sum_{i=1} \lambda_i \log(\lambda_i), \tag{8}$$

where λ_is are the eigenvalues of $\rho_{\mathrm{spin}}^{\Lambda}$. As a concrete example, let us assume that in the fixed frame the momentum-spin states in eq. (1) are of the form

$$|\Psi\rangle = \int\int \mathrm{d}^3 p\ \mathrm{d}^3 q a_{++}(\mathbf{p}, \mathbf{q}, +, +)|\mathbf{p}, \mathbf{q}, +, +\rangle + a_{--}(\mathbf{p}, \mathbf{q}, -, -)|\mathbf{p}, \mathbf{q}, -, -\rangle, \tag{9}$$

whose spin state is a Bell one with maximal spin–spin entanglement [13,14]. It is noted that by adjusting the amplitudes, $a_{\pm,\pm}$, it is possible to produce any desired number of qubits to store information [1]. To pursue an analytic expression, we have assumed that both momenta are along the y-axis, normal to the direction of the boost; the x-axis (the normalization of eq. (7) is based on this assumption). For the state of eq. (9), the reduced (spin–spin) density operator is easily calculated, giving,

$$\rho_{\mathrm{spin}}^{\Lambda} = F_1|+,+\rangle\langle +,+| + F_2|+,+\rangle\langle -,-| + F_2^*|-,-\rangle\langle +,+| + (1-F_1)|-,-\rangle\langle -,-|, \tag{10}$$

where the transformed probability amplitudes are,

$$F_1 = A\left(\frac{p_0^{\Lambda} q_0^{\Lambda}}{p_0 q_0}\right) \int\int \mathrm{d}^3 p\ \mathrm{d}^3 q |a_{++}(\mathbf{p}^{\Lambda}, \mathbf{q}^{\Lambda})|^2 f_1(\mathbf{p}, \mathbf{q}, \zeta) \tag{11}$$

and

$$F_2 = A\left(\frac{p_0^{\Lambda} q_0^{\Lambda}}{p_0 q_0}\right) \int\int \mathrm{d}^3 p\ \mathrm{d}^3 q a_{++}(\mathbf{p}^{\Lambda}, \mathbf{q}^{\Lambda}) a_{--}^*(\mathbf{p}^{\Lambda}, \mathbf{q}^{\Lambda}) f_2(\mathbf{p}, \mathbf{q}, \zeta), \tag{12}$$

where $A = [(p_0 + mc)(p_0^{\Lambda} + mc)(q_0 + mc)(q_0^{\Lambda} + mc)]^{-1}$. In eqs (11) and (12) the momentum distributions have been modified, under the Wigner rotation, by the factors $f_1(\mathbf{p}, \mathbf{q}, \zeta)$ and $f_2(\mathbf{p}, \mathbf{q}, \zeta)$ defined by

$$f_1(\mathbf{p}, \mathbf{q}, \zeta) = \prod_{\mathbf{P}=\mathbf{p},\mathbf{q}} [(\mathbf{P}_0 + mc)^2 \cosh^2(\zeta/2) + \mathbf{P}^2 \sinh^2(\zeta/2)] \tag{13}$$

and

$$f_2(\mathbf{p}, \mathbf{q}, \zeta) = \prod_{\mathbf{P}=\mathbf{p},\mathbf{q}} [(\mathbf{P}_0 + mc)\cosh(\zeta/2) + i\mathbf{P}\sinh(\zeta/2)]^2. \tag{14}$$

Diagonalizing the matrix representation of eq. (10), the eigenvalues of the reduced spin density operator are readily obtained as

$$\lambda_{1,2} = \frac{1}{2}[1 \pm \sqrt{1 - 4F_1 + 4F_1^2 + 4|F_2|^2}], \quad \lambda_{3,4} = 0. \tag{15}$$

From eqs (8) and (15) it is observed that maximal spin–momentum entanglement occurs for $F_1 = \frac{1}{2}$ and $F_2 = 0$ and vanishes for either $F_1 = 1$ or 0, $F_2 = 0$, or $F_1 = F_2 = \frac{1}{2}$. In the next section we employ eqs (11) and (12) to investigate the effect of the momentum probability distributions, as seen by the fixed observer, on the spin–momentum entanglement as viewed by the moving observer.

3. Gaussian momentum distributions

In this section, the result of the previous section is examined for Gaussian distributions. Suppose, in the fixed frame

$$a_{\pm,\pm}(\mathbf{p},\mathbf{q}) = (2\pi w_{1(2)}^2)^{-1/4} \exp\left(-\frac{p^2+q^2}{2w_{1(2)}^2}\right) \tag{16}$$

gives the probability distributions for the occurrence of $|\mathbf{p},\mathbf{q},+,+\rangle$ and $|\mathbf{p},\mathbf{q},-,-\rangle$, respectively [13,19]. It is emphasized that the widths, $w_{1(2)}$, in effect, select momentum states with large contributions (through partial tracing) to the spin–momentum entanglement. Substituting eq. (16) into eqs (11) and (12), one easily finds,

$$F_1 = \frac{D_1[2(p_0+mc)^2\cosh^2(\zeta/2) + w_1^2\sinh^2(\zeta/2)]^2}{\sum_{i=1,2} D_i[2(p_0+mc)^2\cosh^2(\zeta/2) + w_i^2\sinh^2(\zeta/2)]^2} \tag{17}$$

and

$$F_2 = \frac{8\sqrt{D_1D_2}\frac{w_1w_2}{w_1^2+w_2^2}\left[(p_0+mc)^2\cosh^2(\zeta/2) - \frac{w_1^2w_2^2}{w_1^2+w_2^2}\sinh^2(\zeta/2)\right]^2}{\sum_{i=1,2} D_i[2(p_0+mc)^2\cosh^2(\zeta/2) + w_i^2\sinh^2(\zeta/2)]^2}, \tag{18}$$

where

$$D_j = \exp\left[-\frac{2p_0^2\sinh^2(\zeta)}{w_j^2}\right], \quad j = 1, 2.$$

It is observed that the expressions for F_1 and F_2, which in turn determine the degree of spin–momentum entanglement (through eqs (8) and (15)) strongly depend on the distribution widths and the rapidity, ζ. Particularly, in the fixed frame, $\zeta = 0$, eq. (17) yields $F_1 = \frac{1}{2}$ and the eigenvalues in eq. (15) solely depend on $F_2 = w_1w_2/(w_1^2 + w_2^2)$ (see eq. (15)). It is then obvious that as the ratio of the widths approaches zero ($F_2 = 0$), the degree of spin–momentum entanglement approaches the maximal value of unity. On the other hand, in the limit $\zeta \to \infty$, F_1 approaches zero or one, depending on the ratio of the widths, while F_2 approaches zero. It is then seen from eqs (8) and (15) that, regardless of the widths, spin–momentum entanglement diminishes as seen by an observer moving close to c [13,14]. Moreover, the eigenvalues in eq. (15), as functions of rapidity, exhibit minima at certain ζ_0 which depend on the ratio of widths. As the rapidity increases beyond ζ_0, the eigenvalues also increase approaching unity. The behaviour of λ (eq. (15)) vs. ζ, for different ratios of widths, is depicted in figure 1. As a result, it is concluded that the spin–momentum entanglement also exhibits a maximum and decreases to zero as the rapidity increases [13,14]. This conclusion is again valid regardless of the widths. Moreover, as either of the widths tends to zero F_1 and F_2, the spin–momentum entanglement vanishes, irrespective of the rapidity. This, of course, is due to the fact that in the lab frame only one of the spin states, either $|+,+\rangle$ or $|-,-\rangle$, is present and is not affected by a Wigner rotation about the z-axis. Moreover, the graphs and, particularly, the inset of figure 1 indicate that, as the ratio of the widths increases spin–momentum states become more entangled for every rapidity. The reason behind this behaviour is that for larger ratio of widths, more momentum states are present in the system. As for Gaussian distributions of equal widths, $w_1 = w_2 = w$, one easily finds that $F_1 = F_2 = \frac{1}{2}$ (see eqs (17) and

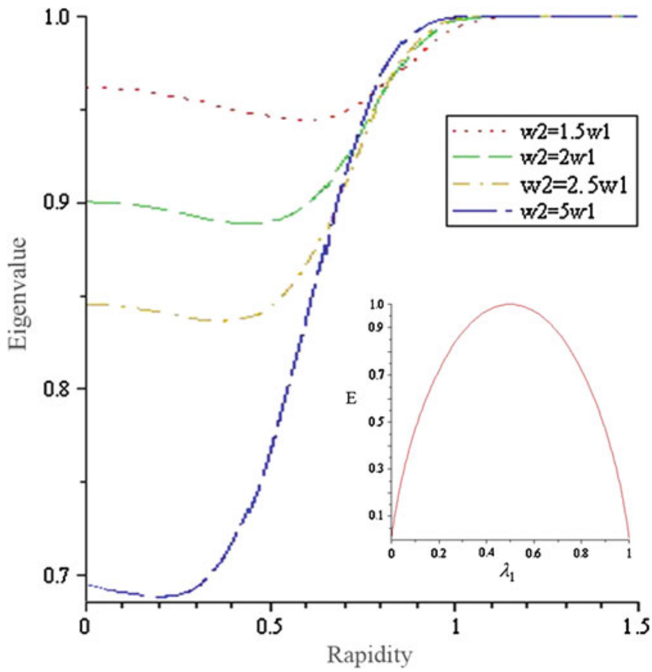

Figure 1. Behaviour of eigenvalues of the reduced density matrix vs. the rapidity, for different width ratios. The top inset identifies the curves for different widths, in units of mc. The bottom inset illustrates the general behaviour of entanglement vs. the eigenvalues [1].

(18)), with vanishing spin–momentum entanglement in the fixed frame $\zeta = 0$. However, in view of the moving observer, F_1 still equals $\frac{1}{2}$ (independent of the rapidity) and

$$F_2 = \left[\frac{1 - (w^2 \tanh^2(\zeta/2)/2(p_0 + mc)^2)}{1 + (w^2 \tanh^2(\zeta/2)/2(p_0 + mc)^2)} \right]^2 \leq 1. \tag{19}$$

Here again the eigenvalues in eq. (15) depend on F_2 alone. As $\zeta \to 0, \infty$, it is evident that $F_2 \to \frac{1}{2}$ which gives a vanishing spin–momentum entanglement. This, in turn, leads to the conclusion that, depending on the widths, the spin–momentum entanglement exhibits a maximum at a certain rapidity. To support the preceding conclusions, the variation of von Neumann entropy with respect to the rapidity is illustrated in figures 2 and 3. Figure 2 is drawn for different (unequal) width ratios, while figure 3 represents the same but for equal widths. In short, the work presented in this section indicates that, generally, spin–momentum entanglement, as measured by the von Neumann entropy, is a frame-dependent quantity. Accordingly, information would flow from one subsystem to the other as viewed by a moving observer. As a vivid consequence, separable states in the fixed frame may, and indeed does, turn into an entangled state as viewed in moving frames.

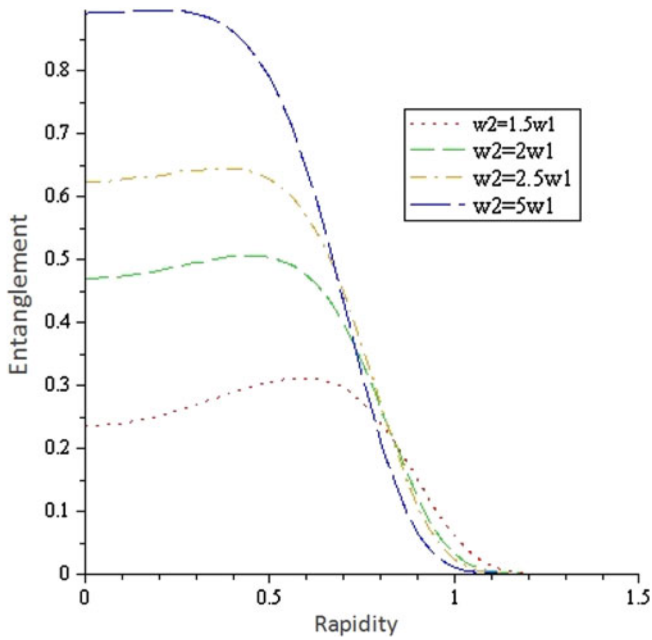

Figure 2. Spin–momenta entanglement, as measured by von Neumann entropy, vs. rapidity for unequal widths (in units of mc). Each curve is identified in the inset.

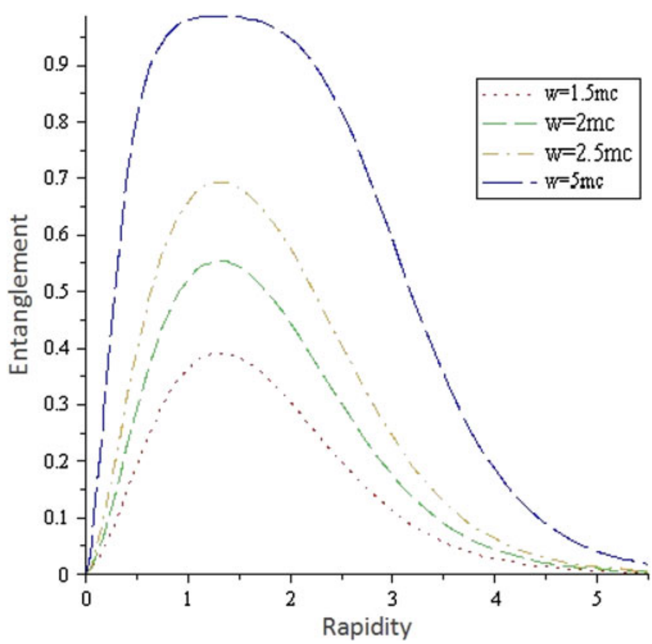

Figure 3. Spin–momenta entanglement, as measured by von Neumann entropy, vs. rapidity for equal widths (in units of mc). Each curve is identified in the inset.

4. Conclusion

In this article we have systematically investigated the properties of von Neumann entropy, as a measure of spin–momentum entanglement, relative to moving observers, for a system of two massive spin-$\frac{1}{2}$ particles. We have assumed that the spin–momentum states of a system occur with Gaussian probability distributions. Although the behaviour of the entanglement is thoroughly discussed in the preceding section, in what follows we outline the more important aspects of this report.

(1) The reduced spin–spin density matrix, which indicates the degree of entanglement between the two spin-$\frac{1}{2}$ particles and their momenta, similar to refs [13,14], is a frame-dependent quantity.
(2) The spin–momentum entanglement, as a function of rapidity, exhibits a maximum at a certain rapidity and diminishes, approaching zero, thereafter. This certain rapidity increases as the ratio of Gaussian widths is increased.
(3) The degree of spin–momentum entanglement, for fixed rapidity, is increased as the ratio of the widths of the momentum probability distributions increases.
(4) For probability distributions of equal widths, the spin–momentum entanglement vanishes in the fixed frame and increases to a maximum, again at a certain rapidity. This behaviour is in sharp contrast with that of spin–spin entanglement. The maximum values of the entanglement also increase and occur at a higher rapidity. As a result and as was observed in refs [15,29], no invariant meaning can be attached to the von Neumann entropy (reduced spin–momentum density matrix).

In short, the work presented in this paper clarifies the relation between storage of information and moving observers with direct applications in the development of quantum information processing schemes.

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