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Three-body interactions and the Landau levels using Nikiforov–Uvarov method

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Abstract. In this article, the eigenvalues for the three-body interactions on the line and the Landau levels in the presence of topological defects have been regenerated by the Nikiforov–Uvarov (NU) method. Two exhaustive lists of such exactly solvable potentials are given.

Keywords. Nikiforov–Uvarov (NU) method; three-body problems; Landau levels; tables of eigenvalues.

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Recently, solving simple quantum mechanical systems within the framework of Nikiforov–Uvarov (NU) method is drawing a lot of interest [1]. This algebraic technique is used for solving the second-order linear differential equations, which have been used successfully to solve Schrödinger, Dirac, Klein–Gordon and Duffin–Kemmer–Petiau wave equations in the presence of some well-known central and non-central potentials ([2–5] and references therein).

The motivation behind this work is to solve Schrödinger equation for the three-body interactions on the line and Landau levels in the presence of topological defects algebraically by the NU method.

The Nikiforov–Uvarov (NU) [1] method is used for reducing the second-order differential equation to a generalized hypergeometric-type equation. In this sense, the Schrödinger equation, after employing an appropriate coordinate transformation $s = s(r)$, transforms to the following form:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second-degree and $\tilde{\tau}(s)$ is a first-degree polynomial. To find the particular solution of eq. (1), one can use the following transformation:

$$\psi_n(s) = \phi_n(s) y_n(s) \tag{2}$$

leading to a hypergeometric-type equation like

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0, \tag{3}$$

where

$$\sigma(s) = \pi(s)\frac{\phi_n(s)}{\phi_n'(s)}, \tag{4}$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \qquad \tau'(s) < 0. \tag{5}$$

Here, $\tau(s)$ is with the parameter s and the most significant point at this stage is that prime factor of $\tau(s)$ shows the differentials at first degree and must be negative to reproduce physically acceptable λ-values which are defined as

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \qquad n = 0, 1, 2, 3, \ldots . \tag{6}$$

It is to be noted that λ or λ_n is obtained from a particular solution of the form $y_n(s)$ which is a polynomial of degree n. Here, the function $\pi(s)$ and the parameter λ are defined as

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}, \tag{7}$$

$$\lambda = \lambda_n = k + \pi'(s), \tag{8}$$

where $\pi(s)$ obviously is a polynomial depending on the transformation function $s(r)$.

In the late sixties, Calogero [6] gave the complete solution of the Schrödinger equation for three particles in one dimension, interacting pairwise by two-body harmonic and inverse-square potentials. There has been a renewed interest in the one-dimensional many-body problem of the Calogero and the Sutherland types [7].

The Schrödinger equation ($2m = \hbar = 1$) for the three-body interactions on the line is

$$\left(-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + V(x_1, x_2, x_3)\right)\Psi(x_1, x_2, x_3) = E\Psi(x_1, x_2, x_3), \tag{9}$$

with the interactions $V(x_1, x_2, x_3)$ with which the three particles are interacting. Using the notation of Calogero and applying Jacobi and polar coordinates successively and a little algebra yields the radial equation [7]

$$\left(-\frac{d^2}{dr^2} + V(r) + \frac{B_l^2 - \frac{1}{4}}{r^2}\right)u_{n,l}(r) = E_{n,l}u_{n,l}(r), \tag{10}$$

where B_l^2 is the eigenvalue of the angular equation

$$\left(-\frac{d^2}{d\phi^2} + U(\phi)\right)F_{n,l}(\phi) = B_l^2 F_l(\phi). \tag{11}$$

Here, the two constants of motion are the eigenvalues $E_{n,l}$ and B_l^2.

In order to apply the NU method, we rewrite eq. (10) using a new variable of the form $x = \frac{3}{8}\omega^2 r^2$ as

$$\frac{d^2u_{n,l}(x)}{dx^2} + \frac{1}{2x}\frac{du_{n,l}(x)}{dx} + \frac{1}{4x^2}\left(-\epsilon^2 x - ax^2 - b\right)u_{n,l}(x) = 0 \tag{12}$$

which leads to a hypergeometric-type equation. Here, we put $-\epsilon^2 = 8E/3\omega^2, a = 8/3\omega^2, b = B_l^2 - 1/4$. After comparing eq. (12) with eq. (1), we obtain $\tilde{\tau}(x) = 1, \sigma(x) = 2x, \tilde{\sigma}(x) = -\epsilon^2 x - ax^2 - b$. Applying the algebraic technique of the NU method, we obtain the values of parameter λ from eq. (6) as

$$\lambda_n = 2n\sqrt{a} \tag{13}$$

and from eq. (8) as

$$\lambda = -\frac{\epsilon^2}{2} - \sqrt{a}B_l - \sqrt{a}. \tag{14}$$

From eqs (13) and (14), we get eigenvalues as

$$E_{n,l} = \sqrt{\frac{3}{2}}\omega\big(2n + B_l + 1\big), \qquad n = 0, 1, 2, 3, ..., \qquad l = 0, 1, 2, ... \tag{15}$$

This algebraic technique was also used to solve eq. (11) for different potentials $U(\phi)$ [8] which are given in table 1. In table 1, the first column gives the list of potentials $U(\phi)$ that are separable algebraically in the ϕ-coordinate. The second, third and fourth columns give the values of λ, λ_l and the eigenvalue B_l^2 of eq. (11) for the angular potential.

Table 1. List of λ, λ_l and eigenvalues B_l^2 of eq. (11) for different angular potential $U(\phi)$ [8]. Here $\Delta = \sqrt{\tilde{B}_l^2 - F^2}$, $F = if_1/2$.

$U(\phi)$	λ	λ_l	B_l^2
$\frac{9g}{2\sin^2(3\phi)}$	$\frac{\tilde{B}_l^2}{2} - \frac{1}{2}(a+1)^2$, $\tilde{B}_l = \frac{B_l}{3}$	$2l^2 + 2l(a + \frac{1}{2})$	$9(l + a + \frac{1}{2})^2$, $a = \frac{1}{2}\sqrt{1+2g}$,
$\frac{9f}{2\cos^2(3\phi)}$	$\frac{\tilde{B}_l^2}{2} - \frac{1}{2}(b+1)^2$	$2l^2 + 2l(b + \frac{1}{2})$	$9(l + b + \frac{1}{2})^2$, $b = \frac{1}{2}\sqrt{1+2f}$
$\frac{9g}{2\sin^2(3\phi)} + \frac{9g}{2\cos^2(3\phi)}$	$\frac{\tilde{B}_l^2}{2} - \frac{1}{2}(a+b+1)^2$	$2l^2 + 2l(a+b+1)$	$9(2l + a + b + \frac{1}{2})^2$
$\frac{9g}{2\sin^2(3\phi)} + \frac{9f_1}{2\tan(3\phi)}$	$\frac{g}{2} - \frac{1}{2}(\tilde{B}_l^2 + \Delta) - \frac{1}{\sqrt{2}}\sqrt{\tilde{B}_l^2 + \Delta}$	$l^2 + l + 2l\sqrt{\frac{1}{2}(\tilde{B}_l^2 + \Delta)}$	$9(l + a + \frac{1}{2})^2 - \frac{9}{16}\frac{f_1^2}{(l+a+\frac{1}{2})^2}$
$\frac{9f}{2\cos^2(3\phi)} - \frac{9f_1}{2\cot(3\phi)}$	$\frac{f}{2} - \frac{1}{2}(\tilde{B}_l^2 + \Delta) - \frac{1}{\sqrt{2}}\sqrt{\tilde{B}_l^2 + \Delta}$	$l^2 + l + 2l\sqrt{\frac{1}{2}(\tilde{B}_l^2 + \Delta)}$	$9(l + b + \frac{1}{2})^2 - \frac{9}{16}\frac{f_1^2}{(l+b+\frac{1}{2})^2}$

In an interesting paper, Marques *et al* [9] have determined Landau levels in a medium with defects. In order to find the eigenvalues of the Landau levels in the presence of topological defects [9], we have solved a radial differential equation as

$$\frac{\mathrm{d}^2R_n(x)}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}R_n(x)}{\mathrm{d}x}+\frac{1}{x^2}\left(\frac{A}{2}x-\frac{a^2}{4}x^2-\frac{L^2}{4}\right)R_n(x)=0, \tag{16}$$

leading to a hypergeometric-type equation which also contains different types of information of defects through the parameters A, a and L. Comparing eqs (1) and (16), we obtain $s=x$, $\psi_n(x)=R_n(x)$, $\tilde{\tau}(x)=1$, $\sigma(x)=x$, $\tilde{\sigma}(x)=A/2x-a^2/4x^2-L^2/4$. After some algebraic calculation, we get the values of parameter λ from eq. (6) as

$$\lambda_n=na \tag{17}$$

and from eq. (8) as

$$\lambda=\frac{A}{2}-\frac{a|L|}{2}-\frac{a}{2}. \tag{18}$$

From eqs (17) and (18), we get a quantized condition as

$$A=a\big(2n+|L|+1\big),\qquad n=0,1,2,3,\dots. \tag{19}$$

This algebraic technique is used to solve eq. (16) for different types of topological defects which are given in table 2. In table 2, the first column gives the types of defects [9], the

Table 2. List of different types of topological defects [9], A, a, L and eigenvalues $E_{n,l}$ of eq. (16)

Type of defect	A	a	L	$E_{n,l}$
Continuous distribution of disclinations [8]	$\frac{2mE}{\hbar^2}-k^2+\frac{qB}{\hbar c}\left(\frac{l}{\alpha^2}\right)$	$\frac{qB}{\alpha\hbar c}$	$\frac{l}{\alpha}$	$\frac{\hbar\omega_H}{2\alpha}\left(2n+\frac{\lvert l\rvert}{\alpha}\pm\frac{l}{\alpha}+1\right)+\frac{\hbar^2k^2}{2m}$, $\omega_H=\frac{\lvert q\rvert B}{mc}$, $n=0,1,2,\dots$
Magnetic screw dislocation [8]	$\frac{2mE}{\hbar^2}-k^2-\frac{qB}{2\hbar c}\left(l-\beta k+\frac{\Phi}{2\pi}\right)$	$\frac{qB}{\hbar c}$	$l-\beta k+\frac{\Phi}{2\pi}$	$\hbar\omega_B\left(n+\frac{\lvert l-\beta k+\frac{\Phi}{2\pi}\rvert}{2}-\frac{(l-\beta k+\frac{\Phi}{2\pi})}{2}+\frac{1}{2}\right)+\frac{\hbar^2k^2}{2m}$, $\omega_B=\frac{qB}{mc}$, $n=0,1,2,3,\dots$
Dispiration [8]	$\frac{2mE}{\hbar^2}-k^2-\frac{qB}{2\hbar c\alpha}(l-\beta k)$	$\frac{qB}{\alpha\hbar c}$	$l-\beta k$	$\frac{\hbar\omega_B}{\alpha}\left(n+\frac{\lvert l-\beta k\rvert}{2\alpha}-\frac{(l-\beta k)}{2\alpha}+\frac{1}{2}\right)+\frac{\hbar^2k^2}{2m}$, $\omega_B=\frac{qB}{mc}$, $n=0,1,2,3,\dots$
Kaluza–Klein theory [8]	$2mE-k^2-Q^2+\frac{BQ}{2\alpha^2}(l-\beta k)$	$\frac{BQ}{\alpha}$	$\frac{l-\beta k}{\alpha}$	$\frac{BQ}{m\alpha}\left(n+\frac{\lvert l-\beta k\rvert}{2\alpha}-\frac{(l-\beta k)}{2\alpha}+\frac{1}{2}\right)+\frac{k^2}{2m}+\frac{Q^2}{2m}$, $n=0,1,2,\dots$

second, third and fourth columns give the values of A, a, L and the eigenvalues $E_{n,l}$ of eq. (16).

Recently, a new general technique [10–12] was developed for solving linear differential equations of arbitrary order to construct new representations of the polynomial solutions of the known second-order linear differential equations with and without source. We hope that this new technique [10–12] may play a crucial role in the study of quantum mechanical problems, in connection with Schrödinger equation. So, it will be very interesting to study the three-body interactions and the Landau levels by the new technique [10–12]. These studies need separate considerations which we hope to investigate in future. Now, we conclude that the NU method is an elegant and powerful technique which generates closed forms for the energy eigenvalues as well as the corresponding eigenfunctions for different quantum mechanical problems in physics. Here, the formalism systematically recovers known results in a natural way.

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