PRAMANA
— journal of
physics

© Indian Academy of Sciences

Vol. 81, No. 2
August 2013
pp. 247–260

# Coxeter groups $A_4$, $B_4$ and $D_4$ for two-qubit systems

RAMAZAN KOÇ[1,*], M YAKUP HACIIBRAHIMOĞLU[1]
and MEHMET KOCA[2]

[1]Department of Physics, Faculty of Engineering, University of Gaziantep, 27310 Gaziantep, Turkey
[2]Department of Physics, College of Science, Sultan Qaboos University, P.O. Box 36, Al-Khoud 123, Muscat, Oman
*Corresponding author. E-mail: koc@gantep.edu.tr

MS received 12 March 2013; revised 20 April 2013; accepted 29 April 2013

**Abstract.** The Coxeter–Weyl groups $W(A_4)$, $W(B_4)$ and $W(D_4)$ have proven very useful for two-qubit systems in quantum information theory. A simple technique is employed to construct the unitary matrix representations of the groups, based on quaternionic transformation of the usual reflection matrices. The von Neumann entropy of each reduced density matrix is calculated. It is shown that these unitary matrix representations are naturally related to various universal quantum gates and they lead to entangled states. Canonical decomposition of generators in terms of fundamental gate representations is given to construct the quantum circuits.

**Keywords.** Quantum computation; quantum information; group theory in quantum mechanics.

**PACS Nos** **03.67.Lx; 03.67.−a; 03.65.Fd**

## 1. Introduction

The Coxeter groups [1,2] have played substantial roles in various fields of mathematics and physics. Their affine extensions describe the lattice structures in arbitrary Euclidean space [3–6]. A quasicrystalline structure can be described by orthogonal projection of higher-dimensional crystals determined by the affine Coxeter–Weyl groups [6–9]. The finite Coxeter groups are constructed from finite Euclidean reflections. These belong to four infinite series, $A_{\rm n}$, $B_{\rm n}$, $D_{\rm n}$ and $I_2(n)$ and exceptional types, $H_3$, $H_4$, $F_4$, $E_6$, $E_7$ and $E_8$. The Coxeter groups describe the symmetry of the polyhedra and higher-dimensional polytopes [1,9]. A subset of these groups consists of various non-crystallographic groups and they describe symmetries of quasicrystal structures in two, three and four dimensions [7,10].

Although many applications of the Coxeter groups in various fields, such as, symmetries of molecular structures, icosahedral viruses and crystal symmetries in three dimensions have been sufficiently studied, their connections to entanglement [11,12], quantum information and quantum computation theory [13] have not yet been fully explored.

The main source of the quantum information theory is quantum entanglement. It takes place at the centre of the quantum information processing. If two or more quantum systems (qubits) cannot be written as a product of superposition of states of the corresponding system, the quantum system is said to be entangled. Entangled states can be generated by quantum gates. Quantum gates operate on qubits and perform unitary transformation. Although various group theoretical models [13–21] of quantum information and quantum computation theory are, in particular, based on Pauli matrices and Clifford algebras, a little is known about the application of Coxeter groups in quantum information theory [22–24]. In this paper we show that the Coxeter groups form a basis for quantum computation leading to the symmetries of the qubit systems.

To support our approach, let us mention that, unitary reflection groups can be associated with Braid groups [25] and they take part in anionic as well as qubit symmetries. Similarly, in this paper we show that there is a new bridge between Coxeter groups and quantum information theory.

It is most probable that the link between different fields leads to a new insight in these areas. Recently, an explicit connection has been established between n-qubit generalized Pauli group and quantum information theory [26,27]. Mermin's pentagram [28] has been associated with three-qubit Pauli group [29]. We have already emphasized that, the relation between quantum information theory and Clifford groups is well known. In this paper, we establish a relation between the Coxeter groups and the $2^2$quantum systems. In this connection, quaternions play a crucial role. Unitary representations of the rank-4 Coxeter group generators can be obtained by constructing the root system of the Coxeter diagram in terms of quaternions [30,31].

The paper is organized as follows. In §2, the basic background of the unitary matrix representations of the quantum gates and their relation to entanglement are explained. In §3, the reflection matrix generators of the Coxeter group $A_4$ are constructed and their unitary matrix representations are given on the basis of quaternions. Their relation to two-qubit quantum information systems and the role in the entangled circuits are studied. The relation between the Coxeter group $A_4$ and the quantum information theory is established. Section 4 is devoted to the study of similar relations between the Coxeter group $B_4$ and the quantum information theory. In §5, the Coxeter group $D_4$ is studied and its relevance to quantum information theory is pointed out. Finally, in §6, some concluding remarks are given.

## 2. Preliminary studies

In this section, some properties of single and two-qubit quantum gates are reviewed. Universal set of quantum gates can be constructed using single qubit and two-qubit logic gates. Entanglements of qubits can be generated using two-qubit gates [13,28–34].

### 2.1 *Quantum gates*

We begin by introducing two-qubit quantum gates that can be represented by generators of the corresponding group. Let us describe some useful single and two-qubit quantum gates and their matrix representations. Single qubit quantum gates are represented by $2 \times 2$ unitary matrices. They are given by the Pauli matrices $\sigma_1$, $\sigma_2$, $\sigma_3$, and rotation matrices $R_x$, $R_y$ and $R_z$, respectively:

$$\text{Identity } I\colon\ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{NOT or } X\colon\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$Y\colon\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z\colon \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$R_x(\alpha) = \exp\left(-\frac{i\alpha}{2}\sigma_1\right), \qquad R_y(\beta) = \exp\left(-\frac{i\beta}{2}\sigma_2\right),$$
$$R_z(\gamma) = \exp\left(-\frac{i\gamma}{2}\sigma_3\right). \tag{1}$$

The unitary matrices $R_x$, $R_y$ and $R_z$ represent the standard Bloch sphere rotations. One can also construct some useful additional gates using linear combinations of the gates (1):

$$\text{PHASE}, \phi\colon \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{i\phi} \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2}$$

The phase gate $\phi$ and Hadamard gate $H$ are useful in constructing quantum circuits.

These gates act on single qubits $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in quantum circuits and output of the circuit can be obtained by acting matrix representation of the gate on input.

In a similar manner we introduce those mostly used two-qubit gates and their matrix representations

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{CNOT} = \sigma_0 \oplus \sigma_1, \quad \text{CZ} = \sigma_0 \oplus \sigma_3, \tag{3}$$

where CNOT gate is known as the controlled NOT gate and CZ is the controlled Z gate. Parallel connections of gates are associated with the direct product of their matrix representations. If $U$ and $V$ are arbitrary $2 \times 2$ unitary gates, their parallel connection in a circuit is defined by $U \otimes V$. Circuit diagram of various gates are illustrated in figure 1.

The two-qubit states can be represented by a set of orthogonal unit vectors in the four-dimensional Euclidean space:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{4}$$

where $|ij\rangle$ is the tensor product of two states

$$|ij\rangle = |i\rangle \otimes |j\rangle.$$

The output of a two-qubit circuit can be obtained by the product of two-qubit input with matrix representation of the gate. Icons of the gates (1) through (3) are illustrated in

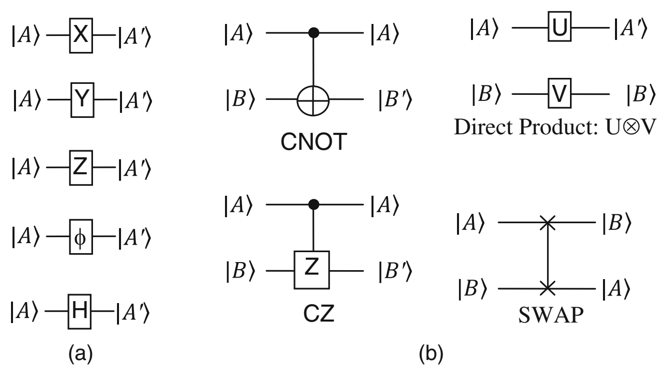


**Figure 1.** Icon of quantum gates. (**a**) Single qubit gates. X-gate is displayed sometimes by and icon '⊕'. (**b**) Two-qubit gates. Controlled Z gate is denoted by CZ.

figure 1. Note that Controlled-NOT (CNOT) and SWAP gates play dominant roles in the quantum information and quantum computation theories.

### 2.2 *Quantum entanglement*

As we mentioned before, we shall also investigate the relation between entanglement and Coxeter groups. Entanglement [35–37] is one of the most striking phenomena of quantum information theory because it has been the core of many applications in quantum computing [38], quantum cryptography [39], quantum teleportation, as well as philosophically oriented discussions concerning quantum theory.

Mathematical description of entangled state is straightforward. If a vector state $|\Psi\rangle$ is not a product of vector state $|\psi_A\rangle \otimes |\psi_B\rangle$ it is called entangled. For example, Bell's singlet state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is entangled because it cannot be written as a direct product of two states.

For an entangled state we introduce a density operator related to entropy measure of the state, which is defined as [37]

$$p = \sum_i p_i \, |\psi_i\rangle \, \langle\psi_i|, \tag{5}$$

where $p_i$ is the probability density. The density operator representing a pure state is $\mathrm{Tr}(\rho^2) = 1$, otherwise it is a mixed state. A consistent method to quantify the entanglement of pure states is provided by the entropy measure of entanglement,

$$E(\psi) = S(\rho_A) = S(\rho_B)\,, \tag{6}$$

where

$$S(\rho_i) = \mathrm{Tr}(\rho_i) \log_2 \rho_i = -\sum_i \lambda_i \log_2 |\lambda_i| \tag{7}$$

with $S(\rho)$ representing the von Neumann entropy and $\lambda_i$ representing the eigenvalues of the reduced density matrix $\rho_i$. For a separable state, the entropy measure $S(\rho) = 0$. Note that the value of $E$ changes from 0 to 1, where 0 corresponds to separable (unentangled) state and 1 corresponds to maximally entangled state. In order to obtain the entropy measure of the entanglement we define the density operator $\rho_A$ as

$$\begin{aligned}\rho_A &= \mathrm{Tr}_B\left(|\psi_i\rangle\langle\psi_i|\right) = \mathrm{Tr}_B \sum_{i,j} |ij\rangle\langle ij| \\ &= \sum_{i,j} |ij\rangle\, \mathrm{Tr}\,\langle ji| = \sum_{i,j} |i\rangle\langle i|\langle j\,|\,j\rangle\,. \end{aligned} \tag{8}$$

This describes the probability of the locally measured state $A$. Implication of the partial trace operation is that local measurements of quantum systems $A$ and $B$ that cannot give any information about preparation of the state.

## 3. The Coxeter–Weyl group $W(A_4)$

In this section, we construct unitary matrix generators of the group $W(A_4)$, and study the relation between the corresponding group and $2^2$ qubit systems.

### 3.1 *Unitary representation of $W(A_4)$*

Matrix generators of the Coxeter reflection groups can be obtained from the Coxeter–Dynkin diagram. The diagram for $A_4$ is shown in figure 2. The generators of the Coxeter group $A_4$ are obtained as reflection matrices with respect to the hyperplanes orthogonal to the simple roots.

If $\Lambda = a_i\omega_i$ $(i = 1, \ldots, 4)$, represents an arbitrary lattice vector then the reflection with respect to one of the simple roots is given by $r_i\Lambda = \Lambda - (\alpha_i, \Lambda)\alpha_i$, where $r_i$ are the reflection matrices, $\alpha_i$ are the simple root vectors and $\omega_i$ are the weight vectors in the dual basis. The following relations between vectors hold:

$$\begin{aligned}&(\alpha_i, \alpha_j) = C_{ij}, \qquad (\omega_i, \omega_j) = (C^{-1})_{ij}, \\ &(\omega_i, \alpha_j) = \delta_{ij}, \qquad i, j = 1, 2, 3, 4.\end{aligned} \tag{9}$$

The simple roots and weight vectors can be written as linear combinations of each other:

$$\omega_i = (C^{-1})_{ij}\alpha_j, \quad \alpha_i = (C)_{ij}\,\omega_j, \tag{10}$$

where $C$ is the Cartan matrix of $A_4$ and is given by

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{11}$$

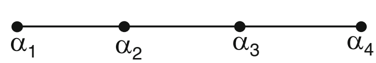


**Figure 2.** Coxeter–Dynkin diagram of $A_4$.

As an illustration, one of the reflection matrixes of $A_4$ can be obtained as follows:

$$\begin{aligned} r_1\alpha_1 &= \alpha_1 - (\alpha_1, \alpha_1)\,\alpha_1 = \alpha_1 - (2\alpha_1) = -\alpha_1 \\ r_1\alpha_2 &= \alpha_2 + \alpha_1 \\ r_1\alpha_3 &= \alpha_3 \\ r_1\alpha_4 &= \alpha_4. \end{aligned} \tag{12}$$

Thus, the reflection generator associated with the first root of the Coxeter–Weyl group $W(A_4)$ can be written in the matrix form on the basis of simple roots:

$$r_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly, we can obtain the other three generators of $A_4$ as:

$$r_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$r_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is obvious that these matrices are not written in the orthogonal basis since simple roots satisfy a non-orthogonal relation as evident from the Cartan matrix (11). In order to establish a relation between qubits and the Coxeter system, the reflection matrices must be constructed in the orthogonal basis.

Recently, quaternionic representations of the root system of rank-4 Coxeter groups have been introduced by Koca *et al* [8,40,42]. Scaled quaternionic simple roots of $A_4$ are given by

$$\begin{aligned} \alpha_1 &= -1, \qquad \alpha_2 = \frac{1}{2}\left(1 + e_1 + e_2 + e_3\right), \\ \alpha_3 &= -e_1, \qquad \alpha_4 = \frac{1}{2}\left(e_1 - \sigma e_2 - \tau e_3\right), \end{aligned} \tag{13}$$

where $\sigma = \frac{1}{2}(1 - \sqrt{5})$ and $\tau = \frac{1}{2}(1 + \sqrt{5})$ and imaginary quaternionic units $e_i$ ($i = 1, 2, 3$) obey the following relation:

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \;\; (i, j, k = 1, 2, 3), \tag{14}$$

where $\delta$ is the Kronecker delta and $\epsilon_{ijk}$ is the Levi–Civita symbol. Formally, the quaternionic representation (13) can be written as

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = T \begin{pmatrix} 1 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{15}$$

where the transformation matrix $T$ is given by

$$T = \frac{1}{2} \begin{pmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & -\sigma & -\tau \end{pmatrix}. \tag{16}$$

Then it is straightforward to obtain a unitary representation of the generators. The similarity transformation

$$g_i = T^{-1} r_i T \tag{17}$$

leads to the unitary generators $g_i$ of $W(A_4)$:

$$g_1^A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2^A = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

$$g_3^A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_4^A = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & \sigma & \tau \\ 0 & \sigma & \tau & 1 \\ 0 & \tau & 1 & \sigma \end{pmatrix}. \tag{18}$$

In the next subsection we show how the unitary matrix representations are related to two-qubit quantum gates.

### 3.2 *Two-qubit systems and Coxeter–Weyl group W($A_4$)*

The four generators generate a group of order 120. Our task now is to decompose the generators $g_i^A (i = 1, \ldots, 4)$ in terms of matrix representations of quantum gates given in eqs (1)–(3) and circuits in figure 1. The first three generators generate the Coxeter–Weyl group of $W(A_3)$ of order 24, the maximal subgroup of $W(A_4)$. This group is relevant for the quantum cryptography and entanglement. In practice, quantum entanglement or cryptosystems are represented by quantum circuits that are constructed using quantum gates.

The first generator $g_1^A$ in (18) just changes the phase of the first qubit and acts as a matrix representation of the phase gate. It can be expressed in terms of the matrix realization of the quantum gates:

$$g_1^A = \mathrm{CZ}(R_z(\pi) \otimes R_z(\pi)). \tag{19}$$

The circuit representation of (15) is illustrated in figure 3.

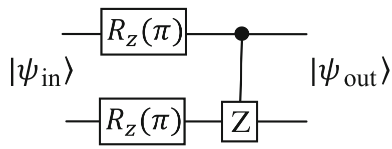


**Figure 3.** Circuit representation of $g_1^A$. Output of the circuit is given by $|\psi_{\text{out}}\rangle = g_1^A\,|\psi_{\text{out}}\rangle$.

The generator $g_2^A$ is associated with a circuit that entangles an unentangled state. In order to construct its circuit representations, we have to obtain its canonical decomposition. After some straightforward treatments one can obtain the decomposition,

$$g_2^A = \mathrm{CZ}\left(R_z\left(\frac{\pi}{2}\right) \otimes R_z\left(\frac{\pi}{2}\right)\right) \mathrm{CNOT}\left(R_x\left(\frac{\pi}{2}\right) \otimes R_z\left(-\frac{\pi}{2}\right)\right) \times \mathrm{CNOT}\left(R_x\left(\frac{\pi}{2}\right) \otimes R_x\left(\frac{\pi}{2}\right)\right). \tag{20}$$

Corresponding quantum circuit is given in figure 4. Let us consider the action of circuit on unentangled states. As an example, when the circuit acting on the state $|\psi_{\text{in}}\rangle = |00\rangle = |0\rangle \otimes |0\rangle$, it produces an entangled output,

$$|\psi_{\text{out}}\rangle = g_2^A|00\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle - |11\rangle). \tag{21}$$

Density matrix of this state can easily be obtained using the relation (5) and it is given by

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}. \tag{22}$$

In order to obtain entropy measurement of the entanglement, we calculate the reduced density matrix, using (8):

$$\rho_A = \mathrm{Tr}\rho_B|\psi_{\text{out}}\rangle\,\langle\psi_{\text{out}}| = \frac{1}{2}\,(|0\rangle\,\langle 0| + |1\rangle\,\langle 1|) \tag{23}$$

then the entropy of the measurement of entanglement is

$$E(\psi_{\text{out}}) = \mathrm{S}(\rho) = -\mathrm{Tr}(\rho_A)\log_2 \rho_A = 1. \tag{24}$$

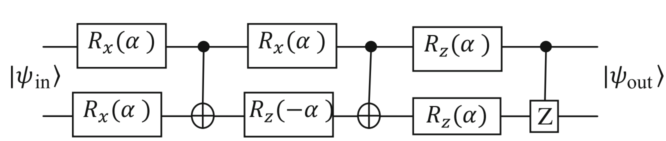


**Figure 4.** Quantum circuit representation of $g_2^A$ given in eq. (20) (the angle $\alpha = \pi/2$).

Therefore, the state is maximally entangled. Strictly speaking, the circuit in figure 4 is called perfect entangler, because it produces maximally entangled state from an appropriate unentangled input. Similar to $g_1^A$, the third generator $g_3^A$ also acts as a matrix representation of phase gate.

Now, we investigate the action of $g_4^A$ on the states. It is obvious that it does not change $|00\rangle$ state. It is interesting that the generator $g_4^A$ produces a non-maximal Bell-type entangled state [14]. When it acts on the separable state

$$|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{3}}(\tau|0\rangle + \sigma|1\rangle) \otimes \frac{1}{\sqrt{\sqrt{5}}}\left(-\sqrt{\tau}\,|0\rangle + \frac{1}{\sqrt{\tau}}\,|1\rangle\right) \tag{25}$$

the output is a non-maximal state

$$|\psi_{\text{out}}\rangle = \frac{1}{\sqrt{3\sqrt{5}}}\left(\frac{2}{\sqrt{\tau}}\,|00\rangle - \frac{\sqrt{\tau}}{\sigma}\,|11\rangle\right). \tag{26}$$

Recently, Kossakowski and Ohya [41] proposed a new scheme of teleportation, based on non-maximal entangled state. Therefore, this type of quantum gate representation may also be useful in quantum information theory.

To determine the entangling capability of $g_4^A$, we calculate the reduced density matrix. The calculation of (8) gives the following result:

$$\rho_A = \text{Tr}\rho_B|\psi_{\text{out}}\rangle\langle\psi_{\text{out}}| = \frac{1}{\sqrt{3\sqrt{5}}}\left(\tau^2|0\rangle\langle 0| + 4\sigma^2|1\rangle\langle 1|\right). \tag{27}$$

Then entropy of measurement or entangling power of the unitary operator $g_4^A$ can be calculated using the relation (7) and the result is

$$E(\psi_{\text{out}}) = S(\rho) = -\text{Tr}(\rho_A)\log_2\rho_A = 0.9495. \tag{28}$$

We would also like to add that the operator $g_4^A$ produces the following three terms of non-maximal entangled states:

$$\begin{aligned} &\frac{1}{2}(|01\rangle + \sigma|10\rangle + \tau|11\rangle), \\ &\frac{1}{2}(\sigma|01\rangle + \tau|10\rangle + |11\rangle), \\ &\frac{1}{2}(\tau|01\rangle + |10\rangle + \sigma|11\rangle), \end{aligned} \tag{29}$$

from the inputs $|01\rangle$, $|10\rangle$ and $|11\rangle$, with entropy measurements 0.823, 0.808 and 0.900, respectively.

This result implies that the Coxeter–Weyl group $W(A_4)$ is strongly related to quantum information theory.

## 4. The Coxeter–Weyl group $W(B_4)$

The procedure presented in the previous section brings a link between the Coxeter–Weyl groups and quantum information theory. Using the same procedure we discuss the role of the group $W(B_4)$ in quantum information theory.

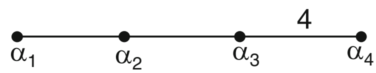


**Figure 5.** Coxeter–Dynkin diagram of $B_4$.

### 4.1 *Unitary representation of W(B$_4$)*

The Coxeter–Dynkin diagram of $B_4$ is shown in figure 5. The generators of $W(B_4)$ can be obtained by using the Coxeter–Dynkin diagram $B_4$.

Let us express the simple roots of $B_4$ in terms of quaternions as [40,42].

$$\alpha_1 = \frac{1}{\sqrt{2}}(1-e_1), \qquad \alpha_2 = \frac{1}{\sqrt{2}}(e_1-e_2),$$
$$\alpha_3 = \frac{1}{\sqrt{2}}(e_2-e_3), \qquad \alpha_4 = e_3. \tag{30}$$

In the quaternionic basis, the generators $g_i$ of $W(A_4)$ can be written as

$$g_1^B = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \quad g_2^B = \begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix},$$
$$g_3^B = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}, \quad g_4^B = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&-1 \end{pmatrix}. \tag{31}$$

Clearly, these generators are associated with some important quantum gates and play a dominant roles in the quantum computation theory.

### 4.2 *Two-qubit systems and Coxeter–Weyl group W(B$_4$)*

The generators generate the Coxeter–Weyl group of $W(B_4)$ of order 384. This group consists of some well-known operators used in quantum entangled circuits. It is interesting

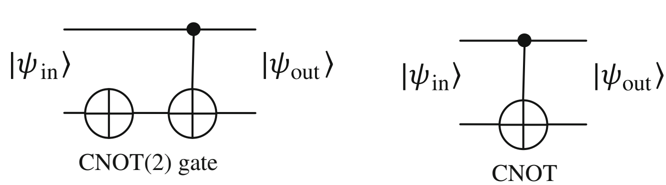


**Figure 6.** Circuit representation of $g_1^B$ and $g_3^B$ generators.

**Table 1.** Input–output relations of CNOT(2) and CNOT gates.

| $\|\psi_{\rm in}\rangle$ | $\|\psi_{\rm out}\rangle$(CNOT(2)) | $\|\psi_{\rm out}\rangle$(CNOT) | $E(\rho)$ |
|---|---|---|---|
| $\frac{1}{\sqrt{2}}(\|0\rangle \pm \|1\rangle)\otimes\|0\rangle$ | $\frac{1}{\sqrt{2}}(\|01\rangle \pm \|10\rangle)$ | $\frac{1}{\sqrt{2}}(\|00\rangle \pm \|11\rangle)$ | 1 |
| $\frac{1}{\sqrt{2}}(\|0\rangle \pm \|1\rangle)\otimes\|1\rangle$ | $\frac{1}{\sqrt{2}}(\|00\rangle \pm \|11\rangle)$ | $\frac{1}{\sqrt{2}}(\|01\rangle \pm \|10\rangle)$ | 1 |

to observe that the generators are associated with CNOT(2), SWAP, CNOT and CZ gates respectively. The generator $g_1^B$ can be expressed in terms of CNOT gate as

$$g_1^B = \text{CNOT}(2) \equiv (\sigma_0 \otimes \sigma_1)\, \text{CNOT}. \tag{32}$$

Circuit representations of the $g_1^B$ and $g_3^B$ are illustrated in figure 6.

Our task is not to investigate the entanglement properties of the generators. As seen in table 1, the generators $g_1^B$ and $g_3^B$ are associated with the most important gates CNOT(2) and CNOT, and they produce the well-known Bell states. This result motivates us to find the relation between the Coxeter–Weyl groups and quantum information theory, because the gates are directly represented by the perfect entangled circuit.

Second generator $g_2^B$ is the matrix representation of SWAP gate and the last generator corresponds to the controlled Z gate, CZ, as mentioned in §3. Both of them are very important in quantum information and computation theory.

## 5. The Coxeter–Weyl group $W(D_4)$

The technique introduced in §2, now leads to a new relation between the Coxeter–Weyl groups and quantum information theory. Now, we explain the relations of the group $W(D_4)$ and the quantum information theory.

### 5.1 *Unitary representation of $W(D_4)$*

The Coxeter diagram for $D_4$ group of order 192 is shown in figure 7. Its matrix generators can be obtained from the simple roots of the Coxeter–Dynkin diagram.

This can be done by expressing the simple roots in terms of quaternions as [43]

$$\alpha_1 = \frac{1}{\sqrt{2}}(e_2 - e_3)\,, \qquad \alpha_2 = \frac{1}{\sqrt{2}}(e_1 + e_3)\,,$$
$$\alpha_3 = \frac{1}{\sqrt{2}}(-e_3 - 1)\,, \quad \alpha_4 = \frac{1}{\sqrt{2}}(1 - e_1)\,. \tag{33}$$

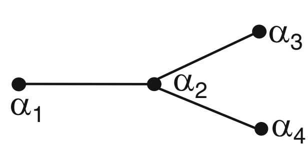


**Figure 7.** Coxeter–Dynkin diagram of $D_4$.

**Table 2.** Action of generators $g_1^D$, $g_3^D$ and $g_4^D$ of $W(D_4)$ on states.

| $\|\psi_{\text{in}}\rangle$ | $\|\psi_{\text{out}}\rangle(g_1^D)$ | $\|\psi_{\text{out}}\rangle(g_3^D)$ | $\|\psi_{\text{out}}\rangle(g_4^D)$ | $E(\rho)$ |
|---|---|---|---|---|
| $\frac{1}{\sqrt{2}}(\|0\rangle \pm \|1\rangle) \otimes \|0\rangle$ | $\frac{1}{\sqrt{2}}(\|00\rangle \pm \|11\rangle)$ | $\frac{1}{\sqrt{2}}(\|00\rangle \mp \|11\rangle)$ | $\frac{1}{\sqrt{2}}(\|01\rangle \pm \|10\rangle)$ | 1 |
| $\frac{1}{\sqrt{2}}(\|0\rangle \pm \|1\rangle) \otimes \|1\rangle$ | $\frac{1}{\sqrt{2}}(\|01\rangle \pm \|10\rangle)$ | $\frac{1}{\sqrt{2}}(\|01\rangle \mp \|10\rangle)$ | $\frac{1}{\sqrt{2}}(\|00\rangle \pm \|11\rangle)$ | 1 |

In the quaternionic basis, the unitary generators $g_i$ of $W(D_4)$ can be written as

$$g_1^D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad g_2^D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$g_3^D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad g_4^D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{34}$$

We note that all of these generators are associated with the matrix representations of CNOT gates such that

$$\begin{aligned} g_1^D &\equiv \text{CNOT}, \\ g_2^D &= (I \otimes \sigma_3) \cdot \text{CNOT} \cdot \text{SWAP} \cdot \text{CNOT}, \\ g_3^D &= (I \otimes \sigma_3) \cdot \text{CNOT}, \\ g_4^D &\ (I \otimes \sigma_1) \cdot \text{CNOT}. \end{aligned} \tag{35}$$

### 5.2 *Two-qubit systems and Coxeter–Weyl group W($D_4$)*

In this section, we investigate the action of the generators on separable states to express their relations to the entangled states. The actions of the generators $g_1^D$, $g_3^D$ and $g_4^D$ in (35) on the separable states are summarized in table 2 and $g_2^D$ are given in (36).

The second generator $g_2^D$ exhibits a Bell state by flipping input. The result is given by

$$\begin{aligned} g_2^D \frac{1}{\sqrt{2}} |0\rangle (|0\rangle \pm |1\rangle) &= \frac{1}{\sqrt{2}} (|00\rangle \mp |11\rangle), \\ g_2^D \frac{1}{\sqrt{2}} |1\rangle (|0\rangle \pm |1\rangle) &= \frac{1}{\sqrt{2}} (|10\rangle \mp |01\rangle). \end{aligned} \tag{36}$$

The entanglement typically arises from the generators of the Coxeter–Weyl groups.

## 6. Conclusion

In this paper, we have worked out the roles of the rank-4 Coxeter–Weyl groups, such as $W(A_4)$, $W(B_4)$ and $W(D_4)$ in two-qubit quantum information systems. We have shown that these groups are associated with maximal and non-maximal entangled gates. It is important to note that the generators of these groups are related to the prominent quantum gates, such as CNOT, SWAP etc. and produce Bell states. In a forthcoming paper we study the relations of the quantum information systems with the Coxeter–Weyl group $W(F_4)$ and the noncrystallographic Coxeter group $W(H_4)$ [43]. As a future work, we shall extend the method presented here to the rank-8 Coxeter–Weyl groups to explore their connections to the 3-qubit systems.

**Acknowledgements**

The authors would like to thank Dr Nazife O Koca for reading the manuscript and for giving suggestions for the improvement of the paper.

**References**

[1] H M Coxeter, *Ann. Math.* **35**, 588 (1933)
[2] H M Coxeter, *J. London Math. Soc.* **10**, 21 (1935)
[3] R Rouquier, Weyl groups, affine Weyl groups and reflection groups, in: *Representations of reductive groups* edited by R W Carter and M Geck (Cambridge Univ. Press, New York, 1998) p. 21
[4] J E Humphreys, *Reflection groups and Coxeter groups* (Cambridge University Press, Cambridge, 1990) p. 105
[5] M W Davis, *The geometry and topology of Coxeter groups* (Princeton University Press, USA, 2007).
[6] H M Coxeter, *Math. Z.* **46**, 380 (1940)
[7] M Baake, P Kramer, M Schlottmann and D Zeidler, *Int. J. Mod. Phys. B* **4**, 2217 (1990)
M Baake, D Joseph, P Kramer and M Schlottmann, *J. Phys. A: Math. Gen.* **23**, L1037 (1990)
M Baake, A guide to mathematical quasicrystals, in: *Quasicrystals: An introduction to structure, physical properties and applications* edited by J B Suck, M Screiber and P Haussler (Springer-Verlag, Berlin Heidelberg, 2010) p. 17
[8] M Koca, N O Koca and M Al-Ajmi, *Int. J. Geom. Meth. Mod. Phys.* **9**, 4 (2012)
M Koca, N O Koca and R Koç, *Affine $A_4$, quaternions, and decagonal quasicrystals*, preprint arXiv:1209.1878v1 (2012)
[9] M Koca, M Al-Ajmi and R Koç, *J. Math. Phys.* **48**, 113515 (2007)
[10] M Koca, N O Koca and R Koç, Non-crystallographic Coxeter groups from higher dimensional lattices and quasicrystallography, to be published in the *Proceedings of the Workshop and Conference on Groups and Geometry* (Indian Statistical Institute, Bangalore, December 10–21, 2012)
[11] R Horodecki, P Horodecki, M Horodecki and K Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009)
[12] R Mosseri and R Dandoloff, *J. Phys. A: Math. Gen.* **34**, 10243 (2001)
[13] M A Nielsen and I L Chuang, *Quantum computation and quantum information* (Cambridge University Press, Cambridge, 2000)
[14] A R P Rau, *J. Phys. A: Math. Theor.* **42**, 412002 (2009)
[15] M Planat, *Gen. Math. Notes* **2**, 96 (2011)
[16] M Planat, *Phys. Scr.* **T147**, 014025 (2012)
[17] M Planat and P Jorrand, *J. Phys. A: Math. Theor.* **41**, 182001 (2008)
[18] A R P Rau, G Selvaraj and D Uskov, *Phys. Rev. A* **71**, 062316 (2005)
[19] M R Kibler, *J. Phys. A: Math. Theor.* **41**, 375302 (2008)

[20] M R Kibler, *Commun. Nonlinear Sci. Numer. Simulat.* **15**, 752 (2010)
[21] M Saniga and M Planat, *J. Phys. A: Math. Theor.* **44**, 225305 (2011)
[22] B L Cerchiai and B van Geemen, *J. Math. Phys.* **51**, 122203 (2010)
[23] M Planat, *Int. J. Theor. Phys.* **49**, 1044 (2010)
[24] M Planat and M R Kibler, *Unitary reflection groups for quantum fault tolerance*, preprint: arXiv:0807.3650v3 (2008)
[25] Z Fan and H de Garis, *Eur. Phys. J. B* **74**, 419 (2010)
[26] P Vrana and P Lévay, *J. Phys. A: Math. Theor.* **43**, 125303 (2010)
[27] A R P Rau, *Phys. Rev. A* **79**, 042323 (2009)
[28] N D Mermin, *Rev. Mod. Phys.* **65**, 803 (1993)
[29] M Saniga and P Lévay, *Europhys. Lett.* **97**, 50006 (2012)
[30] H S M Coxeter, *Am. Math. Monthly* **53**, 136 (1946)
[31] M Koca, *J. Phys.: Conf. Ser.* **284**, 012040 (2011)
[32] M W Coffey and R Deiotte, *Quantum Inf. Process* **9**, 681 (2010)
[33] Y Makhlin, *Quantum Inf. Process* **1**, 681 (2002)
[34] G Cybenko, *Comp. Sci. Eng.* **3**, 27 (2001)
[35] A T Rezakhani, *Phys. Rev. A* **70**, 052313 (2004)
[36] Z Dong-Lin and K Le-Man, *Chin. Phys. B* **18**, 1328 (2009)
[37] W K Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998)
[38] S Felloni and G Strini, *Int. J. Math. Comp. Sci.* **5**, 202 (2009)
[39] T Durt, T Kurtsiefer, A Lamas-Linares and A Ling, *Phys. Rev. A* **78**, 042338 (2008)
[40] M Koca, R Koç and M Al-Barwani, *J. Math. Phys.* **44**, 3123 (2003)
[41] A Kossakowski and M Ohya, *Infin. Dimens. Anal. Quantum. Probab. Relat. Top.* **10**, 411 (2007)
[42] M Koca, R Koç and M Al-Barwani, *J. Phys. A: Math. Gen.* **34**, 11201 (2001)
[43] R Koç, O N Koca and M Koca, in preparation