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A quantum Hamilton–Jacobi proof of the nodal structure of the wave functions of supersymmetric partner potentials

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Abstract. Quantum Hamilton–Jacobi formalism is used to give a proof for Gozzi's criterion, which states that for eigenstates of the supersymmetric partners, corresponding to the same energy, the difference in the number of nodes is equal to one when supersymmetry (SUSY) is unbroken and is zero when SUSY is broken. We also show that this proof is also applicable to the case, where isospectral deformation is involved.

Keywords. Quantum Hamilton–Jacobi formalism; supersymmetry; Gozzi's criterion; exactly solvable models; bound states.

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1. Introduction

The quantum Hamilton–Jacobi (QHJ) formalism, developed by Leacock and Padgett [1,2] and Gozzi [3], has been successfully used to analyse different types of models in one dimension (1D) [4,5]. It is a straightforward and elegant method, which uses the singularity structure information of the quantum momentum function (QMF),

$$p(x) = -i\hbar \frac{\psi_n'(x)}{\psi_n(x)}, \tag{1}$$

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to obtain the eigenvalues and eigenfunctions for a given potential, without solving the differential equation. Here, $\psi_n(x)$ is the solution of the Schrödinger equation. Using this method one can tackle complicated potentials, after suitable point canonical transformations. The use of the singularity structure information has provided interesting insights into the models studied [6–10], which include the new rational potentials with exceptional polynomials as solutions [11–13]. For supersymmetric one-dimensional ES potentials [14,15], the singularity structure of the QMF provided a link to the exactness of the SWKB integral [13,16]. In addition, for potentials exhibiting the two phases of supersymmetry (SUSY), the parameter regimes where SUSY is unbroken and broken arose naturally within this formalism [17]. Rasinariu *et al* [18] have studied the relation between the QHJ formalism and the shape invariance (SI) property exhibited by all ES models in one dimension. They showed that the SI condition translates to fractional linear relations among the QMFs.

In this paper, we examine the phases of SUSY and provide a proof for Gozzi's criterion [19] using the QHJ machinery. Gozzi showed that when SUSY is unbroken, the difference of nodes corresponding to isospectral eigenfunctions of the partner potentials is one and it is equal to zero when SUSY is broken [19]. Given the eigenfunctions of the partner potentials, it is difficult to see this relation between the nodes of isospectral eigenfunctions. We give a proof for Gozzi's criterion when SUSY is unbroken. This general proof breaks down when SUSY is broken and hence we are forced to analyse explicit potentials with broken SUSY [20] and demonstrate that the difference of nodes is zero for each case individually. This proof has interesting implications for the pair of potentials, where one potential is constructed by the isospectral deformation of an existing potential [15]. These potentials are isospectral partners, but are in general non-shape invariant.

In the next section, we give a brief summary of the QHJ formalism and SUSY. This is followed by the proof for Gozzi's criterion and its implications for the partners, involving isospectral deformation. Subsequently, we take explicit example of potentials, having broken SUSY between them and show how to apply our proof to such cases.

2. Quantum Hamilton–Jacobi formalism

The QMF defined by (1) satisfies the Riccati equation

$$p^2(x) - i\hbar p'(x) = 2m(E - V(x)), \tag{2}$$

and the quantization condition,

$$\frac{1}{2\pi}\oint_C p(x)\mathrm{d}x = n\hbar, \tag{3}$$

defined in terms of $p(x)$ is exact. Here, the contour C encloses the n moving poles of $p(x)$, located in the classical region, corresponding to the nodes of the wave function $\psi_n(x)$ prescribed by the oscillation theorem. By calculating the above contour integral in terms of the singularities of $p(x)$ located outside C, one can obtain the energy eigenvalue. These other singularities include fixed poles, which originate from the potential, and the point at infinity that is assumed to be an isolated singularity. This assumption has worked for all the models studied. The residues at all the poles are double valued and one can use the boundary condition,

$$\lim_{\hbar \to 0} p \to p_{\rm cl}, \tag{4}$$

to choose the right value of residue [1,6]. Here, $p_{\rm cl}(x) = \sqrt{2m(E - V(x))}$ is the classical momentum and is defined to be positive just below the branch cut.

In order to obtain the eigenfunctions, one can write $p(x)$ as a meromorphic function in terms of its singularities. For all the ES models studied, the meromorphic form of QMF $(q(x) \equiv ip(x))$ turned out to be of the form

$$q(x) = -W(x) + \frac{P_n'(x)}{P_n(x)}, \tag{5}$$

where $W(x)$ is the superpotential corresponding to the potential $V(x)$ and $P_n(x)$ is an nth degree polynomial. Note that (5) is a valid solution of the QHJ equation, only if $W(x)$ is related to a normalizable ground state. One can write the QMF in this form, only if the potential has poles as singularities. This follows from the property of the Riccati equation that the only singularities of the QMF are the moving and fixed poles. The residues at the poles can be computed using (2). Hence, the well-known theorems from the theory of analytic functions [21] allow us to determine the above form of the QMF. In general, to be able to complete this programme, a suitable change of variable may have to be performed. For all the ES solvable models studied including the new rational potentials with exceptional polynomial solutions, one could write $q(x)$ in the above meromorphic form and $P_n(x)$ turned out to be one of the classical orthogonal polynomials. Note that (5) is a valid solution of the QHJ equation only if it gives rise to normalizable solutions for the ground state wave function when $n = 0$. Exploiting the relation between the QMF and $\psi_n(x)$ given in (1), one can obtain exact expressions for the eigenfunctions. For more details, we refer the reader to [7–9].

3. Supersymmetric quantum mechanics

Supersymmetry turned out to be a successful method to construct new ES potentials in one dimension [14,15,22]. One method is to use the property of shape invariance to construct an isospectral partner $V_+(x)$ for a given potential $V_-(x)$, whose ground state energy is made zero. One can also construct the partners from the superpotential $W(x)$ using the following relations. Setting $\hbar = 2m = 1$, we have

$$V_-(x) = W^2(x) - W'(x); \qquad V_+(x) = W^2(x) + W'(x). \tag{6}$$

The corresponding Hamiltonians are

$$H_-(x) = \frac{p^2}{2m} + V_-(x); \qquad H_+(x) = \frac{p^2}{2m} + V_+(x). \tag{7}$$

Denoting the eigenfunctions of the partners $V_-(x)$ and $V_+(x)$ as $\psi_E(x)$ and $\chi_E(x)$ respectively, the Schrödinger equations for $V_-(x)$ and $V_+(x)$ are

$$-\frac{\mathrm{d}^2\psi_E(x)}{\mathrm{d}x^2} + V_-(x)\psi_E(x) = E\psi_E(x) \tag{8}$$

and

$$-\frac{\mathrm{d}^2\chi_E(x)}{\mathrm{d}x^2} + V_+(x)\chi_E(x) = E\chi_E(x). \tag{9}$$

The intertwining operators A and $A^\dagger$ are defined as

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x); \qquad A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x) \tag{10}$$

respectively. The two Hamiltonians in terms of these operators are

$$H_-(x) = A^\dagger A; \qquad H_+(x) = AA^\dagger. \tag{11}$$

The isospectral eigenfunctions of $H_-(x)$ are related to those of $H_+(x)$ by the following equations:

$$\psi_E(x) = CA^\dagger\chi_E(x), \qquad \chi_E(x) = DA\psi_E(x), \tag{12}$$

where C and D are constants. SUSY is unbroken when the ground state energy of $V_-(x)$ is zero and $A\psi_0(x) = 0$, with $\psi_0(x)$ being the ground state of $V_-(x)$. Here, $\psi_0(x)$ is normalizable, while $\chi_0(x)$ is non-normalizable and the partners are isospectral except for the ground state energy. When SUSY is spontaneously broken, the partners are isospectral including the ground states and the superpotential gives rise to non-normalizable ground state solutions of the Schrödinger equations corresponding to both the partners. In the following section, we consider the case where SUSY is unbroken and show that the eigenfunctions with same energy differ by one node.

4. Relation between the logarithmic derivatives

We introduce the logarithmic derivatives of the eigenfunctions as

$$q_E(x) = \frac{1}{\psi_E(x)}\frac{\mathrm{d}}{\mathrm{d}x}\psi_E(x), \qquad k_E(x) = \frac{1}{\chi_E(x)}\frac{\mathrm{d}}{\mathrm{d}x}\chi_E(x) \tag{13}$$

such that

$$\psi_E(x) = \alpha \exp\left(\int q_E(x)\mathrm{d}x\right), \qquad \chi_E(x) = \beta \exp\left(\int k_E(x)\mathrm{d}x\right), \tag{14}$$

where α and β are the normalization constants. Using (12) one can obtain a relation between the two wave functions as

$$\begin{aligned}\psi_E(x) &= CA^\dagger\chi_E(x)\\ &= C\left(-\frac{\mathrm{d}}{\mathrm{d}x}\chi_E(x) + W(x)\chi_E(x)\right).\end{aligned} \tag{15}$$

Dividing by $\chi_E(x)$ on both sides and using (13), we obtain

$$\frac{\psi_E(x)}{\chi_E(x)} = C(-k_E(x) + W(x)). \tag{16}$$

Similarly, using the equation for $\chi_E(x)$, we can obtain

$$\frac{\chi_E(x)}{\psi_E(x)} = D(q_E(x) + W(x)). \tag{17}$$

Multiplying (16) and (17), we obtain

$$CD[q_E(x) + W(x)][-k_E(x) + W(x)] = 1. \tag{18}$$

Assuming that the eigenfunctions are normalized, one can write

$$\begin{aligned}(\chi_E(x), \chi_E(x)) &= (\chi_E(x), DA\psi_E(x)) \\ &= (\chi_E(x), DC\,AA^\dagger \chi_E(x)) = CDE = 1\end{aligned} \tag{19}$$

which implies

$$CD = \frac{1}{E}. \tag{20}$$

Therefore, (18) becomes

$$[q_E(x) + W(x)][-k_E(x) + W(x)] = E. \tag{21}$$

Using the QHJ equation for $q_E(x)$,

$$q_E^2(x) + q_E'(x) + E - V_-(x) = 0, \tag{22}$$

one obtains

$$E = -[(q_E^2(x) - W^2(x)) + (q_E'(x) + W'(x))], \tag{23}$$

where $V_-(x)$ in terms of the superpotential is used. Substituting this expression of E in (21) and after algebraic manipulation, one finds,

$$k_E(x) = q_E(x) + \left(\frac{q_E'(x) + W'(x)}{q_E(x) + W(x)}\right) \tag{24}$$

with $E = E_n$, which implies that for the log derivative, $q_E(x)$, the point at infinity is an isolated singularity, and we can write $q_E(x)$ as a meromorphic function. Substituting the meromorphic form of $q_E(x)$ from (5) in the second term of the above equation we get

$$k_E(x) - q_E(x) = \frac{q_E'(x) + W'(x)}{q_E(x) + W(x)} = \frac{P_n''(x)}{P_n(x)} - \frac{P_n'(x)}{P_n(x)}. \tag{25}$$

Integrating over the contour C, which encloses n nodes on the real line, we obtain

$$\oint_C k_E(x)\mathrm{d}x = \oint_C q_E(x)\mathrm{d}x + \oint_C \left(\frac{P_n''(x)}{P_n(x)} - \frac{P_n'(x)}{P_n(x)}\right)\mathrm{d}x. \tag{26}$$

Using the the argument principle [21], we obtain

$$\oint_C k_E(x)\mathrm{d}x = \oint_C q_E(x)\mathrm{d}x + (n-1) - n, \tag{27}$$

which gives

$$\oint_C q_E(x)\mathrm{d}x - \oint_C k_E(x)\mathrm{d}x = 1. \tag{28}$$

Thus, we see that log derivatives corresponding to $\psi_E(x)$ and $\chi_E(x)$ have a difference of one pole. This implies that the eigenfunctions of the partner potentials with the same energy, have a difference of one node between them. This completes the proof of Gozzi's criterion. We point out here that the above proof breaks down if $W(x)$ does not correspond to a normalizable ground state. This is exactly what happens when SUSY is broken, which is analysed later.

4.1 *Isospectral deformation*

The technique of isospectral deformation is employed to generate a new family of strictly isospectral potentials. In this family the partners are isospectral including the ground state and they are not, in general, shape invariant. Consider the partners $V_-(x)$, $V_+(x)$ and the corresponding superpotential $W(x)$. We want to find the most general superpotential, $\tilde{W}(x)$, where

$$\tilde{W}(x) = W(x) + \phi(x) \tag{29}$$

such that $\tilde{V}_+(x) = (\tilde{W}(x))^2 + (\mathrm{d}/\mathrm{d}x)\tilde{W}(x)$ is the same as $V_+(x)$. The form of $\phi(x)$ is obtained by demanding that

$$V_+(x) = \tilde{V}_+(x, \lambda) = \tilde{W}^2(x) + \tilde{W}'(x), \tag{30}$$

which leads to the Bernoulli's equation for $\phi(x)$:

$$\phi^2(x) + 2W(x)\phi(x) + \phi'(x) = 0. \tag{31}$$

The solutions of this equation are of the form

$$\phi(x) = \frac{\mathrm{d}}{\mathrm{d}x}(I_0(x) + \lambda), \tag{32}$$

where

$$I_0(x) = \int_0^x (\psi_0^-(x))^2 \mathrm{d}x, \tag{33}$$

with $\psi_0^-(x)$ being the ground state of $V_-(x)$ and λ taking values in the ranges $\lambda > 0$ and $\lambda < -1$. Thus, the most general superpotential $\tilde{W}(x)$ turns out to be

$$\tilde{W}(x) = W(x) + \frac{\mathrm{d}}{\mathrm{d}x}(I_0(x) + \lambda), \tag{34}$$

and the expression for the partner $\tilde{V}_-(x, \lambda)$ is given by

$$\tilde{V}_-(x, \lambda) = V_-(x) - \frac{4\psi_0^-(x)\psi_0^{-\prime}(x)}{(I_0(x) + \lambda)} + \frac{2(\psi_0^-(x))^4}{(I_0(x) + \lambda)^2}. \tag{35}$$

From the above equation, it is clear that $V_-(x)$ and $\tilde{V}_-(x, \lambda)$ are not shape invariant, but are isospectral including the ground state. When one looks at the expression for the new potential, the singularity structure of the QMF corresponding to this potential is not transparent. Therefore, it is not clear from the QHJ point of view that the QHJ condition gives a spectrum identical to the original potential. But, the fact that $\tilde{W}(x)$ corresponds to a normalized wave function allows our proof of Gozzi's criterion to be extended to this case also, and show that the spectrum of the new potential is identical to that of the old potential. Proof of Gozzi's index is applicable to these potentials as well. Normalized ground state wave function of $\tilde{V}_+(x,\lambda)$ is given as

$$\tilde{\psi}_0^+(x) = \frac{\sqrt{\lambda(1+\lambda)}\psi_0^-(x)}{I_0(x)+\lambda}.$$

4.2 *Broken SUSY case*

The correspondence between the nodes of the partners with broken SUSY will be established using the following example [20]. Consider the superpotential

$$W(r,l,\omega) = \frac{\omega r}{2} - \frac{l+1}{r}; \qquad l < -1, \tag{36}$$

giving partner potentials

$$V_1(r,l,\omega) = \frac{\omega^2 r^2}{4} + \frac{l(l+1)}{r^2} - \left(l+\frac{3}{2}\right)\omega, \tag{37}$$

$$V_2(r,l,\omega) = \frac{\omega^2 r^2}{4} + \frac{(l+1)(l+2)}{r^2} - \left(l+\frac{1}{2}\right)\omega, \tag{38}$$

which have broken SUSY between them [20]. They are shape invariant and are related to each other through the relation

$$V_2(r,l,\omega) = V_1(r,l+1,\omega) + 2\omega. \tag{39}$$

From (36), it is seen that the 'ground state solutions', $\exp(\pm\int W(r)\,\mathrm{d}r)$, for the partners are non-normalizable and hence one cannot use the intertwining operators to obtain the eigenfunctions. Therefore, our earlier proof for Gozzi's criterion breaks down.

In order to show the relation between the nodes of the eigenfunctions of the same energy in the case of broken SUSY, we look for potentials $V_1^-(r,l,\omega)$ and $V_2^-(r,l,\omega)$ constructed such that the SUSY is unbroken between the pairs $V_1(r,l,\omega)$, $V_1^-(r,l,\omega)$ and $V_2(r,l,\omega)$, $V_2^-(r,l,\omega)$. The relationship between various potentials is illustrated in figure 1.

In [20], the partner of $V_1(r,l,\omega)$, which has normalizable ground state wave function is given as

$$V_1^-(r,l,\omega) = \frac{\omega^2 r^2}{4} + \frac{(l+1)(l+2)}{r^2} + \left(l+\frac{1}{2}\right)\omega. \tag{40}$$

The corresponding superpotential is

$$W_1 = \frac{\omega r}{2} + \frac{l+1}{r}; \qquad l < -1. \tag{41}$$

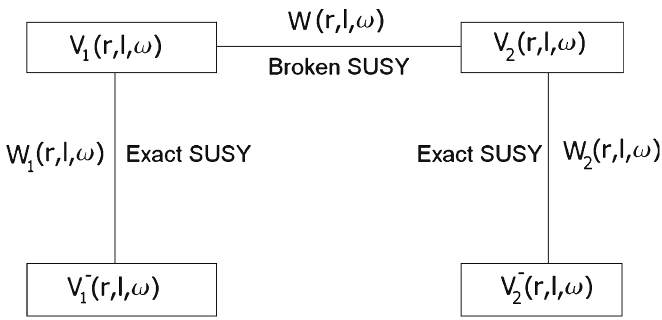

Figure 1. Potentials V_1 and V_2 and their partners with which they share unbroken SUSY.

The shape invariance is obtained through

$$V_1(r,l,\omega) = V_1^-(r,l-1,\omega) - \omega(2l-4). \tag{42}$$

The ground state of $V_1^-(r,l,\omega)$ is normalizable and SUSY is unbroken between them. Therefore, we can say that the two states with the same energy corresponding to $V_1^-(r,l,\omega)$ and $V_1(r,l,\omega)$ have n and $n-1$ nodes respectively.

Similarly for the other potential $V_2(r,l,\omega)$, the partner with which it has unbroken SUSY is given as [20]

$$V_2^-(r,l) = \frac{\omega^2 r^2}{4} + \frac{(l+2)(l+3)}{r^2} + \left(l+\frac{3}{2}\right)\omega. \tag{43}$$

The corresponding superpotential is

$$W_2 = \frac{\omega r}{2} + \frac{l+2}{r}; \qquad l < -1. \tag{44}$$

The shape invariance relation between these partners is

$$V_2(r,l,\omega) = V_2^-(r,l-1,\omega) + \omega(2l+1). \tag{45}$$

Since SUSY is unbroken between them, we can assert that the nth excited state of $V_2^-(r,l,\omega)$ has n nodes and the corresponding state of $V_2(r,l,\omega)$ has $n-1$ nodes. Thus, we have shown that the number of nodes corresponding to the nth excited states of $V_1(r,l,\omega)$ and $V_2(r,l,\omega)$ are $n-1$. Therefore, the difference of nodes between states of the partners, with broken SUSY, corresponding to the same energy is zero.

5. QHJ construction of exact SUSY partners

Given exactly solvable broken SUSY partner potentials $V_i(x)$, $i=1,2$, an explicit construction of new potentials $V_i^-(x)$ can be generally achieved within the QHJ framework. To find the potentials $V_i^-(x)$, so that SUSY is exact for the pairs $V_i(x)$ and $V_i^-(x)$, we look for solutions of the Riccati equation

$$W_i^2(x) + W_i'(x) = V_i(x) - C_i, \qquad i=1,2 \tag{46}$$

for the suprepotentials $W_i(x)$, where C_i are constants. The superpotential $W_i(x)$ equals the QMF apart from a numerical factor. In general, QMF corresponding to an excited state will have moving poles. For the ground state these moving poles are absent. The QMF, and hence $W_i(x)$, are completely determined by the fixed singular points of the potential $V_i(x)$. The quadratic nature of (46) gives two solutions for $W_i(x)$. We choose $W_i(x)$ such that

$$\exp\left(-\int W_i(x)\mathrm{d}x\right) = \psi_0(x) \tag{47}$$

is normalizable and therefore becomes the zero energy ground state solution of the potential $V_i^-(x) = W_i^2(x) - W_i'(x)$. With $W_i^2(x) + W_i'(x) + C_i = V_i(x)$ SUSY is exact between $V_i(x)$ and $V_i^-(x)$. Thus, for potentials supporting the bound states, we can always construct a $W_i(x)$ which is associated with a normalizable ground state.

Applying the above process to the potentials $V_1(x)$ and $V_2(x)$, we shall arrive at the familiar results of [20] already used above.

6. Conclusions

Making use of the QHJ machinery, we have shown that the nodes of eigenfunctions of the SUSY partners, corresponding to the same energy, have a difference of one node when SUSY is unbroken and zero nodes when SUSY is broken. This furnishes a formal and independent proof of Gozzi's criterion.

The process of isospectral deformation from $W(x)$ to $\tilde{W}(x)$ leads to strictly isospectral potentials $V_+(x)$ and $\tilde{V}_+(x)$. The proof of SUSY being unbroken in this case is completed via the proof of Gozzi's criterion.

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