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Exact solutions of some nonlinear partial differential equations using functional variable method

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Abstract. The functional variable method is a powerful solution method for obtaining exact solutions of some nonlinear partial differential equations. In this paper, the functional variable method is used to establish exact solutions of the generalized forms of Klein–Gordon equation, the (2 + 1)-dimensional Camassa–Holm Kadomtsev–Petviashvili equation and the higher-order nonlinear Schrödinger equation. By using this useful method, we found some exact solutions of the above-mentioned equations. The obtained solutions include solitary wave solutions, periodic wave solutions and combined formal solutions. It is shown that the proposed method is effective and general.

Keywords. Functional variable method; (2 + 1)-dimensional Camassa–Holm Kadomtsev–Petviashvili equation; generalized forms of Klein–Gordon equation; higher-order nonlinear Schrödinger equation.

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1. Introduction

Nonlinear partial differential equations (NPDEs) are encountered in various disciplines, such as physics, mechanics, chemistry, biology, mathematics and engineering. The study of exact solutions of these equations plays a major role in the study of the propagation of waves. Nonlinear wave equations involve one or more of dispersion, dissipation, diffusion, reaction and convection. The balance between nonlinearity and linear dispersion generates solitons. However, the balance between nonlinearity and genuinely nonlinear dispersion gives rise to the so-called compactons: solitons free of exponential wings.

Recently, many new approaches for finding exact solutions to NPDEs have been proposed, such as ansatz method and topological solitons [1–8], tanh method [9,10], multiple

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exp-function method [11], simplest equation method [12–15], Hirota's direct method [16,17], transformed rational function method [18] and so on.

The functional variable method, which is a direct and effective algebraic method for computing exact travelling wave solutions, was first proposed by Zerarka *et al* [19]. This method was further developed by others [20,21].

The aim of this paper is to construct exact solutions of the generalized forms of Klein–Gordon equation, the (2 + 1)-dimensional Camassa–Holm Kadomtsev–Petviashvili equation (CH–KP equation) and the higher-order nonlinear Schrödinger equation by using the functional variable method.

The Klein–Gordon equation (KGE) and the higher-order nonlinear Schrödinger equation play significant roles in many areas of science such as solid-state physics, nonlinear optics and quantum field theory.

The generalized KGE (gKGE) [22] is modelled by the equation

$$(q^m)_{tt} - k^2(q^m)_{xx} + F(q) = 0, \tag{1}$$

where the dependent variable $q(x,t)$ represents the wave profile. Also, k is a constant and m is a positive integer with $m \geq 1$. In fact, if $m = 1$, eq. (1) reduces to the regular KGE. In this paper, the following three forms of the function $F(q)$ will be considered:

$$F(q) = aq^m - bq^{2m}, \tag{2}$$

$$F(q) = aq^m - bq^{3m}, \tag{3}$$

$$F(q) = aq^m - bq^n. \tag{4}$$

These three cases will be respectively labelled as Forms I–III. In all these three forms, a and b are real constants.

The paper is arranged as follows. In §2, the functional variable method is described briefly. In §3, this method is applied to the generalized forms of Klein–Gordon equation, the CH–KP equation and the higher-order nonlinear Schrödinger equation.

2. The functional variable method

Zerarka *et al* [19] introduced the functional variable method for constructing exact solutions of the nonlinear wave equations. Zerarka and Ouamane [20] proposed the functional variable method to solve the generalized form of the Boussinesq system and the regularized long-wave (RLW) equation. Cevikel *et al* [21] used the functional variable method to obtain exact solutions of the Zakharov–Kuznetsov-modified equal-width (ZK-MEW), the modified Benjamin–Bona–Mahony (mBBM) and the modified KdV–Kadomtsev–Petviashvili (KdV–KP) equations.

Consider a nonlinear evolution equation

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, ...) = 0, \tag{5}$$

where P is a polynomial in u and its partial derivatives.

To find the travelling wave solution of eq. (5) we introduce the wave variable $\xi = x - ct$ so that

$$u(x,t) = U(\xi). \tag{6}$$

The nonlinear partial differential equation can be converted to an ordinary differential equation (ODE) as

$$Q(U, U', U'', ...) = 0, \tag{7}$$

where Q is a polynomial in U and its total derivatives and $' = \mathrm{d}/\mathrm{d}\xi$.

Let us make a transformation in which the unknown function $U(\xi)$ is considered as a functional variable in the form

$$U_\xi = F(U) \tag{8}$$

and some successive derivatives of U are

$$\begin{aligned} U_{\xi\xi} &= \frac{1}{2}(F^2)', \\ U_{\xi\xi\xi} &= \frac{1}{2}(F^2)''\sqrt{F^2}, \\ U_{\xi\xi\xi\xi} &= \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'], \end{aligned} \tag{9}$$

where $' = \mathrm{d}/\mathrm{d}U$.

The ODE (7) can be reduced in terms of $U,\ F$ and its derivatives upon using the expressions of eq. (9) into eq. (7) gives

$$R(U, F, F', F'', F''', ...) = 0. \tag{10}$$

This particular eq. (10) is of special interest because it admits analytical solutions for a large class of nonlinear wave-type equations. After integration, eq. (10) provides the expression of F, and this together with eq. (8) give relevant solutions to the original problem. In order to illustrate how the method works, we examine some examples treated by other approaches. This matter is exposed in the following section.

3. Applications

In this section, three examples are presented to illustrate the applicability of the functional variable method to solve nonlinear partial differential equations.

Example 3.1. Let us consider the CH–KP equation [23]

$$(u_t + 2ku_x - u_{xxt} - au^n u_x)_x + u_{yy} = 0 \tag{11}$$

and

$$(w_t + 2kw_x - w_{xxt} + aw^n(w^n)_x)_x + w_{yy} = 0, \tag{12}$$

where n is the strength of nonlinearity, and $a > 0,\ \ k \in R$.

Now, applying the transformations

$$u(x,y,t)=U(\xi), \qquad w(x,y,t)=W(\xi), \qquad \xi=x+ly+ct, \tag{13}$$

to eqs (12) and (13) and integrating the resultant equations twice, we get

$$(c+2k+l^2)U-cU''-\frac{a}{n+1}U^{n+1}=0 \tag{14}$$

and

$$(c+2k+l^2)W-cW''+\frac{a}{2}W^{2n}=0. \tag{15}$$

The constants of integration are zero since the solitary wave solution and its derivatives are zero as $\xi\to\pm\infty$.

Then we use the transformations

$$U_\xi=F(U), \qquad W_\xi=F(W), \tag{16}$$

that will convert eqs (14) and (15) to

$$\frac{c+2k+l^2}{c}U-\frac{a}{c(n+1)}U^{n+1}=\frac{(F^2(U))'}{2} \tag{17}$$

and

$$\frac{c+2k+l^2}{c}W+\frac{a}{2c}W^{2n}=\frac{(F^2(W))'}{2}. \tag{18}$$

According to eq. (9), we get from eqs (17) and (18) the expressions of the functions $F(U)$ and $F(W)$ as

$$F(U)=\sqrt{\frac{c+2k+l^2}{c}}U\sqrt{1-\frac{2a}{(c+2k+l^2)(n+1)(n+2)}U^n} \tag{19}$$

and

$$F(W)=\sqrt{\frac{c+2k+l^2}{c}}W\sqrt{1+\frac{a}{(c+2k+l^2)(2n+1)}W^{2n-1}}. \tag{20}$$

After changing the variables

$$Z_1=\frac{2a}{(c+2k+l^2)(n+1)(n+2)}U^n \tag{21}$$

and

$$Z_2=-\frac{a}{(c+2k+l^2)(2n+1)}W^{2n-1}, \tag{22}$$

and using the relation (8), the solutions of eqs (14) and (15) are in the following forms:

$$U(\xi) = \left\{ -\frac{(c+2k+l^2)(n+1)(n+2)}{2a}\operatorname{csch}^2\left(\frac{n}{2}\sqrt{\frac{c+2k+l^2}{c}}\xi\right)\right\}^{1/n}, \tag{23}$$

$$W(\xi) = \left\{ \frac{(c+2k+l^2)(2n+1)}{a}\operatorname{csch}^2\left(\frac{2n-1}{2}\sqrt{\frac{c+2k+l^2}{c}}\xi\right)\right\}^{1/(2n-1)}. \tag{24}$$

We can easily obtain the following hyperbolic solutions:

$$u_1(x,y,t) = \left\{ -\frac{(c+2k+l^2)(n+1)(n+2)}{2a} \times \operatorname{csch}^2\left(\frac{n}{2}\sqrt{\frac{c+2k+l^2}{c}}(x+ly+ct)\right)\right\}^{1/n}, \tag{25}$$

$$w_1(x,y,t) = \left\{ \frac{(c+2k+l^2)(2n+1)}{a} \times \operatorname{csch}^2\left(\frac{2n-1}{2}\sqrt{\frac{c+2k+l^2}{c}}(x+ly+ct)\right)\right\}^{1/(2n-1)} \tag{26}$$

and

$$u_2(x,y,t) = \left\{ \frac{(c+2k+l^2)(n+1)(n+2)}{2a} \times \operatorname{sech}^2\left(\frac{n}{2}\sqrt{\frac{c+2k+l^2}{c}}(x+ly+ct)\right)\right\}^{1/n}, \tag{27}$$

$$w_2(x,y,t) = \left\{ -\frac{(c+2k+l^2)(2n+1)}{a} \times \operatorname{sech}^2\left(\frac{2n-1}{2}\sqrt{\frac{c+2k+l^2}{c}}(x+ly+ct)\right)\right\}^{1/(2n-1)}. \tag{28}$$

For $(c+2k+l^2)/c < 0$, it is easy to see that solutions (25)–(28) can reduce to periodic solutions as follows:

$$u_3(x,y,t) = \left\{ \frac{(c+2k+l^2)(n+1)(n+2)}{2a} \times \csc^2\left(\frac{n}{2}\sqrt{-\frac{c+2k+l^2}{c}}(x+ly+ct)\right)\right\}^{1/n}, \tag{29}$$

$$w_3(x,y,t) = \left\{ -\frac{(c+2k+l^2)(2n+1)}{a} \times \csc^2\left(\frac{2n-1}{2}\sqrt{-\frac{c+2k+l^2}{c}}(x+ly+ct)\right)\right\}^{1/(2n-1)} \tag{30}$$

and

$$u_4(x,y,t) = \left\{\frac{(c+2k+l^2)(n+1)(n+2)}{2a} \times \sec^2\left(\frac{n}{2}\sqrt{-\frac{c+2k+l^2}{c}}(x+ly+ct)\right)\right\}^{1/n}, \tag{31}$$

$$w_4(x,y,t) = \left\{-\frac{(c+2k+l^2)(2n+1)}{a} \times \sec^2\left(\frac{2n-1}{2}\sqrt{-\frac{c+2k+l^2}{c}}(x+ly+ct)\right)\right\}^{1/(2n-1)}. \tag{32}$$

Example 3.2. Here, the generalized forms of Klein–Gordon equation are investigated.

Form I

In this case, by virtue of eqs (1) and (2), the gKGE is given by

$$(q^m)_{tt} - k^2(q^m)_{xx} + aq^m - bq^{2m} = 0. \tag{33}$$

The wave variable $\xi = x - ct$ carries (33) into the ODE

$$(c^2 - k^2)(q^m)'' + aq^m - bq^{2m} = 0. \tag{34}$$

We use the transformation

$$q(\xi) = V^{1/m}(\xi), \tag{35}$$

that will reduce (34) into the ODE

$$(c^2 - k^2)V'' + aV - bV^2 = 0. \tag{36}$$

According to eq. (9), we get from eq. (36) the expression of the function $F(V)$ as

$$F(V) = \sqrt{\frac{a}{k^2 - c^2}}V\sqrt{1 - \frac{2b}{3a}V}. \tag{37}$$

Using transformation (8), and then setting the constants of integration to zero, we can obtain the following result:

$$U(\xi) = -\frac{3a}{2b}\operatorname{csch}^2\left(\sqrt{\frac{a}{4(k^2-c^2)}}\xi\right). \tag{38}$$

When $a/4(k^2 - c^2) > 0$, we have the following hyperbolic solutions:

$$q_1(x,t) = \left(-\frac{3a}{2b}\right)^{1/m}\operatorname{csch}^{2/m}\left(\sqrt{\frac{a}{4(k^2-c^2)}}(x-ct)\right), \tag{39}$$

$$q_2(x,t) = \left(\frac{3a}{2b}\right)^{1/m}\operatorname{sech}^{2/m}\left(\sqrt{\frac{a}{4(k^2-c^2)}}(x-ct)\right). \tag{40}$$

When $a/4(k^2 - c^2) < 0$, we can obtain the following periodic solutions:

$$q_3(x,t) = \left(\frac{3a}{2b}\right)^{1/m} \csc^{2/m}\left(\sqrt{\frac{a}{4(c^2-k^2)}}(x-ct)\right), \tag{41}$$

$$q_4(x,t) = \left(\frac{3a}{2b}\right)^{1/m} \sec^{2/m}\left(\sqrt{\frac{a}{4(c^2-k^2)}}(x-ct)\right). \tag{42}$$

Form II

In this case, eqs (1) and (3) together gives

$$(q^m)_{tt} - k^2(q^m)_{xx} + aq^m - bq^{3m} = 0. \tag{43}$$

Using the wave transformation

$$q(x,t) = q(\xi), \qquad \xi = x - ct \tag{44}$$

eq. (43) is converted to the following ODE:

$$(c^2 - k^2)(q^m)'' + aq^m - bq^{3m} = 0. \tag{45}$$

We propose a transformation denoted by

$$q(\xi) = V^{1/m}(\xi), \tag{46}$$

then eq. (45) is converted to

$$(c^2 - k^2)V'' + aV - bV^3 = 0. \tag{47}$$

Following eq. (9), it is easy to deduce from eq. (47) the expression of the function $F(V)$ which reads as

$$F(V) = \sqrt{\frac{a}{k^2-c^2}}V\sqrt{1 - \frac{b}{2a}V^2}. \tag{48}$$

The solution of eq. (47) is obtained as

$$V(\xi) = i\sqrt{\frac{2a}{b}}\operatorname{csch}\left(\sqrt{\frac{a}{k^2-c^2}}\xi\right). \tag{49}$$

Using the transformation (46), we can easily obtain the following hyperbolic solutions:

$$q_1(x,t) = \left(-\frac{2a}{b}\right)^{1/2m} \operatorname{csch}^{1/m}\left(\sqrt{\frac{a}{k^2-c^2}}(x-ct)\right), \tag{50}$$

$$q_1(x,t) = \left(\frac{2a}{b}\right)^{1/2m} \operatorname{sech}^{1/m}\left(\sqrt{\frac{a}{k^2-c^2}}(x-ct)\right). \tag{51}$$

For $a/(k^2 - c^2) < 0$, it is easy to see that solutions (50) and (51) can reduce to periodic solutions as follows:

$$q_3(x,t) = \left(\frac{2a}{b}\right)^{1/2m} \csc^{1/m}\left(\sqrt{\frac{a}{c^2-k^2}}(x-ct)\right), \tag{52}$$

$$q_4(x,t) = \left(\frac{2a}{b}\right)^{1/2m} \sec^{1/m}\left(\sqrt{\frac{a}{c^2-k^2}}(x-ct)\right). \tag{53}$$

Form III

Here, eqs (1) and (4) together imply

$$(q^m)_{tt} - k^2(q^m)_{xx} + aq^m - bq^n = 0. \tag{54}$$

This is the generalized form of the nonlinear KGE. The special case $m = 1$ reduces to the first type of nonlinear KGE that was studied along with its perturbation terms [24]. In particular, the case $m = 1$ with $n = 3$ is called the Φ^6 model that appears in solid-state physics, condensed matter physics as well as quantum field theory [25]. More details are presented in [22].

Assume eq. (54) has the travelling wave solution as

$$q(x,t) = q(\xi), \qquad \xi = x - ct, \tag{55}$$

where c is the wave velocity. Substituting (55) into eq. (54) yields

$$(c^2 - k^2)(q^m)'' + aq^m - bq^n = 0. \tag{56}$$

By making the following transformation

$$q(\xi) = V^{1/m}(\xi), \tag{57}$$

eq. (56) becomes

$$(c^2 - k^2)V'' + aV - bV^{n/m} = 0. \tag{58}$$

Following eq. (9), it is easy to deduce from eq. (58) the expression of the function $F(V)$ as

$$F(V) = \sqrt{\frac{a}{k^2 - c^2}}V\sqrt{1 - \frac{2bm}{a(m+n)}V^{(n/m)-1}}. \tag{59}$$

After changing the variables

$$Z = \frac{2bm}{a(m+n)}V^{(n/m)-1}, \tag{60}$$

and using the relation (8), the solution of eq. (58) is in the following form:

$$V(\xi) = \left\{-\frac{a(n+m)}{2bm}\operatorname{csch}^2\left(\frac{n-m}{2m}\sqrt{\frac{a}{k^2-c^2}}\xi\right)\right\}^{m/(n-m)}. \tag{61}$$

Using the transformation (57), we can get the following hyperbolic solutions of eq. (54):

$$q_1(x,t) = \left\{-\frac{a(n+m)}{2bm}\operatorname{csch}^2\left(\frac{n-m}{2m}\sqrt{\frac{a}{k^2-c^2}}(x-ct)\right)\right\}^{1/(n-m)} \tag{62}$$

and

$$q_2(x,t) = \left\{\frac{a(n+m)}{2bm}\operatorname{sech}^2\left(\frac{n-m}{2m}\sqrt{\frac{a}{k^2-c^2}}(x-ct)\right)\right\}^{1/(n-m)}, \tag{63}$$

for $a/(k^2 - c^2) > 0$. It is easy to see that solutions (62) and (63) can reduce to periodic solutions as follows:

$$q_3(x,t) = \left\{\frac{a(n+m)}{2bm}\csc^2\left(\frac{n-m}{2m}\sqrt{\frac{a}{c^2-k^2}}(x-ct)\right)\right\}^{1/(n-m)} \tag{64}$$

and

$$q_4(x,t) = \left\{\frac{a(n+m)}{2bm}\sec^2\left(\frac{n-m}{2m}\sqrt{\frac{a}{c^2-k^2}}(x-ct)\right)\right\}^{1/(n-m)}, \tag{65}$$

for $a/(k^2-c^2) < 0$.

By comparing our results with Biswas *et al*'s results [22], it can be seen that the results are the same.

Example 3.3. The higher-order nonlinear Schrödinger equation in the form

$$q_z = ia_1q_{tt} + ia_2q|q|^2 + a_3q_{ttt} + a_4(q|q|^2)_t + a_5q(|q|^2)_t, \tag{66}$$

describes propagation of ultrashort pulses in nonlinear optical fibres, where the complex function $q = q(z,t)$ is a slowly varying envelope of the electric field, the subscripts z and t are spatial and temporal partial derivatives in retard time coordinates. a_1, a_2, a_3, a_4 and a_5 are respectively the real parameters related to the group velocity, self-phase modulation, third-order dispersion, self-steepening and self-frequency shift arising from stimulated Raman scattering. Recently, some exact solitary wave solutions of eq. (66) have been successfully obtained by the generally projective Riccati method, the F-expansion method and the extended F-expansion method, etc. [26–28].

Since $q = q(z,t)$ in eq. (66) is a complex function we suppose that

$$q(z,t) = f(\xi)\exp[i(kz+\omega t)], \qquad \xi = t + \lambda z, \tag{67}$$

where $f(\xi)$ is a real function, k, ω and λ are constants, all of which are to be determined.

Substituting the travelling wave variables (67) into eq. (66), removing the common factor $\mathrm{e}^{i(kz+\omega t)}$, we have the overdetermined ODEs for $f(\xi)$

Im:

$$(a_1+3a_3\omega)f''(\xi) - (a_3\omega^3 + a_1\omega^2 + k)f(\xi) + (a_2+a_4\omega)f^3(\xi) = 0, \tag{68}$$

Re:

$$a_3f'''(\xi) - (2a_1\omega + 3a_3\omega^2 + \lambda)f'(\xi) + (3a_4+2a_5)f^2(\xi)f'(\xi) = 0. \tag{69}$$

Integrating eq. (69) once and setting the integration constants to be zero, we obtain

$$a_3f''(\xi) - (2a_1\omega + 3a_3\omega^2 + \lambda)f(\xi) + \frac{1}{3}(3a_4+2a_5)f^3(\xi) = 0. \tag{70}$$

It can be proved that the following conclusion holds: the necessary and sufficient condition for a non-constant function $f(\xi)$ satisfying both eqs (68) and (70) is that the coefficients of eqs (68) and (70) satisfy the proportional relation as follows:

$$\frac{a_3}{a_1+3a_3\omega} = \frac{2a_1\omega + 3a_3\omega^2 + \lambda}{a_3\omega^3 + a_1\omega^2 + k} = \frac{3a_4+2a_5}{3(a_2+a_4\omega)}, \tag{71}$$

from which it is derived that

$$\omega = \frac{3a_2a_3 - a_1(3a_4 + 2a_5)}{6a_3(a_4 + a_5)}, \tag{72}$$

$$k = 8a_3\omega^3 + 8a_1\omega^2 + \frac{2a_1^2 + 3\lambda a_3}{a_3}\omega + \frac{a_1}{a_3}\lambda. \tag{73}$$

Based on this conclusion, we only solve eq. (70) (or eq. (68)), instead of both eqs (68) and (70), provided that ω and k appearing in eqs (68) and (70) are replaced by (72) and (73) respectively.

Rewrite second-order ordinary differential equation (70) as follows:

$$f''(\xi) - \left(\frac{2a_1\omega + 3a_3\omega^2 + \lambda}{a_3}\right) f(\xi) + \left(\frac{3a_4 + 2a_5}{3a_3}\right) f^3(\xi) = 0. \tag{74}$$

Then we use the transformation

$$f_\xi = F(f) \tag{75}$$

that will convert eq. (74) to

$$\frac{(F^2(f))'}{2} - \left(\frac{2a_1\omega + 3a_3\omega^2 + \lambda}{a_3}\right) f + \left(\frac{3a_4 + 2a_5}{3a_3}\right) f^3 = 0. \tag{76}$$

According to eq. (9), we get from eq. (76) the expression for function $F(f)$ as

$$F(f) = \sqrt{\frac{2a_1w + 3a_3w^2 + \lambda}{a_3}}\, f \sqrt{1 - \frac{3a_4 + 2a_5}{6(2a_1w + 3a_3w^2 + \lambda)} f^2}. \tag{77}$$

The solution of eq. (74) is obtained as

$$f(\xi) = \pm\sqrt{-6\frac{2a_1w + 3a_3w^2 + \lambda}{3a_4 + 2a_5}} \operatorname{csch}\left(\sqrt{\frac{2a_1w + 3a_3w^2 + \lambda}{a_3}}\xi\right). \tag{78}$$

When $(2a_1w + 3a_3w^2 + \lambda)/a_3 > 0$, we have the following hyperbolic solutions:

$$q_1(z,t) = \pm\sqrt{-6\left(\frac{2a_1w + 3a_3w^2 + \lambda}{3a_4 + 2a_5}\right)} \times \operatorname{csch}\left(\sqrt{\frac{2a_1w + 3a_3w^2 + \lambda}{a_3}}(t + \lambda z)\right) \exp[i(kz + \omega t)], \tag{79}$$

$$q_2(z,t) = \pm\sqrt{6\left(\frac{2a_1w + 3a_3w^2 + \lambda}{3a_4 + 2a_5}\right)} \times \operatorname{sech}\left(\sqrt{\frac{2a_1w + 3a_3w^2 + \lambda}{a_3}}(t + \lambda z)\right) \exp[i(kz + \omega t)]. \tag{80}$$

When $(2a_1w + 3a_3w^2 + \lambda)/a_3 < 0$, we can obtain the following periodic solutions:

$$q_3(z,t) = \pm\sqrt{6\left(\frac{2a_1w + 3a_3w^2 + \lambda}{3a_4 + 2a_5}\right)} \times \csc\left(\sqrt{-\frac{2a_1w + 3a_3w^2 + \lambda}{a_3}}(t + \lambda z)\right)\exp[i(kz + \omega t)], \quad (81)$$

$$q_4(z,t) = \pm\sqrt{6\left(\frac{2a_1w + 3a_3w^2 + \lambda}{3a_4 + 2a_5}\right)} \times \sec\left(\sqrt{-\frac{2a_1w + 3a_3w^2 + \lambda}{a_3}}(t + \lambda z)\right)\exp[i(kz + \omega t)]. \quad (82)$$

4. Conclusion

In this work, we obtained exact solutions of the generalized forms of Klein–Gordon equation, the $(2+1)$-dimensional Camassa–Holm Kadomtsev–Petviashvili equation (CH–KP equation) and the higher-order nonlinear Schrödinger equation by using the functional variable method. The obtained solutions may be significant and important for analysing the nonlinear phenomena arising in applied physical sciences. This method definitely can be applied to nonlinear evolution equations which can be converted to a second-order ODE through the travelling wave transformation. The results show that the proposed method is direct, effective and can be applied to many other nonlinear evolution equations in mathematical physics.

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