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Single-peak solitary wave solutions for the variant Boussinesq equations

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Abstract. This paper presents all possible smooth, cusped solitary wave solutions for the variant Boussinesq equations under the inhomogeneous boundary condition. The parametric conditions for the existence of smooth, cusped solitary wave solutions are given using the phase portrait analytical technique. Asymptotic analysis and numerical simulations are provided for smooth, cusped solitary wave solutions of the variant Boussinesq equations.

Keywords. Variant Boussinesq equations; single-peak solitary wave; solitary wave; cuspon.

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1. Introduction

It is well known that the exact solutions for the nonlinear wave equations can help people to understand the described process properly. So finding the exact solutions of nonlinear equation is very important. Travelling wave solution is an important type of solution for the nonlinear partial differential equation and many nonlinear partial differential equations are found to have a variety of travelling wave solutions. For instance, the Camassa–Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

was proposed by Camassa and Holm [1] as a model equation for unidirectional nonlinear dispersive waves in shallow water. This equation has attracted a lot of attention over the past decade due to its interesting mathematical properties. The Camassa–Holm equation has been found to have peakons, cuspons and compactons solutions [2,3]. To our knowledge, cuspons have never been found for the variant Boussinesq equations (1.2).

$$\begin{cases} H_t + HH_x + gu_x + \frac{1}{3}au_{ttx} = 0 \\ u_t + Hu_x + uH_x = 0 \end{cases} \tag{1.2}$$

which is a model for water waves ($a \neq 0$), where $u(x, t)$ is the velocity, $H(x, t)$ is the total depth and the subscripts denote partial derivatives.

In fact, it is important to consider various boundary conditions of travelling wave solutions. Qiao and Zhang [4] investigated the Camassa–Holm equation under inhomogeneous boundary condition, and they obtained all possible single-peak soliton solutions of the Camassa–Holm equation. Recently, Chen and Li [5] studied the osmosis $K(2, 2)$ equation under inhomogeneous boundary condition and obtained smooth, peaked and cusped solitary wave solutions of the osmosis $K(2, 2)$ equation. Zhang and Chen [6] obtained new types of cusped solitons of a partial differential equation by setting the partial differential equation under inhomogeneous boundary condition.

In this paper, we study the single-peak solitary wave solutions of eq. (1.2) under the inhomogeneous boundary condition

$$\lim_{\xi \to \pm\infty} u(\xi) = A. \tag{1.3}$$

The conditions of existence of the smooth and cusped solitary wave solutions are given using the phase portrait analytical technique, which was developed by Li *et al* [7–13]. We obtain smooth, cusped solitary wave solutions of the variant Boussinesq equations (1.2) and analyse their analytical and dynamical behaviour.

The paper is organized as follows. In §2, we discuss the asymptotic behaviour of solutions of the variant Boussinesq equations (1.2). In §3, we give all single-peak solitary wave solutions of the variant Boussinesq equations (1.2) under the inhomogeneous boundary condition (1.3).

2. Asymptotic behaviour of solutions

To find all travelling wave solutions of (1.2), we set

$$H(x, t) = \phi(\xi), \qquad u(x, t) = \psi(\xi), \qquad \xi = x - \lambda t, \tag{2.1}$$

where λ ($\lambda \neq 0$) are constants to be determined.

Substituting (2.1) into systems (1.2), the systems (1.2) are reduced to

$$\begin{cases} -\lambda\phi' + \phi\phi' + g\psi' + \frac{1}{3}a\lambda^2\psi''' = 0, \\ -\lambda\psi' + \phi\psi' + \psi\phi' = 0, \end{cases} \tag{2.2}$$

where $'$ is the derivative with respect to ξ. Integrating (2.2) with respect to ξ leads to

$$\begin{cases} -\lambda\phi + \frac{1}{2}\phi^2 + g\psi + \frac{1}{3}a\lambda^2\psi'' = c_1, \\ -\lambda\psi + \phi\psi = c_2, \end{cases} \tag{2.3}$$

where c_1 and c_2 are integral constants.

Substituting the second formulation of (2.3) into the first formulation of (2.3), we have

$$-\frac{\lambda^2}{2} + \frac{c_2^2}{2\psi^2} + g\psi + \frac{1}{3}a\lambda^2\psi'' = c_1. \tag{2.4}$$

Further, we get

$$(\psi')^2 = \frac{-3g\psi^3 + 6(c_1 + (\lambda^2/2))\psi^2 + 6c_3\psi + 3c_2^2}{a\lambda^2\psi}, \tag{2.5}$$

where c_3 is also an integration constant.

Let us solve (2.5) using the following inhomogeneous boundary condition:

$$\lim_{\xi \to \pm\infty} \psi(\xi) = A, \tag{2.6}$$

where A is a constant. Equation (2.5) can be cast into the following ordinary differential equation:

$$(\psi')^2 = \frac{-3g(\psi - A)^2(\psi - B)}{a\lambda^2\psi}, \tag{2.7}$$

if $c_2 g \neq 0$. If $g = 0$, eq. (2.5) can be cast into the following ordinary differential equation:

$$(\psi')^2 = \frac{6(c_1 + (\lambda^2/2))(\psi - A)^2}{a\lambda^2\psi}. \tag{2.8}$$

If $c_2 = 0$, eq. (2.5) can be cast into the following ordinary differential equation:

$$(\psi')^2 = \frac{-3g(\psi - A)^2}{a\lambda^2}. \tag{2.9}$$

DEFINITION 2.1

A function $\psi(\xi)$ is said to be a single-peak solitary wave solution for eq. (2.4) if $\psi(\xi)$ satisfies the following conditions:

- (A1) $\psi(\xi)$ is continuous on R and has a unique peak point ξ_0, where $\psi(\xi)$ attains its global maximum or minimum value.
- (A2) $\psi(\xi) \in C^3(R - \{\xi_0\})$ satisfies (2.4) on $R - \{\xi_0\}$.
- (A3) $\lim_{\xi \to \pm\infty} \psi(\xi) = A$.

DEFINITION 2.2

A wave function $\psi(\xi)$ is called cuspon if $\psi(\xi)$ is smooth locally on either side of ξ_0 and $\lim_{\xi \uparrow \xi_0} \psi_\xi(\xi) = -\lim_{\xi \downarrow \xi_0} \psi_\xi(\xi) = \pm\infty$.

Without losing the generality, we assume $\xi_0 = 0$.

Theorem 2.3. *Suppose that $\psi(\xi)$ is a single-peak solitary wave solution for eq.* (2.4) *at the peak point $\xi_0 = 0$.*

- (i) *If $c_2 g \neq 0$, then $\psi(0) = 0$ or $\psi(0) = B$.*
- (ii) *If $g = 0$, then $\psi(0) = 0$.*

Proof. If $\psi(0) \neq 0$, then $\psi(\xi) \neq 0$ for any $\xi \in R$ since $\psi(\xi) \in C^3(R - \{0\})$. Differentiating both sides of eq. (2.7) yields $\psi(\xi) \in C^\infty(R)$.

(i) For $c_2 g \neq 0$

When $\psi(0) \neq B$, if $\psi(0) \neq 0$ then $\psi(\xi) \in C^\infty(R)$. By the definition of single-peak solution we have $\psi'(0) = 0$. However, by eq. (2.7) we must have $\psi(0) = A$, which contradicts the fact that 0 is the unique peak point.

If $\psi(0) \neq 0$, by eq. (2.7) we know $\psi'(0)$ exists. According to the definition of peak point, we have $\psi'(0) = 0$. Thus we obtain $\psi(0) = B$ from eq. (2.7), since $\psi(0) = A$ contradicts the fact that 0 is the unique peak point.

(ii) For $g = 0$

If $\psi(0) \neq 0$, by eq. (2.8) we know $\psi'(0)$ exists. According to the definition of peak point, we have $\psi'(0) = 0$. However, by eq. (2.8) we must have $\psi(0) = A$, which contradicts the fact that 0 is the unique peak point. This completes the proof.

By eq. (2.9), it is obvious that the variant Boussinesq equations have no single-peak solitary wave solutions for $c_2 = 0$.

Theorem 2.4. *Suppose that $\psi(\xi)$ is a single-peak solitary wave solution for eq. (2.4) at the peak point $\xi_0 = 0$. Then we have the following solution classifications and asymptotic behaviour.*

(i) *If $\psi(0) \neq 0$, then $\psi(\xi)$ is a smooth solitary wave solution.*
(ii) *If $\psi(0) = 0$ and $c_2 g \neq 0$, then $\psi(\xi)$ is a cusped solitary wave solution and*

$$\psi(\xi) = \mu|\xi|^{2/3} + O(|\xi|^{4/3}), \qquad \xi \to 0, \tag{2.10}$$

$$\psi'(\xi) = \frac{2}{3}\mu|\xi|^{-1/3}\text{sign}(\xi) + O(|\xi|^{1/3}), \qquad \xi \to 0, \tag{2.11}$$

where

$$\mu = \text{sign}(A)\left(\frac{3|A|\sqrt{|B|}}{2}\sqrt{\left|\frac{3g}{a\lambda^2}\right|}\right)^{2/3}.$$

Thus $\psi(\xi) \notin H^1_{\text{loc}}(R)$.

(iii) *If $\psi(0) = 0$ and $g = 0$, then $\psi(\xi)$ is a cusped solitary wave solution as eqs (2.10) and (2.11) where*

$$\mu = \text{sign}(A)\left(\frac{3|A|}{2}\sqrt{\left|\frac{6(c_1 + (\lambda^2/2))}{a\lambda^2}\right|}\right)^{2/3}.$$

Thus $\psi(\xi) \notin H^1_{\text{loc}}(R)$.

Proof. (i) By proving Theorem 2.3, we know that if $\psi(0) = 0$, then $\psi(\xi)$ is a smooth solitary wave solution.

(ii) If $\psi(0) = 0$ and $c_2 g \neq 0$, from eq. (2.7) we obtain

$$\psi_\xi = -\text{sign}(A)\sqrt{\frac{-3g(\psi - B)}{a\lambda^2\psi}}(\psi - A)\text{sign}(\xi). \tag{2.12}$$

Let

$$h(\psi) = \frac{1}{(A - \psi)\sqrt{(\psi - B)\text{sign}(B)}},$$

then

$$h(0) = \frac{1}{A\sqrt{\text{sign}(B)B}}$$

and

$$\int \text{sign}(A)\sqrt{\text{sign}(A)\psi}h(\psi)\text{d}\psi = \sqrt{\frac{3g}{a\lambda^2}\text{sign}\left(\frac{g}{a}\right)} \int \text{sign}(\xi)\text{d}\xi. \tag{2.13}$$

Inserting $h(\psi) = h(0) + O(\psi)$ into eq. (2.13) and using the initial condition $\psi(0) = 0$, we obtain

$$\frac{2}{3}[\text{sign}(A)\psi]^{3/2}|h(0)|(1 + O(\psi)) = \sqrt{\frac{3g}{a\lambda^2}\text{sign}\left(\frac{g}{a}\right)}|\xi|. \tag{2.14}$$

Thus

$$\psi = \text{sign}(A)\left(\frac{3}{2|h(0)|}\sqrt{\frac{3g}{a\lambda^2}\text{sign}\left(\frac{g}{a}\right)}\right)^{2/3} |\xi|^{2/3}(1 + O(\psi))^{-2/3} \tag{2.15}$$

which implies $\psi = O(|\xi|^{3/2})$. Therefore, we have

$$\psi(\xi) = \mu|\xi|^{2/3} + O(|\xi|^{4/3}), \qquad \xi \to 0,$$

$$\psi'(\xi) = \frac{2}{3}\mu|\xi|^{-1/3}\text{sign}(\xi) + O(|\xi|^{1/3}), \qquad \xi \to 0,$$

where

$$\mu = \text{sign}(A)\left(\frac{3|A|\sqrt{|B|}}{2}\sqrt{\left|\frac{3g}{a\lambda^2}\right|}\right)^{2/3}.$$

Thus $\psi(\xi) \notin H^1_{\text{loc}}(R)$.

(iii) Similar to the proof of the above (ii),we omit it here. This completes the proof.

3. Smooth and cusped single-peak solitary wave solutions

Theorem 2.4 gives a classification for all single-peak solitary wave solutions for eq. (2.4). In this section, we shall present all possible single-peak solitary wave solutions and obtain some implicit solutions.

For $c_2 g \neq 0$, using the standard phase portrait analytical technique (see figure 1) and Theorem 2.3, we know that if $\psi(\xi)$ is a single-peak solitary wave solution of eq. (2.4), then

$$\psi_\xi = -\text{sign}(A)\sqrt{\frac{3g(B-\psi)}{a\lambda^2\psi}}(\psi - A)\text{sign}(\xi). \tag{3.1}$$

Taking the integration of both sides of eq. (3.1) leads to

$$\int f(\psi)\text{d}\psi = |\xi|, \tag{3.2}$$

where

$$f(\psi) = \frac{1}{\text{sign}(A)(A-\psi)}\sqrt{\frac{a\lambda^2\psi}{3g(B-\psi)}}.$$

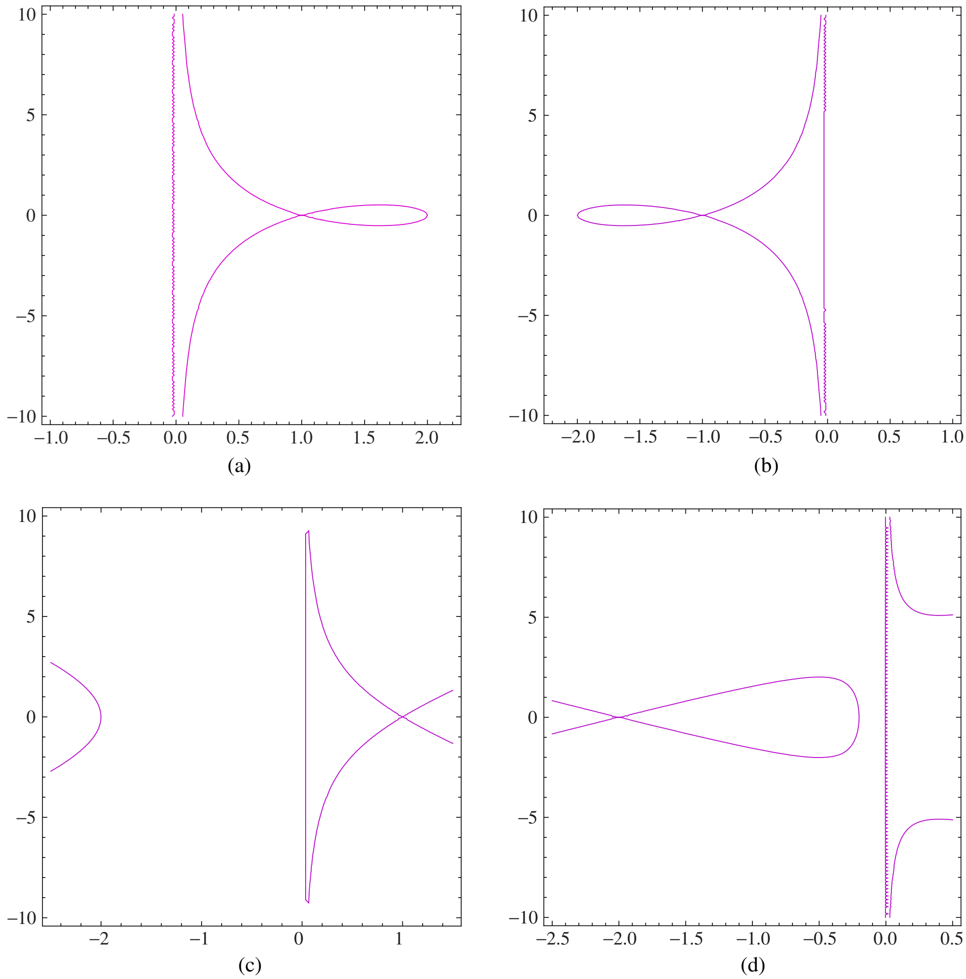

Figure 1. Phase portraits of eq. (2.7) on the (ψ, ψ_ξ) plane for $|\frac{g}{a\lambda^2}| = 1$. (**a**) $A = 1$, $B = 2$, (**b**) $A = -1$, $B = -2$, (**c**) $A = 1$, $B = -2$, (**d**) $A = -2$, $B = -0.5$.

Case I. For eq. (2.7), $B > A > 0$ or $B < A < 0$.

(1) If $B > A > 0$

From eq. (3.2), we obtain the implicit solution $\psi(\xi)$ defined by

$$F(\psi) - K = \sqrt{\frac{3g}{a\lambda^2}}|\xi|, \tag{3.3}$$

where K is an integration constant.

$$F(\psi) = \sqrt{\frac{A}{B-A}} \ln\left|\frac{2k + \beta(\psi - A) + 2\sqrt{k(B\psi - \psi^2)}}{A - \psi}\right| + \sin^{-1}\left(\frac{B - 2\psi}{B}\right) - K = \sqrt{\frac{3g}{a\lambda^2}}|\xi|, \tag{3.4}$$

where $k = A(B - A)$, $\beta = B - 2A$.

For $\psi(0) = 0$, according to standard phase portrait analytical technique, we have $0 \leq \psi < A$, and the integration constant $K = \sqrt{A/(B-A)}\ln B + (\pi/2)$. From $f(\psi) > 0$, we know that $F(\psi)$ is strictly increasing on $[0, A)$ with $F(0) = 0$, $F(A-) = \infty$. Denote

$$F_1(\psi) = F|_{[0,A)}(\psi). \tag{3.5}$$

Then $F_1(\psi)$ has the inverse on the interval $[0, A)$. So, eq. (3.4) can be solved uniquely for ψ on the interval $[0, A)$. Therefore, we define $F_1(\psi)$'s inverse as $\psi_1(\xi) = F_1^{-1}(|\xi|)$ which is a cusped solitary wave solution and its profile is shown in figure 2a.

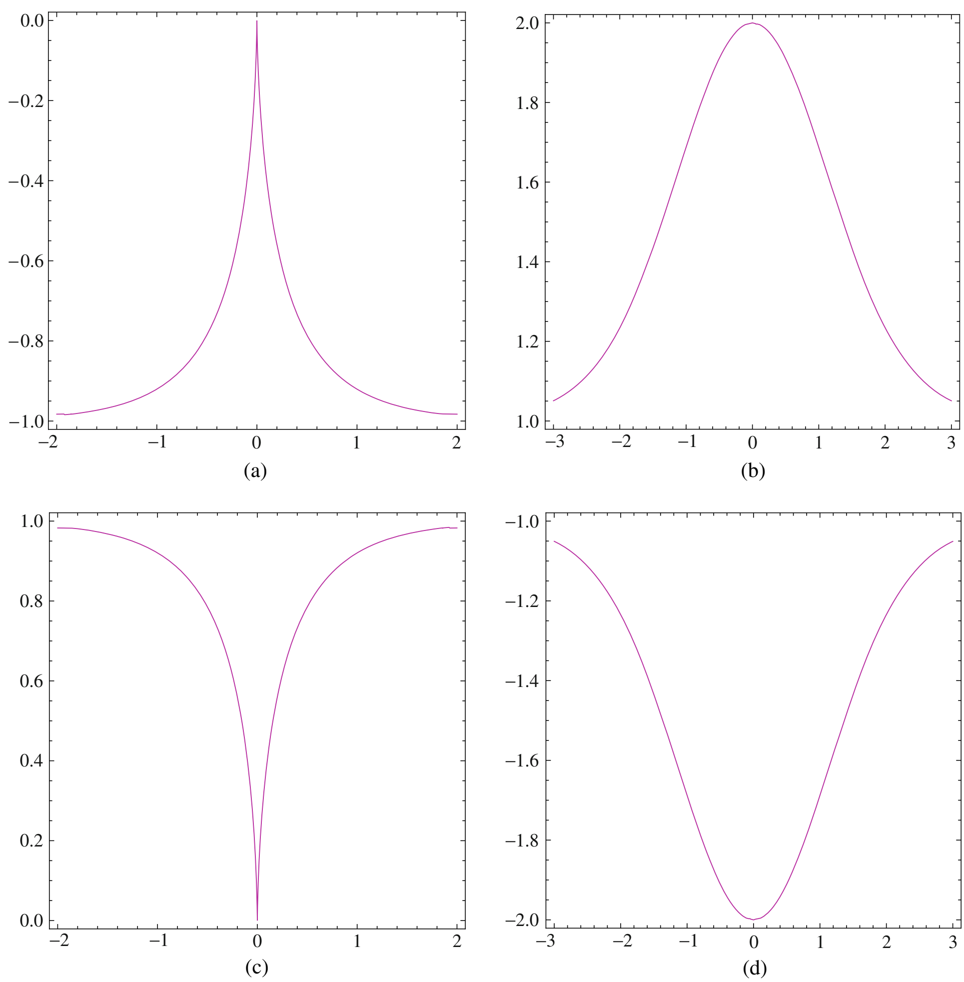

Figure 2. The profiles of waves. (**a**) Cuspon, $A = 1$, $B = 2$, $\psi(0) = 0$, (**b**) smooth solitary wave, $A = 1$, $B = 2$, $\psi(0) = B$, (**c**) anti-cuspon, $A = -1$, $B = -2$, $\psi(0) = 0$, (**d**) smooth solitary wave, $A = -1$, $B = -2$, $\psi(0) = B$.

For $\psi(0) = B$, we have $A < \psi \leq B$, and the integration constant $K = 0$. From $f(\psi) < 0$, we know that $F(\psi)$ is strictly decreasing on $(A, B]$ with $F(B) = 0$, $F(A+) = \infty$. Denote

$$F_2(\psi) = F|_{(A,B]}(\psi). \tag{3.6}$$

Then $F_2(\psi)$ has the inverse on the interval $(A, B]$. So, eq. (3.6) can be solved uniquely for ψ on the interval $(A, B]$. Therefore, we define $F_2(\psi)$'s inverse as $\psi_2(\xi) = F_2^{-1}(|\xi|)$ which is a smooth solitary wave solution and its profile is shown in figure 2b.

(2) If $B < A < 0$, this case is completely similar to the case of $B > A > 0$. The profile of cusped solitary wave solution is shown in figure 2c and the profile of the smooth solitary wave solution is shown in figure 2d.

Case II. For eq. (2.7), $B < 0 < A$ or $A < 0 < B$.

(1) If $B < 0 < A$

In this case, according to Theorem 2.3 and standard phase portrait analytical technique (see figure 1d), we have $\psi(0) = 0$, $0 \leq \psi < A$.

From eq. (3.2), we obtain the implicit solution $\psi(\xi)$ defined by

$$G(\psi) - K = \sqrt{\frac{-3g}{a\lambda^2}}|\xi|, \tag{3.7}$$

where K is an integration constant.

$$G(\psi) = \sqrt{\frac{A}{A-B}}\ln\left|\frac{2k + \beta(A-\psi) + 2\sqrt{k(\psi^2 - B\psi)}}{A-\psi}\right| \\ -\ln\left|\sqrt{\psi^2 - B\psi} + \psi - \frac{B}{2}\right| - K = \sqrt{\frac{-3g}{a\lambda^2}}|\xi|, \tag{3.8}$$

where

$$k = A(A-B), \qquad \beta = B - 2A,$$
$$K = \left(\sqrt{\frac{A}{A-B}} - 1\right)\ln|B| + \ln 2.$$

From $f(\psi) > 0$, we know that $G(\psi)$ is strictly increasing on $[0, A)$ with $G(0) = 0$, $G(A-) = \infty$. Denote

$$G_1(\psi) = G|_{[0,A)}(\psi). \tag{3.9}$$

Then $G_1(\psi)$ has the inverse on the interval $[0, A)$. So, eq. (3.9) can be solved uniquely for ψ on the interval $[0, A)$. Therefore, we define $G_1(\psi)$'s inverse as $\psi_3(\xi) = G_1^{-1}(|\xi|)$ which is a cusped solitary wave solution.

(2) If $A < 0 < B$, this case is completely similar to the case of $B < 0 < A$.

Case III. For eq. (2.7), $A < B < 0$ or $0 < B < A$.

(1) If $A < B < 0$

In this case, according to Theorem 2.3 and standard phase portrait analytical technique (see figure 1d), we have $\psi(0) = B$, $A < \psi \leq B$.

From eq. (3.2), we obtain the implicit solution $\psi(\xi)$ defined by

$$H(\psi) - K = \sqrt{\frac{-3g}{a\lambda^2}}|\xi|, \tag{3.10}$$

where K is an integration constant.

$$H(\psi) = \sqrt{\frac{A}{A-B}}\ln\left|\frac{2k + \beta(A-\psi) - 2\sqrt{k(\psi^2 - B\psi)}}{\psi - A}\right| \\ -\ln\left|\sqrt{\psi^2 - B\psi} + \psi - \frac{B}{2}\right| - K = \sqrt{\frac{-3g}{a\lambda^2}}|\xi|, \tag{3.11}$$

where

$$k = A(A-B), \qquad \beta = B - 2A,$$
$$K = \left(\sqrt{\frac{A}{A-B}} - 1\right)\ln|B| + \ln 2.$$

From $f(\psi) > 0$, we know that $H(\psi)$ is strictly increasing on $(A, B]$ with $H(B) = 0$, $H(A+) = \infty$. Denote

$$H_1(\psi) = H|_{(A,B]}(\psi). \tag{3.12}$$

Then $H_1(\psi)$ has the inverse on the interval $(A, B]$. So, eq. (3.11) can be solved uniquely for ψ on the interval $(A, B]$. Therefore, we define $H_1(\psi)$'s inverse as $\psi_4(\xi) = H_1^{-1}(|\xi|)$ which is a smooth solitary wave solution.

(2) If $0 < B < A$, this case is completely similar to the case of $A < B < 0$.

Case IV. For eq. (2.8), $A > 0$ or $A < 0$.

For $g = 0$, using the standard phase portrait analytical technique (see figure 3) and Theorem 2.3, we know that if $\psi(\xi)$ is a single-peak solitary wave solution of eq. (2.4), then

$$\psi_\xi = -\text{sign}(A)\sqrt{\frac{3(2c_1 + \lambda^2)}{a\lambda^2\psi}}(\psi - A)\text{sign}(\xi). \tag{3.13}$$

Integrating both sides of eq. (3.13) leads to

$$\int g(\psi)\mathrm{d}\psi = |\xi|, \tag{3.14}$$

where

$$g(\psi) = \frac{1}{\text{sign}(A)(A-\psi)}\sqrt{\frac{a\lambda^2\psi}{3(2c_1 + \lambda^2)}}.$$

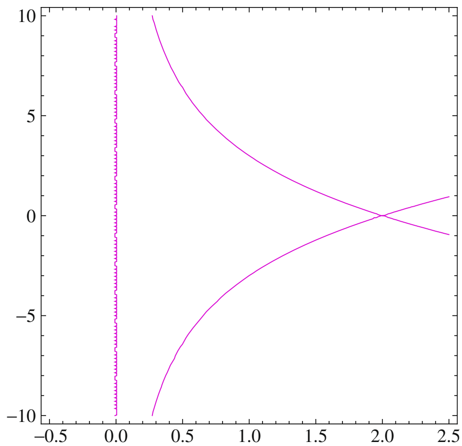

Figure 3. Phase portraits of eq. (2.8) on the (ψ, ψ_ξ) plane for $A = 2$, $g = 0$, $(c_1 + \frac{\lambda^2}{2})/(a\lambda^2) = \frac{3}{2}$.

(1) If $A > 0$

In this case, according to Theorem 2.3 and standard phase portrait analytical technique (see figure 3), we have $\psi(0) = 0$, $0 \leq \psi < A$.

From eq. (3.14), we obtain the implicit solution $\psi(\xi)$ defined by

$$J(\psi) = -2\sqrt{\psi} + \sqrt{A}\ \ln\frac{\sqrt{A} + \sqrt{\psi}}{\sqrt{A} - \sqrt{\psi}} = \sqrt{\frac{3(2c_1 + \lambda^2)}{a\lambda^2}}|\xi|. \tag{3.15}$$

From $g(\psi) > 0$, we know that $J(\psi)$ is strictly increasing on $[0, A)$ with $J(0) = 0$, $J(A-) = \infty$. Denote

$$J_1(\psi) = J|_{[0,A)}(\psi). \tag{3.16}$$

Then $J_1(\psi)$ has the inverse on the interval $[0, A)$. So, eq. (3.15) can be solved uniquely for ψ on the interval $[0, A)$. Therefore, we define $J_1(\psi)$'s inverse as $\psi_5(\xi) = J_1^{-1}(|\xi|)$ which is a cusped solitary wave solution.

(2) If $A < 0$, this case is completely similar to the case of $A > 0$.

Theorem 2.3. *Suppose that $\psi(\xi)$ is a single-peak solitary wave solution for eq. (2.4) at the peak point $\xi_0 = 0$, which satisfies the boundary condition (2.6). Then we have the following conclusions:*

(1) *If $c_2 g \neq 0$*

(i) *If $0 < A < B$, eq. (2.4) has the cusped solitary wave solution*

$$\psi(\xi) = F_1^{-1}\left(\sqrt{\frac{3g}{a\lambda^2}}|x - \lambda t|\right),$$

with the properties:

$$\psi(0) = 0, \qquad \psi(\pm\infty) = A, \qquad \psi'(0+) = +\infty, \qquad \psi'(0-) = -\infty.$$

At the same time, *eq.* (2.4) *has the smooth solitary wave solution*

$$\psi(\xi) = F_2^{-1}\left(\sqrt{\frac{3g}{a\lambda^2}}|x - \lambda t|\right),$$

with the properties:

$$\psi(0) = B, \qquad \psi(\pm\infty) = A, \qquad \psi'(0+) = +\infty, \qquad \psi'(0-) = -\infty.$$

(ii) *If* $B < 0 < A$, *eq.* (2.4) *has the cusped solitary wave solution*

$$\psi(\xi) = G_1^{-1}\left(\sqrt{\frac{-3g}{a\lambda^2}}|x - \lambda t|\right),$$

with the properties:

$$\psi(0) = 0, \qquad \psi(\pm\infty) = A, \qquad \psi'(0+) = +\infty, \qquad \psi'(0-) = -\infty.$$

(iii) *If* $A < B < 0rm$, *eq.* (2.4) *has the smooth solitary wave solution*

$$\psi(\xi) = H_1^{-1}\left(\sqrt{\frac{-3g}{a\lambda^2}}|x - \lambda t|\right),$$

with the properties:

$$\psi(0) = B, \qquad \psi(\pm\infty) = A, \qquad \psi'(0+) = -\infty, \qquad \psi'(0-) = +\infty.$$

(2) *If* $g = 0$ *and* $A > 0$, *eq.* (2.4) *has the cusped solitary wave solution*

$$\psi(\xi) = J_1^{-1}\left(\sqrt{\frac{3(2c_1 + \lambda^2)}{a\lambda^2}}|x - \lambda t|\right),$$

with the properties:

$$\psi(0) = 0, \qquad \psi(\pm\infty) = A, \qquad \psi'(0+) = +\infty, \qquad \psi'(0-) = -\infty.$$

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