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Construction of an exact solution of time-dependent Ginzburg–Landau equations and determination of the superconducting–normal interface propagation speed in superconductors

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Abstract. A new approach is taken to calculate the speed of front propagation at which the interface moves from a superconducting to a normal region in a superconducting sample. Using time-dependent Ginzburg–Landau (TDGL) equations we have calculated the speed by constructing a new exact solution. This approach is based on a method given by Di Bartolo and Dorsey. Our result for the speed agrees with the result of Di Bartolo and Dorsey.

Keywords. Front propagation in superconductor; Ginzburg–Landau equation; velocity selection; exact solution.

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1. Introduction

In recent years the phenomenon of magnetic field penetration or its expulsion from superconducting sample is a great attraction to different research groups. To determine the velocity at which the interface moves from a superconducting to normal phase is an interesting problem. Several methods such as marginal stability hypothesis [1,2], variational speed selection [3,4], structural stability [5] and construction of exact solutions [6] are proposed for the analysis of dynamical velocity selection.

The nonlinear diffusion equations are used to describe several problems arising in fluid dynamics, population growth, pulse propagation in nerves, and many other biological, chemical and physical phenomena. In these problems, it is found that if the system is suddenly made unstable, the subsequent dynamics is characterized by the propagation of fronts. The prototype of such equations is the parabolic reaction diffusion equation [7,8]

$$u_t = u_{xx} + F(u) \quad \text{with} \quad F(0) = F(1) = 0 \text{ and } F > 0,$$

where the subscripts denote derivatives. The nonlinear term $F(u)$ is such that there exist two steady states, one stable and the other one unstable. As shown rigorously in the classic work of Aronson and Weinberger [9], for positive reaction terms $F(u) > 0$ sufficiently localized initial conditions evolve into travelling wave fronts of the form $u(x, t) = U(x - vt)$ connecting the stable state $u = 1$ at $x = -\infty$ to the unstable state $u = 0$ at $x = -\infty$. The speed at which the front propagates in the minimum speed v is bounded below and above is estimated as $2\sqrt{F'(0)} \leq v \leq 2\,\sup\sqrt{F(u)/u}$, so that for the special case of the Fisher–Kolmogorov equation, $F(u) = u - u^3$, the selected speed $v = 2$. It is found that the velocity of the selected front also depends on the discretization of the continuous partial differential equation [10].

For the evolutionary description of the system between two homogeneous equilibrium states, it is necessary to embed a superconducting sample in a stationary applied magnetic field equal to the critical field H_C. The magnetic field is rapidly removed. Then the planar superconducting–normal interface is dynamically unstable and propagates toward the normal phase, thereby expelling any trapped magnetic flux [11]. As a result, the sample is in the Meissner state. It has been considered that the interface remains planar during all the processes. For type-I superconductor, in the spinodal regime the spinodal growth proceeds by a phase-slip process, i.e., the leakage of the order parameter occurs in the spinodal regime, resulting in flux trapping at intermediate times. The speed of the superconducting–normal front is controlled by this trapped magnetic flux in the front.

The purpose of this paper is to develop an insight into the front propagation speed in superconductors. In §2, we have presented the detailed procedure for constructing an exact solution of the time-dependent Ginzburg–Landau (TDGL) equations for superconducting fronts. The starting points are the TDGL equations comprising a coupled equation for the density of superconducting electrons and the local magnetic field. Section 3 provides the results of the calculations for superconducting order parameter, vector potential and magnetic field and presents some properties of the solutions.

2. Theory

Dorsey [12] has given an interesting account of the dynamics of interfaces in superconductors. One of the important aspects of this phenomenon [13,14] is that there is a competition between dynamic instability [15] which promotes the growth of a highly ramified interface, and surface tension, which favours a smooth interface. An important point that has emerged is that surface tension anisotropy has a major effect on the morphology of this phenomenon [13,14].

In dimensionless units [12] the one-dimensional TDGL equations in this case reduce to

$$\partial_t f = \frac{1}{\kappa^2}\partial_x^2 f - q^2 f + f - f^3 \tag{2.1}$$

$$\bar{\sigma}\,\partial_t q = \partial_x^2 q - f^2 q. \tag{2.2}$$

Here, the quantity f is the magnitude of the superconducting order parameter ψ, q is the gauge-invariant vector potential connected to the magnetic field as $h = \partial q/\partial x$, $\bar{\sigma}$ is the dimensionless normal state conductivity which is the ratio of the order parameter diffusion constant, $D_\psi = \hbar/2m\gamma$ (γ is the order parameter relaxation time, m is the mass of a Cooper pair) to the magnetic field diffusion constant $D_h = 1/4\pi\sigma^{(n)}$ and κ is the Ginzburg–Landau parameter. The steady travelling wave solution for the TDGL equations can be written in the forms

$$f(x,t) = F(x - vt) = F(X)$$

and

$$q(x,t) = Q(x - vt) = Q(X),$$

where $X = x - vt$ with $v > 0$. Then eqs (2.1) and (2.2) become

$$\frac{1}{\kappa^2}F'' + vF' - Q^2F + F - F^3 = 0 \tag{2.3}$$

$$Q'' + \bar{\sigma}vQ' - F^2Q = 0. \tag{2.4}$$

Di Bartolo and Dorsey [6] were interested in calculating the propagating speed, and they used the ansatz like:

$$F + Q = 1. \tag{2.5}$$

This ansatz is satisfied only for $\kappa = 1/\sqrt{2}$ and $\sigma = 1/2$. Upon substituting this ansatz in eqs (2.3) and (2.4), they have

$$F'' + \frac{v}{2}F' + F^2 - F^3 = 0. \tag{2.6}$$

By using the reduction of order method given by Saarloos [2] they constructed an exact solution for a particular set of parameters giving front speed v =1.4142. These parameters are the order parameter and the vector potential given by

$$F(X) = \frac{1}{\mathrm{e}^{X/\sqrt{2}} + 1}, \qquad Q(X) = \frac{1}{\mathrm{e}^{-X/\sqrt{2}} + 1}. \tag{2.7}$$

For our next step, we make a minor contribution to the interface problem by considering the following generalized Ginzburg–Landau equation:

$$\left(1 + \beta^2\right)F'' + vF' - Q^2F + a_1F + a_2F^2 + a_3F^3 = 0, \tag{2.8}$$

$$Q'' + \sigma_0 Q' - F^2Q = 0, \tag{2.9}$$

(taking $1/\kappa^2 = 1 + \beta^2$ and $\bar{\sigma}v = \sigma_0$). Here, a_1, a_2, and a_3 are constant parameters to be determined.

As we shall show, these parameters can be chosen to lead to an exact solution for F and Q, which is physically acceptable. Let us write

$$F(X) = \frac{1}{1 + \alpha\mathrm{e}^{\xi X}} \quad \text{and} \quad Q(X) = \frac{\alpha\beta\mathrm{e}^{\xi X}}{1 + \alpha\mathrm{e}^{\xi X}}, \tag{2.10}$$

where α is any arbitrary constant in front solution.

We show that the solutions satisfy eqs (2.8) and (2.9) if the order parameter and vector potential satisfy eq. (2.10).

$$F' = \frac{-\alpha\xi\mathrm{e}^{\xi X}}{(1 + \alpha\mathrm{e}^{\xi X})^2} \tag{2.10a}$$

$$F'' = \frac{\alpha^2\xi^2 e^{2\xi X} - \alpha\xi^2 e^{\xi X}}{(1+\alpha e^{\xi X})^3} \tag{2.10b}$$

$$Q' = \frac{\alpha\beta\xi e^{\xi X}}{(1+\alpha e^{\xi X})} - \frac{\alpha^2\beta\xi e^{2\xi X}}{(1+\alpha e^{\xi X})^2} \tag{2.10c}$$

and

$$Q'' = \frac{\alpha\beta\xi^2 e^{\xi X}}{(1+\alpha e^{\xi X})} - \frac{(\alpha\beta\xi e^{\xi X})(\alpha\xi e^{\xi X})}{(1+\alpha e^{\xi X})^2} - \frac{2\alpha^2\beta\xi^2 e^{2\xi X}}{(1+\alpha e^{\xi X})^2} + \frac{2\alpha^2\beta\xi(\alpha\xi e^{\xi X})e^{2\xi X}}{(1+\alpha e^{\xi X})^3}. \tag{2.10d}$$

Substitution of (2.10) and (2.10a–2.10d) into (2.8) implies

$$\begin{aligned}
&(1+\beta^2)\left[\frac{\alpha^2\xi^2 e^{2\xi X} - \alpha\xi^2 e^{\xi X}}{(1+\alpha e^{\xi X})^3}\right] + v\left[\frac{-\alpha\xi e^{\xi X}}{(1+\alpha e^{\xi X})^2}\right] \\
&\quad - \left[\frac{\alpha^2\beta^2 e^{2\xi X}}{(1+\alpha e^{\xi X})^2}\right]\frac{1}{(1+\alpha e^{\xi X})} + \frac{a_1}{(1+\alpha e^{\xi X})} \\
&\quad + \frac{a_2}{(1+\alpha e^{\xi X})^2} + \frac{a_3}{(1+\alpha e^{\xi X})^3} = 0 \\
&\left(1+\beta^2\right)\left[\alpha^2\xi^2 e^{2\xi X} - \alpha\xi^2 e^{\xi X}\right] - v\alpha\xi e^{\xi X}(1+\alpha e^{\xi X}) - \alpha^2\beta^2 e^{2\xi X} \\
&\quad + a_1(1+\alpha e^{\xi X})^2 + a_2(1+\alpha e^{\xi X}) + a_3 = 0.
\end{aligned} \tag{2.11}$$

Now equating the coefficients of $e^{2\xi X}$, we have

$$\begin{aligned}
&\alpha^2(1+\beta^2)\xi^2 - v\alpha^2\xi - \alpha^2\beta^2 + a_1\alpha^2 = 0, \\
&v\xi = (1+\beta^2)\xi^2 - \beta^2 + a_1, \\
&v = (1+\beta^2)\xi - \frac{\beta^2}{\xi} + \frac{a_1}{\xi}.
\end{aligned} \tag{2.12}$$

Also equating the coefficients of $e^{2\xi X}$ implies

$$\begin{aligned}
&-\alpha\xi^2(1+\beta^2) - v\alpha\xi + 2a_1\alpha + a_2\alpha = 0, \\
&v\xi = 2a_1 + a_2 - \xi^2(1+\beta^2), \\
&v = \frac{2a_1 + a_2}{\xi} - \xi(1+\beta^2).
\end{aligned} \tag{2.13}$$

From (2.12) and (2.13) the new constants take the value

$$(1+\beta^2)\xi - \frac{\beta^2}{\xi} + \frac{a_1}{\xi} = \frac{2a_1 + a_2}{\xi} - \xi(1+\beta^2),$$

$$\sqrt{2}(1+\beta^2) - \sqrt{2}\beta^2 - (a_1 + a_2)\sqrt{2} = 0, \qquad \text{for } \xi = \frac{1}{\sqrt{2}},$$

$$\sqrt{2}\,[1-(a_1+a_2)]=0.$$
$$\therefore a_1+a_2=1. \tag{2.14}$$

From eq. (2.11), by equating the constant coefficient, we have

$$a_1+a_2+a_3=0.$$
$$\therefore\ a_3=-1. \tag{2.15}$$

Now from (2.12)

$$v\ =\xi+\beta^2\left(\xi-\frac{1}{\xi}\right)+\frac{a_1}{\xi}=\frac{1}{\sqrt{2}}-\frac{\beta^2}{\sqrt{2}}+\sqrt{2}a_1.$$
$$\therefore\ v=\sqrt{2}\,(1+a_1)-\frac{\beta^2+1}{\sqrt{2}}. \tag{2.16}$$

The speed in terms of κ for which a superconducting front exists can be obtained as follows:

$$v=2\sqrt{2}-\frac{1}{\sqrt{2}\kappa^2} \tag{2.17}$$

for $a_1=1$.

Again, if we substitute eq. (2.10) into eq. (2.9), the speed can take the value

$$v=\frac{1}{\sqrt{2}\,\bar{\sigma}} \tag{2.18}$$

which is the front speed in terms of dimensionless normal state conductivity.

Again, Di Bartolo *et al* used their exact solutions (2.7) as a starting point for a perturbation approximation of eqs (2.2) and (2.4) to give a more general approach to calculate the speed. In this technique they found the selected velocity for superconducting–normal interface as

$$v=\frac{1}{\sqrt{2}\,\bar{\sigma}\,Q_\infty+\beta_\kappa}, \tag{2.19}$$

where β_κ is the kinetic coefficient which is a function of κ and $\bar{\sigma}$. We see that for $Q_\infty=1$ and $\beta_\kappa=0$, eq. (2.19) is in excellent agreement with our result (2.18).

Now it follows from solution (2.10)

$$\beta F\ =\beta\frac{1}{\left(1+\alpha e^{\xi X}\right)}=\beta\left[\frac{1}{1+\alpha e^{\xi X}}-1\right]+\beta$$
$$=-Q+\beta,$$
$$\beta F+Q=\beta, \tag{2.20}$$

which is a more general relation than $F+Q=1$, the expression (2.5) of Di Bartolo *et al* [6].

3. Result and discussion

It is well known that fronts propagate in superconductor with a continuous order parameter at a unique shape and speed. We know TDGL equation is widely used to calculate the speed in various methods. We find the speed in a different manner by deriving an exact solution of TDGL equation and some properties of the solutions are discussed here.

From eq. (2.20) we show that, if $\beta = 1$, then this equation reduces to eq. (2.5) and we have $\kappa = 1/\sqrt{2}$. Substituting this value of κ in eq. (2.17) gives $v = \sqrt{2}$, which is fully in agreement with Di Bartolo *et al* value of front speed.

Finally, we analyse the profile of the order parameter, vector potential and magnetic field for constant velocity solution. For a set of dimensionless parameters $\kappa = 1/\sqrt{2}$, $\bar{\sigma} = 1/2$ and $v = \sqrt{2}$,we plot a graph for new solution for the constant, $\alpha = 15$. We choose the value of X as $-20 \leq X \leq 20$. This is shown in figure 1.

3.1 *Results for* $\alpha = 15, \kappa = 1/\sqrt{2}, \bar{\sigma} = 1/2, v = \sqrt{2}$

Taking these values, the solutions $F(X)$ and $Q(X)$ of (2.10) become

$$F(X) = \frac{1}{1 + 15\mathrm{e}^{X/\sqrt{2}}}, \tag{2.21}$$

$$Q(X) = \left[\frac{15\mathrm{e}^{X/\sqrt{2}}}{1 + 15\mathrm{e}^{X/\sqrt{2}}}\right], \tag{2.22}$$

and

$$Q'(X) = \frac{15\mathrm{e}^{X/\sqrt{2}}}{\sqrt{2}\left(1 + 15\mathrm{e}^{X/\sqrt{2}}\right)^2}. \tag{2.23}$$

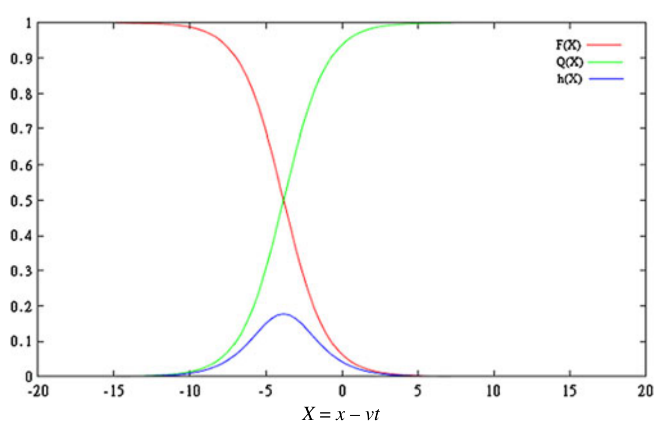

Figure 1. The graph showing the values of order parameter, vector potential and the field for the front moving with speed $v = \sqrt{2}$ while $\kappa = 1/\sqrt{2}$, $\bar{\sigma} = 1/2$ are in dimensionless unit and the constant $\alpha = 15$.

3.2 *Result for* $\alpha = 0$

Here we have, $F(X) = \pm 1$, $Q(x) = 0$ and $h(x) = 0$, which means that the trapped magnetic flux is absent and so the superconducting state becomes stable.

4. Conclusion

We have studied the front propagation in unstable state. $F = 0$ is an unstable state and $F = \pm 1$ are stable states. We consider a situation where initially the order parameter F asymptotically/exponentially decays for large X, or, in particular, one with $F(X, 0) \neq 0$ in a localized region only. The region with $F \neq 0$ expands in time, and a propagating front evolves separating the superconducting and normal phases which are produced after reducing the applied magnetic field to zero. The exact solution (2.10) of the time-dependent Ginzburg–Landau equations has been constructed which shows a duality between order parameter and vector potential (figure 1). Finally, we have obtained the superconducting–normal interface propagation speed ((eqs (2.17)–(2.19)).

It may be possible to compare the above results with suitable experiments, say, for different values of α. The circumstance that the functions are given explicitly and are relatively simple may facilitate comparison. Besides, exact solutions may be obtainable for higher order functions $F(X)$, for example $F(X) = a_1 X + a_3 X^3 + a_5 X^5$, etc. These matters are under investigation, as front propagation in superconductors is an important phenomenon, possibly leading to new insights into superconductivity.

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