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Space-time algebra for the generalization of gravitational field equations

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Abstract. The Maxwell–Proca-like field equations of gravitolectromagnetism are formulated using space-time algebra (STA). The gravitational wave equation with massive gravitons and gravitomagnetic monopoles has been derived in terms of this algebra. Using space-time algebra, the most generalized form of gravitoelectromagnetic Klein–Gordon equation has been obtained. Finally, the analogy in formulation between massive gravitational theory and electromagnetism has been discussed.

Keywords. Gravitoelectromagnetism; field equation; Maxwell–Proca equation; monopole; space-time algebra; Clifford algebra.

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1. Introduction

The term gravitoelectromagnetism (GEM) refers to the formal analogies between Newton's law of gravitation and Coulomb's law of electricity. In Newton's law, the origin of gravitational field is the mass of the body whereas in Coulomb's law, the source of electromagnetic field is the charge of the particle. Similarly, according to general relativity, the mass current produces the gravitomagnetic field just as the electric current produces the magnetic field [1].

Although Maxwell [2] himself has noticed the possibility of formulating the theory of gravitation in a form corresponding to the electromagnetic equations, the theoretical analogy between the electromagnetic and the gravitational field equations has been first suggested by Heaviside [3] in 1893. Cattani [4] has introduced a new field (called Heavisidian field) which depends on the velocities of gravitational charges in the same way as a magnetic field depends on the velocities of electric charges. He has also derived the covariant equations for linear gravitational fields in the same way as for electromagnetic fields.

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The set of governing equations related to linear gravity can be expressed in a similar form with their electromagnetic counterparts (assuming with $G = c = 1$),

$$\nabla \cdot \mathbf{E}_{\mathrm{g}} = -\varrho_e, \tag{1a}$$

$$\nabla \cdot \mathbf{H}_{\mathrm{g}} = 0, \tag{1b}$$

$$\nabla \times \mathbf{E}_{\mathrm{g}} = -\frac{\partial \mathbf{H}_{\mathrm{g}}}{\partial t}, \tag{1c}$$

$$\nabla \times \mathbf{H}_{\mathrm{g}} = -\mathbf{J}_{\mathrm{g}}^{e} + \frac{\partial \mathbf{E}_{\mathrm{g}}}{\partial t}, \tag{1d}$$

where ϱ_e and $\mathbf{J}_{\mathrm{g}}^{e}$ are the gravitoelectric mass density and gravitoelectric mass current density, respectively [5]. The field $\mathbf{E}_{\mathrm{g}}$ is called the gravitoelectric field while $\mathbf{H}_{\mathrm{g}}$ is termed as the gravitomagnetic field. The Newtonian theory of gravitation may be interpreted in terms of a gravitoelectric field.

The fields of GEM in eq. (1) can be defined in close analogy with the classical electrodynamics,

$$\mathbf{E}_{\mathrm{g}} = -\nabla \varphi_e - \frac{\partial \mathbf{A}_{\mathrm{g}}^{e}}{\partial t}, \tag{2a}$$

$$\mathbf{H}_{\mathrm{g}} = \nabla \times \mathbf{A}_{\mathrm{g}}^{e}, \tag{2b}$$

where φ_e and $\mathbf{A}_{\mathrm{g}}^{e}$ are the gravitoelectric scalar potential and vector potential, respectively. Similarly, the Lorenz gauge condition can be written as

$$\nabla \cdot \mathbf{A}_{\mathrm{g}}^{e} + \frac{\partial \varphi_e}{\partial t} = 0. \tag{3}$$

In 1931, Dirac [6] has postulated the existence of magnetic monopoles in order to construct formal symmetry among electromagnetic field equations. Analogously to Dirac's magnetic monopole theory in electromagnetism, the existence of gravitomagnetic mass can be proposed for the GEM. In relevant literature, gravitomagnetic mass is also named as the dual mass, gravitomagnetic charge (monopole) or magnetic mass [7]. By introducing gravitomagnetic mass terms to eq. (1), the Maxwell-type gravitational field equations show more symmetry between the gravitoelectric and gravitomagnetic fields [8],

$$\nabla \cdot \mathbf{E}_{\mathrm{g}} = -\varrho_e, \tag{4a}$$

$$\nabla \cdot \mathbf{H}_{\mathrm{g}} = -\varrho_m, \tag{4b}$$

$$\nabla \times \mathbf{E}_{\mathrm{g}} = \mathbf{J}_{\mathrm{g}}^{m} - \frac{\partial \mathbf{H}_{\mathrm{g}}}{\partial t}, \tag{4c}$$

$$\nabla \times \mathbf{H}_{\mathrm{g}} = -\mathbf{J}_{\mathrm{g}}^{e} + \frac{\partial \mathbf{E}_{\mathrm{g}}}{\partial t}, \tag{4d}$$

where ϱ_m and $\mathbf{J}_g^m$ are the gravitomagnetic mass density and the gravitomagnetic mass current density, respectively. The fields of Dirac–Maxwell-type equations of GEM are invariant under the following duality transformation:

$$\mathbf{E}_g \Rightarrow \mathbf{E}_g \cos\theta + \mathbf{H}_g \sin\theta; \quad \mathbf{H}_g \Rightarrow -\mathbf{E}_g \sin\theta + \mathbf{H}_g \cos\theta \tag{5}$$

with the corresponding matrix representation

$$\begin{pmatrix} \mathbf{E}_g \\ \mathbf{H}_g \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{E}_g \\ \mathbf{H}_g \end{pmatrix}. \tag{6}$$

For a particular value of $\theta = \pi/2$, these expressions reduce to

$$\mathbf{E}_g \to \mathbf{H}_g; \quad \mathbf{H}_g \to -\mathbf{H}_g \tag{7}$$

and

$$\begin{pmatrix} \mathbf{E}_g \\ \mathbf{H}_g \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E}_g \\ \mathbf{H}_g \end{pmatrix}. \tag{8}$$

Since the matrix representation in eq. (6) defines a more general form of duality transformation, the other quantities related to GEM in eq. (4) are invariant under the following duality transformations as well:

$$\begin{pmatrix} \varrho_e \\ \varrho_m \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \varrho_e \\ \varrho_m \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} \mathbf{J}_g^e \\ \mathbf{J}_g^m \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{J}_g^e \\ \mathbf{J}_g^m \end{pmatrix}. \tag{10}$$

Naturally, introduction of gravitomagnetic mass terms to eq. (1) causes some modification in the definitions of the GEM fields in the following manner:

$$\mathbf{E}_g = -\nabla\varphi_e - \frac{\partial \mathbf{A}_g^e}{\partial t} - \nabla \times \mathbf{A}_g^m, \tag{11a}$$

$$\mathbf{H}_g = -\nabla\varphi_m - \frac{\partial \mathbf{A}_g^m}{\partial t} + \nabla \times \mathbf{A}_g^e, \tag{11b}$$

where φ_m and $\mathbf{A}_g^m$ represent the gravitomagnetic scalar potential and the gravitomagnetic vector potential, respectively. Similar to eq. (3), one more Lorenz condition related to the new potential terms must also be satisfied.

$$\nabla \cdot \mathbf{A}_g^m + \frac{\partial \varphi_m}{\partial t} = 0. \tag{12}$$

Although GEM and electromagnetism have been constructed on different backgrounds, in most cases the analogies to the classical equations of electromagnetism are shown. The set of Maxwell-like equations of GEM is based on a framework within Einstein's general relativity in the weak field approximation. The weak gravitational field is naturally treated according to the linearized general relativity theory. As pointed out before, this weak field approximation splits gravitation into components similar to the electric and magnetic fields. In this approximation scheme, one can also discuss the effect of spatial gauge transformations on the GE and GM vector fields. If Ψ is an arbitrary differentiable scalar function, under the following GEM gauge transformation

$$\varphi' = \varphi - \frac{\partial}{\partial t}\Psi, \tag{13a}$$

$$\mathbf{A}'_{\mathrm{g}} = \mathbf{A}_{\mathrm{g}} + \nabla\Psi, \tag{13b}$$

the GE and GM fields remain invariant analogous to the electric and magnetic fields of electromagnetism [9]. By analogy with the covariant laws of electromagnetism in space-time, Clark and Tucker [10] have defined gravitoelectromagnetic potentials and fields to emulate electromagnetic gauge transformations under substitutions belonging to the gauge symmetry group of perturbative gravitation.

The classical Maxwell equations in electromagnetic theory are based on zero rest mass of the photon. However, the effects of finite photon rest mass in the Maxwell equations are expressed by the Proca equations [11,12]. Similarly, in the framework of a generalization of linear gravitation to the case when the gravitons have non-zero rest mass, Argyris and Ciubotariu [13] have shown that a result can be obtained analogous to Proca equation. According to their theory, the Proca-type field equations for GEM can be expressed straightforward from their electromagnetic counterparts as

$$\nabla \cdot \mathbf{E}_{\mathrm{g}} = -\varrho_e - \mu_\gamma^2 \varphi_e, \tag{14a}$$

$$\nabla \cdot \mathbf{H}_{\mathrm{g}} = 0, \tag{14b}$$

$$\nabla \times \mathbf{E}_{\mathrm{g}} = -\frac{\partial \mathbf{H}_{\mathrm{g}}}{\partial t}, \tag{14c}$$

$$\nabla \times \mathbf{H}_{\mathrm{g}} = -\mathbf{J}_{\mathrm{g}}^e + \frac{\partial \mathbf{E}_{\mathrm{g}}}{\partial t} - \mu_\gamma^2 \mathbf{A}_{\mathrm{g}}^e. \tag{14d}$$

Here the fields $\mathbf{E}_{\mathrm{g}}$ and $\mathbf{H}_{\mathrm{g}}$ can be defined in terms of the potentials just as given in eq. (2). Analogously to the Proca equation in electromagnetic theory, the term μ_γ represents the inverse Compton wavelength of the graviton,

$$\mu_\gamma = \frac{m_{\mathrm{g}} c}{\hbar}, \tag{15}$$

where m_{g} is the mass of the graviton.

As emphasized by Argyris and Ciubotariu [13], the presence of massive gravitons breaks gauge invariance in massive linear GEM. In contrast to the Maxwell-type theory of linear gravitation, the potentials φ_e and $\mathbf{A}_{\mathrm{g}}^e$ are directly measurable quantities so that gauge invariance is not possible. On the other hand, the Lorenz gauge condition given in eq. (3) is required in order to conserve mass.

Finally, the Dirac-type field equations in eq. (4) can be combined with the Proca-type gravitational field equations in eq. (14) in the same theory. In this way, the most generalized form of Maxwell-like equations of GEM can be written by the following vector algebra formalism:

$$\nabla \cdot \mathbf{E}_{\mathrm{g}} = -\varrho_e - \mu_\gamma^2 \varphi_e, \tag{16a}$$

$$\nabla \cdot \mathbf{H}_{\mathrm{g}} = -\varrho_m, \tag{16b}$$

$$\nabla \times \mathbf{E}_{\mathrm{g}} = \mathbf{J}_{\mathrm{g}}^m - \frac{\partial \mathbf{H}_{\mathrm{g}}}{\partial t}, \tag{16c}$$

$$\nabla \times \mathbf{H}_{\mathrm{g}} = -\mathbf{J}_{\mathrm{g}}^e + \frac{\partial \mathbf{E}_{\mathrm{g}}}{\partial t} - \mu_\gamma^2 \mathrm{A}_{\mathrm{g}}^e. \tag{16d}$$

Because of the formal correspondence in the expressions of field equations related to gravity and electromagnetism, several mathematical formalisms including quaternions, biquaternions, octonions, sedenions and Clifford numbers have been used to formulate GEM in analogy to electromagnetism.

Real quaternions were first described in 1843 by the Irish mathematician Sir William Rowan Hamilton [14] to extend complex numbers into four dimensions. Similarly, biquaternions (complex quaternions) are composed of two real quaternions and have eight real components. Singh [15] has proposed a quaternionic model to formulate the linear gravitational field equations with Heavisidian monopoles. Similarly, Majernik [16] has used the biquaternion theory to generalize the Maxwell-like gravitational field equations. Introducing gravitodyons (particle carrying gravitational and Heavisidian charges simultaneously), Negi *et al* [17–21] have derived biquaternionic equations of gravitodyons in terms of four potentials and maintained the structural symmetry between the generalized electromagnetic field of dyons (particle carrying both electric and magnetic charges) and those of the generalized gravito-Heavisidian fields of gravitodyons. Similarly, we [22] have enhanced biquaternionic form of linear gravity by evaluating both Proca-type fields and monopole terms. Additionally, by combining the generalized field equations in electromagnetism with the Maxwell-type field equations in gravity, a unified theory for gravi-electromagnetism has been developed [23].

Octonions form the widest normed division algebra after the algebra of quaternions and they can also be used to investigate the analogy between electromagnetism and gravity. Split and hyperbolic octonions differ from classical octonions, since they have hyperbolic basis instead of the complex ones. Using the split octonions, Bisht *et al* [24] have derived relevant field equations for the unified fields of dyons and gravitodyons in compact, simpler and manifestly covariant forms. Köplinger has used hyperbolic octonions to formulate quantum gravity [25].

In order to unify the generalized field equations of electromagnetism and gravity using real components, a sixteen-dimensional mathematical structure (as a sedenion) is needed. In our recent paper [26], the sedenionic formulation has been presented for the unification of generalized field equations of dyons and gravitodyons.

Since geometric (Clifford) algebra combines the dot and cross products into a single geometric product, the Maxwell equations and their most elegant expression can be derived in terms of this algebra. Space-time algebra (STA) is the geometric algebra of Minkowski space-time which is first developed by Hestenes [27]. Cafaro and Ali [28] have formulated the Maxwell equations with massive photons and magnetic monopoles using this algebra. They have also demonstrated how to obtain a single nonhomogeneous multivectorial equation describing the theory. Using the complex Clifford algebra formalism, Ulrych [29] has proposed a linear vector model of gravitation in the context of quantum physics as a generalization of electromagnetism with the help of a hyperbolic unitary gauge symmetry.

In this paper, an alternative formulation of GEM which is based on linear covariant equations in the same way as for the electromagnetic theory has been derived. In relevant literature, using STA, the Proca–Maxwell-type formulation of GEM with gravitomagnetic monopoles has not been formulated yet. In this paper, we have shown that all the field equations of GEM can be expressed by a single equation analogously to the generalized electromagnetism of STA [28]. Furthermore, the most generalized form of gravitational wave equation and Klein–Gordon equation have been obtained in simpler and compact notations.

The lay-out of this paper is as follows: Firstly, a brief introduction to STA is given in §2. Later, the compact formulation of massive GEM with monopoles has been derived. Furthermore, the most elegant expressions related to GEM are presented. Finally, the derived equations for GEM are compared with the generalized equations of electromagnetism in STA.

2. Space-time algebra

Geometric algebra (GA) offers a description of physical laws that is independent of any coordinate frame. The GA of Minkowski space-time is called STA. In order to describe STA, a set of basis vectors $\{\mathbf{e}_\mu\}_{\mu=0,\ldots,3}$ where $\mathbf{e}_0^2 = -1$ and $\mathbf{e}_n^2 = 1$ for $n = 1, 2, 3$, can be introduced. These four basis vectors satisfy the relations,

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \frac{1}{2}(\mathbf{e}_\mu \mathbf{e}_\nu + \mathbf{e}_\nu \mathbf{e}_\mu) = \eta_{\mu\nu} = \mathrm{diag}(-+++), \tag{17a}$$

$$\mathbf{e}_\mu \wedge \mathbf{e}_\nu = \frac{1}{2}(\mathbf{e}_\mu \mathbf{e}_\nu - \mathbf{e}_\nu \mathbf{e}_\mu) = -\mathbf{e}_\nu \wedge \mathbf{e}_\mu, \tag{17b}$$

where $\mu, \nu = 0, \ldots, 3$. The three space basis vectors $\mathbf{e}_n$ share the same algebra as the Pauli spin matrices σ_n from nonrelativistic quantum mechanics. The four ST basis $\mathbf{e}_\mu$ obey the same algebraic relations as Dirac gamma matrices γ_μ from relativistic quantum mechanics. The antisymmetric part of the product defines the outer product which is also known as the wedge product.

The GA combines the dot and cross products into a single product termed as geometric product. This product is associative and distributive with respect to addition. The geometric product between two space-time vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the sum of a symmetric inner product $\mathbf{a} \cdot \mathbf{b}$ and an antisymmetric outer product $\mathbf{a} \wedge \mathbf{b}$.

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \tag{18}$$

When **a** and **b** are space vectors, the inner product $\mathbf{a} \cdot \mathbf{b}$ corresponds to scalar product and the result of a scalar product is a scalar. But the result of outer product is a new entity which is termed as bivector. If a bivector is extended by a third vector of STA, the result is a new element called trivector. In three-dimensional Euclidian space, there is one basis trivector equal to $\mathcal{I} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ that commutes with the basis vectors,

$$\mathbf{e}_1\mathcal{I} = \mathcal{I}\mathbf{e}_1, \quad \mathbf{e}_2\mathcal{I} = \mathcal{I}\mathbf{e}_2, \quad \mathbf{e}_3\mathcal{I} = \mathcal{I}\mathbf{e}_3$$

and satisfies $\mathcal{I}^2 = -1$.

By repeating multiplication of basis vectors given in eq. (17), we arrive at STA with 16 elements. A complete basis for STA is given as

$$\begin{aligned} &1 \text{ scalar} \\ &4 \text{ vectors } \{\mathbf{e}_\mu\} \\ &6 \text{ bivectors } \{\mathbf{e}_\mu\mathbf{e}_\nu\} \\ &4 \text{ trivectors(pseudovector) } \{\mathbf{e}_\mu\mathbf{e}_\nu\mathbf{e}_\alpha\} = \{\mathcal{J}\mathbf{e}_\beta\} \\ &1 \text{ pseudoscalar } \{\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\} = \{\mathcal{J}\}. \end{aligned} \tag{19}$$

On the other hand, unlike the trivector $\mathcal{I}$, the unit pseudoscalar $\mathcal{J} = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ anti-commutes with all basis vectors

$$\mathbf{e}_0\mathcal{J} = -\mathcal{J}\mathbf{e}_0, \quad \mathbf{e}_1\mathcal{J} = -\mathcal{J}\mathbf{e}_1, \quad \mathbf{e}_2\mathcal{J} = -\mathcal{J}\mathbf{e}_2, \quad \mathbf{e}_3\mathcal{J} = -\mathcal{J}\mathbf{e}_3$$

and also satisfies $\mathcal{J}^2 = -1$. Both trivector $\mathcal{I}$ and pseudoscalar $\mathcal{J}$ are not unit imaginary number usually employed in physical applications.

A general element $\mathbb{M}$ of the STA, called a multivector, can be written in the expanded form as

$$\mathbb{M} = \sum_{n=0}^{4} \langle M \rangle_n = \alpha + \mathbf{a} + \mathcal{B} + \mathcal{J}\mathbf{b} + \mathcal{J}\beta, \tag{20}$$

where α and β are real scalars, **a** and **b** are real space-time vectors and $\mathcal{B}$ is a bivector.

Scalars, vectors, bivectors and trivectors represent 0-, 1-, 2- and 3-dimensional subspaces respectively. Therefore, a multivector can be decomposed into sums of elements in different grades. The grade-0 scalar, grade-1 vector, grade-2 bivector, grade-3 trivector (pseudovector) and grade-4 pseudoscalar parts of $\mathbb{M}$ can be explicitly expressed as

$$\begin{aligned} \langle M \rangle_0 &= \alpha, \\ \langle M \rangle_1 &= \mathbf{a} = a_0\mathbf{e}_1 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3, \\ \langle M \rangle_2 &= \mathcal{B} = B_0\mathbf{e}_0\mathbf{e}_1 + B_1\mathbf{e}_0\mathbf{e}_2 + B_2\mathbf{e}_0\mathbf{e}_3 \\ &\quad + B_3\mathbf{e}_1\mathbf{e}_2 + B_4\mathbf{e}_1\mathbf{e}_3 + B_5\mathbf{e}_2\mathbf{e}_3, \\ \langle M \rangle_3 &= \mathcal{J}\mathbf{b} = b_0\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + b_1\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3 + b_2\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 \\ &\quad + b_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \\ \langle M \rangle_4 &= \mathcal{J}\beta = \beta\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \end{aligned} \tag{21}$$

where n represents the grade of a multivector element as $\langle M \rangle_n$.

3. Gravitoelectromagnetism in space-time algebra

In order to generalize field equations of gravitoelectromagnetism, let us define the following generalized potential in terms of STA:

$$\mathbb{A} = -\varphi_e \mathbf{e}_0 - \mathbf{A}_{\mathrm{g}}^e + \mathcal{I}\varphi_m + \mathcal{J}\mathbf{A}_{\mathrm{g}}^m, \tag{22}$$

where

$$\mathbf{A}_{\mathrm{g}}^e = A_x^e \mathbf{e}_1 + A_y^e \mathbf{e}_2 + A_z^e \mathbf{e}_3 \tag{23}$$

and

$$\mathbf{A}_{\mathrm{g}}^m = A_x^m \mathbf{e}_1 + A_y^m \mathbf{e}_2 + A_z^m \mathbf{e}_3 \tag{24}$$

represent gravitoelectric and gravitomagnetic vector potentials, respectively.

Similarly, the four-dimensional space-time differential operator can be introduced as

$$\Box = \mathbf{e}_0 \frac{\partial}{\partial t} - \nabla = \mathbf{e}_0 \frac{\partial}{\partial t} - \mathbf{e}_1 \frac{\partial}{\partial x} - \mathbf{e}_2 \frac{\partial}{\partial y} - \mathbf{e}_3 \frac{\partial}{\partial z}, \tag{25}$$

where ∇ is the usual space vector derivative given in vector algebra

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y} + \mathbf{e}_3 \frac{\partial}{\partial z}. \tag{26}$$

Hence, d'Alembertian operator used in wave theory can be written as

$$\boxdot = \Box\overline{\Box} = \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}, \tag{27}$$

where

$$\overline{\Box} = -\mathbf{e}_0 \frac{\partial}{\partial t} + \nabla = -\mathbf{e}_0 \frac{\partial}{\partial t} + \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y} + \mathbf{e}_3 \frac{\partial}{\partial z}. \tag{28}$$

Since geometric product combines the dot and cross products into a single product, the Maxwell–Proca-like gravitational equations with monopole terms can be expressed elegantly in terms of this algebra. For this, by operating $\overline{\Box}$ on the generalized potential $\mathbb{A}$ given in eq. (22), the following equation can be written:

$$\begin{aligned}\overline{\Box}\mathbb{A} &= \left[-\mathbf{e}_0 \frac{\partial}{\partial t} + \nabla\right][-\varphi_e \mathbf{e}_0 - \mathbf{A}_{\mathrm{g}}^e + \mathcal{I}\,\varphi_m + \mathcal{J}\mathbf{A}_{\mathrm{g}}^m] \\ &= -\frac{\partial \varphi_e}{\partial t} - \left[\frac{\partial \mathbf{A}_{\mathrm{g}}^e}{\partial t} + \nabla\varphi_e\right]\mathbf{e}_0 - \nabla\mathbf{A}_{\mathrm{g}}^e \\ &\quad + \mathcal{I}\left[\frac{\partial \mathbf{A}_{\mathrm{g}}^m}{\partial t} + \nabla\,\varphi_m\right] - \mathcal{J}\left[\frac{\partial \varphi_m}{\partial t} + \nabla\mathbf{A}_{\mathrm{g}}^m\right].\end{aligned} \tag{29}$$

Using the following geometric product equations

$$\nabla\mathbf{A}_{\mathrm{g}}^e = \nabla \cdot \mathbf{A}_{\mathrm{g}}^e + \nabla \wedge \mathbf{A}_{\mathrm{g}}^e = \nabla \cdot \mathbf{A}_{\mathrm{g}}^e + \mathcal{I}\nabla \times \mathbf{A}_{\mathrm{g}}^e, \tag{30}$$

and

$$\nabla\mathbf{A}_{\mathrm{g}}^m = \nabla \cdot \mathbf{A}_{\mathrm{g}}^m + \nabla \wedge \mathbf{A}_{\mathrm{g}}^m = \nabla \cdot \mathbf{A}_{\mathrm{g}}^m + \mathcal{I}\nabla \times \mathbf{A}_{\mathrm{g}}^m, \tag{31}$$

eq. (29) can be rewritten as

$$\overline{\Box}\mathbb{A} = -\left[\frac{\partial\varphi_e}{\partial t} + \nabla\cdot\mathbf{A}_g^e\right] - \left[\nabla\varphi_e + \frac{\partial\mathbf{A}_g^e}{\partial t} + \nabla\times\mathbf{A}_g^m\right]\mathbf{e}_0 + \mathcal{I}\left[\nabla\varphi_m + \frac{\partial\mathbf{A}_g^m}{\partial t} - \nabla\times\mathbf{A}_g^e\right] - \mathcal{J}\left[\frac{\partial\varphi_m}{\partial t} + \nabla\cdot\mathbf{A}_g^m\right].$$

Regarding Lorenz-like gauge conditions in eqs (3) and (12), the Maxwell-like gravitational potential equations in (11) reduce to the single equation

$$\overline{\Box}\mathbb{A} = \mathbb{F}. \tag{32}$$

Here bivector $\mathbb{F}$ is the generalized gravitoelectromagnetic fields

$$\mathbb{F} = \mathbf{E}_g\mathbf{e}_0 - \mathcal{I}\mathbf{H}_g = [\mathbf{E}_g + \mathcal{J}\mathbf{H}_g]\mathbf{e}_0, \tag{33}$$

where

$$\mathbf{E}_g = E_x\mathbf{e}_1 + E_y\mathbf{e}_2 + E_z\mathbf{e}_3 \tag{34}$$

and

$$\mathbf{H}_g = H_x\mathbf{e}_1 + H_y\mathbf{e}_2 + H_z\mathbf{e}_3 \tag{35}$$

represent respectively the gravitoelectric and gravitomagnetic fields of GEM in terms of STA. Using the definition in eq. (18), we can separate eq. (32) into scalar and bivector parts:

$$\overline{\Box}\cdot\mathbb{A} = 0, \tag{36}$$

$$\overline{\Box}\wedge\mathbb{A} = \mathcal{F}, \tag{37}$$

where

$$\mathcal{F} = E_x\mathbf{e}_1\mathbf{e}_0 + E_y\mathbf{e}_2\mathbf{e}_0 + E_z\mathbf{e}_3\mathbf{e}_0 - H_x\mathbf{e}_2\mathbf{e}_3 - H_y\mathbf{e}_3\mathbf{e}_1 - H_z\mathbf{e}_1\mathbf{e}_2. \tag{38}$$

Explicitly, eq. (32) is equivalent to the biquaternion form of Maxwell–Proca-type gravitational equation derived in our former paper [22].

Similarly, if differential operator is applied to the bivector $\mathbb{F}$, we obtain

$$\Box\mathbb{F} = \left[\mathbf{e}_0\frac{\partial}{\partial t} - \nabla\right]\left[\mathbf{E}_g\mathbf{e}_0 - \mathcal{I}\mathbf{H}_g\right] = \frac{\partial\mathbf{E}_g}{\partial t} - \mathcal{J}\frac{\partial\mathbf{H}_g}{\partial t} - \mathbf{e}_0\nabla\mathbf{E}_g + \mathcal{I}\,\nabla\mathbf{H}_g. \tag{39a}$$

This equation may be written in the following expanded form by decomposing the geometric products into dot and cross products:

$$\Box\mathbb{F} = \frac{\partial\mathbf{E}_g}{\partial t} - \mathcal{J}\frac{\partial\mathbf{H}_g}{\partial t} - \mathbf{e}_0\left[\nabla\cdot\mathbf{E}_g + \mathcal{I}\,\nabla\times\mathbf{E}_g\right] + \mathcal{I}\left[\nabla\cdot\mathbf{H}_g + \mathcal{I}\,\nabla\times\mathbf{H}_g\right] \tag{39b}$$

and

$$\square\mathbb{F} = -\mathbf{e}_0\nabla\cdot\mathbf{E}_{\rm g} - \left[\nabla\times\mathbf{H}_{\rm g} - \frac{\partial\mathbf{E}_{\rm g}}{\partial t}\right] + \mathcal{I}\nabla\cdot\mathbf{H}_{\rm g}$$
$$-\mathcal{J}\left[\nabla\times\mathbf{E}_{\rm g} + \frac{\partial\mathbf{H}_{\rm g}}{\partial t}\right]. \tag{39c}$$

Using the definitions in eq. (16), we reach the following expression in terms of ST algebra:

$$\square\mathbb{F} = \mathbf{e}_0\varrho_e + \mathbf{J}^e_{\rm g} - \mathcal{I}\varrho_m - \mathcal{J}\mathbf{J}^m_{\rm g} - \mu^2_\gamma[-\varphi_e\mathbf{e}_0 - \mathbf{A}^e_{\rm g}]. \tag{40}$$

If we define new multivectors to obtain a compact formulation as

$$\mathbb{J} = \varrho + \mathbf{J} = (\mathbf{e}_0\varrho_e - \mathcal{I}\varrho_m) + (\mathbf{J}^e_{\rm g} - \mathcal{J}\mathbf{J}^m_{\rm g}) \tag{41}$$

and

$$\mathbb{A}^e = -\varphi_e\mathbf{e}_0 - \mathbf{A}^e_{\rm g} \tag{42}$$

and rearrange eq. (40), the four Maxwell–Proca-like gravitational field equations with monopole terms are reduced to the single equation

$$\square\mathbb{F} + \mu^2_\gamma\mathbb{A}^e = \mathbb{J}. \tag{43}$$

This expression is also equivalent to the biquaternionic equation derived by us [22].

On the other hand, operating differential operator $\square$ on potential eq. (32)

$$\square\overline{\square}\mathbb{A} = \square\mathbb{F} \tag{44}$$

and using the compact expression in eq. (43), we arrive at

$$\boxdot\,\mathbb{A} = \mathbb{J} - \mu^2_\gamma\mathbb{A}^e. \tag{45}$$

Collecting potential terms in one side, finally we reach the most generalized gravitational wave equation with Proca and monopole terms,

$$\boxdot\,\mathbb{A} + \mu^2_\gamma\mathbb{A}^e = \mathbb{J}. \tag{46}$$

This compact formulation actually contains eight subequations:

$$\frac{\partial^2\varphi_e}{\partial t^2} - \frac{\partial^2\varphi_e}{\partial x^2} - \frac{\partial^2\varphi_e}{\partial y^2} - \frac{\partial^2\varphi_e}{\partial z^2} + \mu^2_\gamma\varphi_e = -\varrho_e, \tag{47a}$$

$$\frac{\partial^2\varphi_m}{\partial t^2} - \frac{\partial^2\varphi_m}{\partial x^2} - \frac{\partial^2\varphi_m}{\partial y^2} - \frac{\partial^2\varphi_m}{\partial z^2} = -\varrho_m, \tag{47b}$$

$$\frac{\partial^2 A^e_x}{\partial t^2} - \frac{\partial^2 A^e_x}{\partial x^2} - \frac{\partial^2 A^e_x}{\partial y^2} - \frac{\partial^2 A^e_x}{\partial z^2} + \mu^2_\gamma A^e_x = -J^e_x, \tag{47c}$$

$$\frac{\partial^2 A^e_y}{\partial t^2} - \frac{\partial^2 A^e_y}{\partial x^2} - \frac{\partial^2 A^e_y}{\partial y^2} - \frac{\partial^2 A^e_y}{\partial z^2} + \mu^2_\gamma A^e_y = -J^e_y, \tag{47d}$$

$$\frac{\partial^2 A_z^e}{\partial t^2} - \frac{\partial^2 A_z^e}{\partial x^2} - \frac{\partial^2 A_z^e}{\partial y^2} - \frac{\partial^2 A_z^e}{\partial z^2} + \mu_\gamma^2 A_z^e = -J_z^e, \tag{47e}$$

$$\frac{\partial^2 A_x^m}{\partial t^2} - \frac{\partial^2 A_x^m}{\partial x^2} - \frac{\partial^2 A_x^m}{\partial y^2} - \frac{\partial^2 A_x^m}{\partial z^2} = -J_x^m, \tag{47f}$$

$$\frac{\partial^2 A_y^m}{\partial t^2} - \frac{\partial^2 A_y^m}{\partial x^2} - \frac{\partial^2 A_y^m}{\partial y^2} - \frac{\partial^2 A_y^m}{\partial z^2} = -J_y^m, \tag{47g}$$

$$\frac{\partial^2 A_z^m}{\partial t^2} - \frac{\partial^2 A_z^m}{\partial x^2} - \frac{\partial^2 A_z^m}{\partial y^2} - \frac{\partial^2 A_z^m}{\partial z^2} = -J_z^m. \tag{47h}$$

In the absence of gravitomagnetic monopoles in GEM theory, the terms ϱ_m and $\mathbf{J}_g^m$ in eq. (16) vanish and the Proca-type expressions of GEM must be defined by eq. (14). According to this new situation, the derived eqs (43) and (46) must be revised in the following manner, respectively,

$$\Box\mathbb{F} + \mu_\gamma^2 \mathbb{A}^e = \mathbb{J}^e \tag{48}$$

and

$$\left[\boxdot + \mu_\gamma^2\right] \mathbb{A}^e = \mathbb{J}^e. \tag{49}$$

Here, the space-time vector $\mathbb{J}^e$ is defined as

$$\mathbb{J}^e = \mathbf{e}_0 \varrho_e + \mathbf{J}^e \tag{50}$$

and termed as the generalized gravitoelectric current density. In this situation, hereafter eqs (48) and (49) respectively represent the Proca equation and wave equation of GEM without monopole terms.

For the source-free fields of GEM the space-time vector $\mathbb{J}^e$ also vanishes, $\mathbb{J}^e = 0$, and eq. (49) should be represented in the following new form:

$$\left[\boxdot + \mu_\gamma^2\right] \mathbb{A}^e = 0. \tag{51}$$

This formula is essentially the compact form of Klein–Gordon equation of GEM for the graviton. We can also rewrite this equation in terms of its components in the following manner:

$$\frac{\partial^2 A_x^e}{\partial t^2} - \frac{\partial^2 A_x^e}{\partial x^2} - \frac{\partial^2 A_x^e}{\partial y^2} - \frac{\partial^2 A_x^e}{\partial z^2} + \mu_\gamma^2 A_x^e = 0, \tag{52a}$$

$$\frac{\partial^2 A_y^e}{\partial t^2} - \frac{\partial^2 A_y^e}{\partial x^2} - \frac{\partial^2 A_y^e}{\partial y^2} - \frac{\partial^2 A_y^e}{\partial z^2} + \mu_\gamma^2 A_y^e = 0, \tag{52b}$$

$$\frac{\partial^2 A_z^e}{\partial t^2} - \frac{\partial^2 A_z^e}{\partial x^2} - \frac{\partial^2 A_z^e}{\partial y^2} - \frac{\partial^2 A_z^e}{\partial z^2} + \mu_\gamma^2 A_z^e = 0, \tag{52c}$$

$$\frac{\partial^2 \varphi_e}{\partial t^2} - \frac{\partial^2 \varphi_e}{\partial x^2} - \frac{\partial^2 \varphi_e}{\partial y^2} - \frac{\partial^2 \varphi_e}{\partial z^2} = 0. \tag{52d}$$

Finally, if we take $\mu_\gamma = 0$ in eq. (48), a situation that states zero rest mass of graviton, we obtain the following compact formulation for the classical fields of GEM given in eq. (1):

$$\Box\mathbb{F} = \mathbb{J}^e \tag{53}$$

which is similar to the quaternion form for the Maxwell equations in electromagnetic theory [30].

Re-operation of $\Box$ on eq. (53) enables us to derive the mass conservation equation for GEM,

$$\boxdot\mathbb{F} = \Box\mathbb{J}^e = \Box\cdot\mathbb{J}^e + \Box\wedge\mathbb{J}^e. \tag{54}$$

Since $\boxdot$ is a scalar operator, the left-hand side of this equation is a bivector. Naturally, the right-hand side of eq. (54) must also be bivector. Therefore, the following equations can be written:

$$\boxdot\mathbb{F} = \Box\wedge\mathbb{J}^e \tag{55}$$

and

$$\Box\cdot\mathbb{J}^e = \frac{\partial\varrho_e}{\partial t} + \nabla\cdot\mathbf{J}^e = 0 \tag{56}$$

which is similar to the charge conservation equation of STA in electromagnetic theory [31].

4. Conclusion

In this study, for the first time, a mathematical model has been proposed in order to formulate the Maxwell–Proca-like equations of GEM-based STA. Introducing monopole terms, it has also been demonstrated that a single multivectorial eq. (43) summarizes the linear gravitational field equations given in eq. (16).

The special eq. (43) also proves the analogy between the electromagnetic theory and GEM. This equation is equivalent to the geometric algebra formulation of the fundamental equations of massive classical electrodynamics in the presence of magnetic monopoles proposed by Cafaro and Ali [28].

In general, since geometric (Clifford) algebra combines the dot and cross products into a single geometric product, it has been proved that the most elegant expression of GEM can be derived in terms of this algebra. This paper also shows that the most generalized form of gravitational wave eq. (46) including both Proca and monopole terms can be expressed in a form similar to the biquaternionic formulation of GEM before it was derived by us [22]. Furthermore, the obtained equations in terms of STA prove that basic equations of GEM can be written in the similar form with their electromagnetic counterparts.

The simpler and compact notations are the basic advantages of the present formalism. Since it is easily associated with traditional vector algebra formalism, the STA simplifies the study of the formulation related to GEM. If it is needed, the derived equations can also be represented in terms of simple matrices. Furthermore, the space-time vector derivative and geometric product enable us to unite all the equations of GEM into a single equation.

References

[1] S M Kopeikin, *Int. J. Mod. Phys. D* **15**, 305 (2006)
[2] J C Maxwell, *Philos. Trans.* **155**, 492 (1865)
[3] O Heaviside, *The Electrician* **31**, 282 (1893)
[4] D D Cattani, *Nuovo Cimento B* **60**, 67 (1980)
[5] B Mashhoon, H J Paik and C M Will, *Phys. Rev. D* **39**, 2825 (1989)
[6] P A M Dirac, *Proc. R. Soc. London A* **133**, 60 (1931)
[7] J Q Shen, *Ann. Phys.* **13**, 532 (2004)
[8] B S Rajput, *J. Math. Phys.* **25**, 351 (1984)
[9] B Mashhoon, Gravitoelectromagnetism: A Brief Review (chapter 3), in: *The measurement of gravitomagnetism: A challenging enterprise* edited by L Iorio (NOVA Science, New York, 2007)
[10] S J Clark and R W Tucker, *Class. Quant. Grav.* **17**, 4125 (2000)
[11] A Proca, *Compt. Rend.* **190**, 1377 (1930)
[12] R S Lakes, *Phys. Lett. A* **329**, 298 (2004)
[13] J Argyris and C Ciubotariu, *Aust. J. Phys.* **50**, 879 (1997)
[14] W R Hamilton, *Elements of quaternions* (Chelsea, New York, 1899) Vols I, II
[15] A Singh, *Lett. Nuovo Cimento* **32**, 5 (1981)
[16] V Majernik, *Astrophys. Space. Sci.* **84**, 191 (1982)
[17] P S Bisht, O P S Negi and B S Rajput, *Indian J. Pure Appl. Phys.* **24**, 543 (1993)
[18] P S Bisht, O P S Negi and B S Rajput, *Int. J. Theor. Phys.* **32**, 2099 (1993)
[19] S Dangwal, P S Bisht and O P S Negi, *Russ. Phys. J.* **49**, 1274 (2006)
[20] P S Bisht, G Karnatak and O P S Negi, *Int. J. Theor. Phys.* **49**, 1344 (2010)
[21] A S Rawat and O P S Negi, *Int. J. Theor. Phys.* **51**, 738 (2012)
[22] S Demir and M Tanışlı, *Eur. Phys. J. Plus* **126**, 51 (2011)
[23] S Demir and M Tanışlı, *Eur. Phys. J. Plus* **126**, 115 (2011)
[24] P S Bisht, S Dangwal and O P S Negi, *Int. J. Theor. Phys.* **47**, 2297 (2008)
[25] J Köplinger, *Appl. Math. Comput.* **188**, 942 (2007)
[26] S Demir and M Tanışlı, *Int. J. Theor. Phys.* **51**, 1239 (2012)
[27] D Hestenes, *Space-time algebra* (Gordon and Breach, New York, 1966)
[28] C Cafaro and S A Ali, *Adv. Appl. Clifford Alg.* **17**, 23 (2007)
[29] S Ulrych, *Phys. Lett. B* **633**, 631 (2006)
[30] S Demir, M Tanışlı and N Candemir, *Adv. Appl. Clifford Alg.* **20**, 547 (2010)
[31] C Doran and A Lasenby, *Geometric algebra for physicists* (Cambridge University Press, Cambridge, 2003)