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# Classification of single travelling wave solutions to the generalized Zakharov–Kuznetsov equation

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**Abstract.** By the complete discrimination system for the polynomial method, the classification of single travelling wave solutions to the generalized Zakharov–Kuznetsov equation with $p = 2$ was obtained.

**Keywords.** Complete discrimination system for polynomial; the generalized Zakharov–Kuznetsov equation; travelling wave solution.

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## 1. Introduction

The investigation of exact solutions of nonlinear wave equations plays an important role in the study of nonlinear physical phenomena. Many authors started concentrating on researching the exact solutions of nonlinear wave equations [1,2]. For many nonlinear differential equations which can be directly reduced to the integral forms under the travelling wave transformation, their single wave solutions can be classified by using Liu's complete discrimination system for the polynomial [3–9].

The Zakharov–Kuznetsov equation (ZK) was first derived for describing weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma composed of cold ions and hot isothermal electrons [10]. Consider the generalized ZK equation:

$$u_t + (\alpha u^p + \beta u^2 p)u_x + \gamma u_{xxx} + \delta u_{xyy} = 0, \tag{1}$$

where $\alpha, \beta, \gamma, \delta$ are real constant parameters and $p > 0, \gamma \neq 0$. In ref. [11], some solutions of the generalized Zakharov–Kuznetsov (ZK) equation were given by the bifurcation method of dynamical systems. A number of modified ZK equations have also been studied [12,13], most of which can be written in the form of eq. (1).

In this paper, by using Liu's complete discrimination system for the polynomial method, we considered eq. (1) with $p = 2$, which reads as

$$u_t + (\alpha u^2 + \beta u^4)u_x + \gamma u_{xxx} + \delta u_{xyy} = 0, \tag{2}$$

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where $\alpha, \beta, \gamma, \delta$ are real constant parameters and $\beta \neq 0, \gamma + \delta \neq 0$, and gave the classification of its single travelling wave solutions.

## 2. Classification of the solutions of eq. (2)

Under the travelling wave transformation $u(x, y, t) = v(\xi), \xi = x + y - ct$, we have

$$-cu' + (\alpha u^2 + \beta u^4)u' + (\gamma + \delta)u''' = 0. \tag{3}$$

Integrating eq. (3) twice, yields

$$(u')^2 = c_1 + c_0 u + \frac{c}{\gamma + \delta}u^2 - \frac{\alpha}{6(\gamma + \delta)}u^4 - \frac{\beta}{15(\gamma + \delta)}u^6, \tag{4}$$

where $c_0, c_1$ are arbitrary constants.

*Case* 1. $c_0 = 0$

Let $v = u^2$, we have

$$(v')^2 = f(v), \tag{5}$$

where

$$f(v) = a_4 v^4 - \frac{2\alpha}{3(\gamma + \delta)}v^3 + \frac{4c}{\gamma + \delta}v^2 + 4c_1 v$$

and

$$a_4 = -\frac{4\beta}{15(\gamma + \delta)}.$$

If $a_4 > 0$, taking the variable transformation

$$w = a_4^{1/4}\left(v + \frac{5\alpha}{8\beta}\right), \quad \xi_1 = a_4^{1/4}\xi, \tag{6}$$

we have

$$(w')^2 = f(w), \quad f(w) = w^4 + pw^2 + qw + r, \tag{7}$$

where

$$\begin{aligned} p &= -\frac{\left(-\frac{15\beta}{\gamma+\delta}\right)^{1/2}(5\alpha^2 + 32\beta c)}{16\beta^2}, \\ q &= -\frac{5^{1/4}\left(-\frac{\beta}{3(\gamma+\delta)}\right)^{3/4}(-25\alpha^3 - 240\alpha\beta c + 192\beta^2 c_1(\delta + \gamma))}{16\sqrt{2}\beta^3}, \\ r &= \frac{5\alpha(25\alpha^3 + 320\alpha\beta c - 512\beta^2(\gamma + \delta)c_1)}{1024\beta^3(\gamma + \delta)}. \end{aligned} \tag{8}$$

If $a_4 < 0$, taking the variable transformation

$$w = (-a_4)^{1/4}\left(v + \frac{5\alpha}{8\beta}\right), \quad \xi_1 = (-a_4)^{1/4}\xi, \tag{9}$$

we have

$$(w')^2 = f(w), \quad f(w) = -(w^4 + pw^2 + qw + r), \tag{10}$$

where

$$\begin{aligned} p &= -\frac{\left(\frac{15\beta}{\gamma+\delta}\right)^{1/2}(5\alpha^2 + 32\beta c)}{16\beta^2}, \\ q &= -\frac{5^{1/4}\left(\frac{\beta}{3(\gamma+\delta)}\right)^{3/4}(-25\alpha^3 - 240\alpha\beta c + 192\beta^2 c_1(\delta+\gamma))}{16\sqrt{2}\beta^3}, \\ r &= -\frac{5\alpha(25\alpha^3 + 320\alpha\beta c - 512\beta^2(\gamma+\delta)c_1)}{1024\beta^3(\gamma+\delta)}. \end{aligned} \tag{11}$$

The complete discrimination system of $F(w)$ is given by

$$\begin{aligned} D_1 &= 4, \quad D_2 = -p, \quad D_3 = -2p^3 + 8pr - 9q^2, \\ D_4 &= -p^3q^2 + 4p^4r + 36pq^2r - 32p^2r^2 - \frac{27}{4}q^4 + 64r^3, \\ E_2 &= 9p^2 - 32pr. \end{aligned} \tag{12}$$

*Case* 1.1. $D_2 < 0, D_3 = 0, D_4 = 0$, i.e. $p > 0, q = 0, r = p^2/4$, then $f(w) = (w^2 + (p/2))^2$. If $a_4 > 0$, the solution of eq. (2) is

$$u = \pm\left(\pm a_4^{-1/4}\sqrt{\frac{p}{2}}\tan\left(\sqrt{\frac{p}{2}}\,a_4^{1/4}\xi - \xi_0\right) - \frac{5\alpha}{8\beta}\right)^{1/2}. \tag{13}$$

*Case* 1.2. $D_2 = 0, D_3 = 0, D_4 = 0$, i.e. $p = 0, q = 0, r = 0$, then $f(w) = w^4$. If $a_4 > 0$, the solution of eq. (2) is

$$u = \pm\left(\pm a_4^{-1/2}(\xi - \xi_0)^{-1} - \frac{5\alpha}{8\beta}\right)^{1/2}. \tag{14}$$

*Case* 1.3. $D_2 > 0, D_3 = 0, D_4 = 0, E_2 > 0$, i.e. $p < 0, q = 0, r = p^2/4$, then

$$f(w) = \left(w - \sqrt{-\frac{p}{2}}\right)^2\left(w + \sqrt{-\frac{p}{2}}\right)^2.$$

When $w > \sqrt{-p/2}$ or $w < -\sqrt{-p/2}$, the solution of eq. (2) is

$$u = \pm\left(\pm a_4^{-1/4}\sqrt{-\frac{p}{2}}\coth\left(\sqrt{-\frac{p}{2}}\,a_4^{1/4}\xi - \xi_0\right)\right) - \frac{5\alpha}{8\beta}\Bigg)^{1/2}. \tag{15}$$

If $\beta < w < \alpha$, the solution of eq. (2) is

$$u = \pm\left(\pm a_4^{-1/4}\sqrt{-\frac{p}{2}}\tanh\left(\sqrt{-\frac{p}{2}}\,a_4^{1/4}\xi - \xi_0\right)\right) - \frac{5\alpha}{8\beta}\Bigg)^{1/2}. \tag{16}$$

*Case* 1.4. $D_2 > 0, D_3 = 0, D_4 = 0, E_2 = 0$, i.e. $p = -27/8, q = \pm 27/8, r = -243/256$. If $a_4 > 0$, then $f(w) = (w - \alpha_1)^3(w + 3\alpha_1)$, where $\alpha_1 = \pm 3/4$. When $w > \max(\alpha_1, -3\alpha_1)$ or $w < \min(\alpha_1, -3\alpha_1)$, the solution of eq. (2) is

$$u = \pm\left(a_4^{-1/4}\left(\frac{4\alpha_1}{-1 + 4\alpha_1^2 a_4^{1/2}(\xi - \xi_0)^2} + \alpha_1\right) - \frac{5\alpha}{8\beta}\right)^{1/2}. \tag{17}$$

If $a_4 < 0$, $f(w) = -(w - \alpha_1)^3(w + 3\alpha_1)$, where $\alpha_1 = \pm 3/4$. When $\max(\alpha_1, -3\alpha_1) > w > \min(\alpha_1, -3\alpha_1)$, the solution of eq. (2) is

$$u = \pm\left((-a_4)^{-1/4}\left(-\frac{4\alpha_1}{1 + 4\alpha_1^2(-a_4)^{1/2}(\xi - \xi_0)^2} + \alpha_1\right) - \frac{5\alpha}{8\beta}\right)^{1/2}. \tag{18}$$

*Case* 1.5. $D_2 D_3 < 0, D_4 = 0$, then $f(w) = (w - \sigma)^2((w - \sigma)^2 + s^2)$, where $\sigma, s$ are real and $s > 0$. When $a_4 > 0$, the solution of eq. (2) is

$$u = \pm\left(a_4^{-1/4}\left(\frac{4\sigma^2 + s^2}{-2\sigma \pm s\sinh\left(a_4^{1/4}\sqrt{4\sigma^2 + s^2}\,\xi - \xi_0\right)} + \sigma\right) - \frac{5\alpha}{8\beta}\right)^{1/2}. \tag{19}$$

*Case* 1.6. $D_1 > 0, D_3 > 0, D_4 > 0$. If $a_4 > 0$, $f(w) = (w - \alpha_1)(w - \alpha_2)(w - \alpha_3)(w - \alpha_4)$, where $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. Denote $\delta_2 = \dfrac{\sqrt{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}{2}$. When $\alpha_2 < w < \alpha_1$, the solution of eq. (2) is

$$u = \pm\left(a_4^{-1/4}\frac{\alpha_2(\alpha_1 - \alpha_4)\mathrm{sn}^2(\pm\delta_2 a_4^{1/4}(\xi - \xi_0), m) - \alpha_1(\alpha_2 - \alpha_4)}{(\alpha_1 - \alpha_4)\mathrm{sn}^2(\pm\delta_2 a_4^{1/4}(\xi - \xi_0), m) - (\alpha_2 - \alpha_4)} - \frac{5\alpha}{8\beta}\right)^{1/2}, \tag{20}$$

and when $\alpha_4 < w < \alpha_3$, the solution of eq. (2) is

$$u = \pm\left(a_4^{-1/4}\frac{\alpha_4(\alpha_2 - \alpha_3)\mathrm{sn}^2(\pm\delta_2 a_4^{1/4}(\xi - \xi_0), m) - \alpha_3(\alpha_2 - \alpha_4)}{(\alpha_2 - \alpha_3)\mathrm{sn}^2(\pm\delta_2 a_4^{1/4}(\xi - \xi_0), m) - (\alpha_2 - \alpha_4)} - \frac{5\alpha}{8\beta}\right)^{1/2}, \tag{21}$$

where

$$m^2 = \frac{(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}.$$

If $a_4 < 0$, $f(w) = -(w - \alpha_1)(w - \alpha_2)(w - \alpha_3)(w - \alpha_4)$, where $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. When $\alpha_2 < w < \alpha_1$ the solution of eq. (2) is

$$u = \pm\left((-a_4)^{-1/4}\frac{\alpha_3(\alpha_1 - \alpha_2)\mathrm{sn}^2\left(\pm\delta_2(-a_4)^{1/4}(\xi - \xi_0), m\right) - \alpha_2(\alpha_1 - \alpha_3)}{(\alpha_1 - \alpha_2)\mathrm{sn}^2\left(\pm\delta_2(-a_4)^{1/4}(\xi - \xi_0), m\right) - (\alpha_1 - \alpha_3)} - \frac{5\alpha}{8\beta}\right)^{1/2}, \tag{22}$$

and when $\alpha_4 < w < \alpha_3$, the solution of eq. (2) is

$$u = \pm\left((-a_4)^{-1/4}\frac{\alpha_1(\alpha_3-\alpha_4)\mathrm{sn}^2\big(\pm\delta_2(-a_4)^{1/4}(\xi-\xi_0),m\big)-\alpha_2(\alpha_1-\alpha_3)}{(\alpha_3-\alpha_4)\mathrm{sn}^2\big(\pm\delta_2(-a_4)^{1/4}(\xi-\xi_0),m\big)-(\alpha_3-\alpha_1)}-\frac{5\alpha}{8\beta}\right)^{1/2}, \tag{23}$$

where

$$m^2 = \frac{(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}.$$

*Case* 1.7. $D_2D_3 \geq 0$, $D_4 < 0$, then $f(w) = (w-\alpha_1)(w-\beta_1)((w-\sigma)^2+s^2)$, where $\alpha_1 > \beta_1, s > 0$.

$$u = \pm\left(\big(\mathrm{sign}(a_4)a_4\big)^{-1/4}\frac{b_1\mathrm{cn}\big(\pm\varpi(\mathrm{sign}(a_4)a_4)^{1/4}(\xi-\xi_0),m\big)+b_2}{b_3\mathrm{cn}\big(\pm\varpi(\mathrm{sign}(a_4)a_4)^{1/4}(\xi-\xi_0),m\big)+b_4}-\frac{5\alpha}{8\beta}\right)^{1/2}, \tag{24}$$

where

$$\begin{aligned} &b_1 = \frac{1}{2}(\alpha_1+\beta_1)b_3-\frac{1}{2}(\alpha_1-\beta_1)b_4, \quad b_2=\frac{1}{2}(\alpha_1+\beta_1)b_4-\frac{1}{2}(\alpha_1-\beta_1)b_3,\\ &b_3 = \alpha_1-\sigma-\frac{s}{m_1}, \; b_4 = \alpha_1-\sigma-sm_1, \quad E = \frac{s^2+(\alpha_1-\sigma)(\beta_1-\sigma)}{s(\alpha_1-\beta_1)},\\ &m_1 = E\pm\sqrt{E^2+1}, \quad \varpi = \frac{\sqrt{\mp 2sm_1(\alpha_1-\beta_1)}}{2mm_1}, \end{aligned} \tag{25}$$

and select $m_1$ such that $m_1a_4 < 0$.

*Case* 1.8. $D_2D_3 \leq 0$, $D_4 > 0$, then $f(w) = ((w-\sigma_1)^2+s_1^2)((w-\sigma_2)^2+s_2^2)$, where $s_1 \geq s_2 > 0$. When $a_4 > 0$, the solution of eq. (2) is

$$u = \pm\left(a_4^{-1/4}\frac{b_1\mathrm{sn}\big(\pm\eta {a_4}^{1/4}(\xi-\xi_0),m\big)+b_2\mathrm{cn}\big(\pm\eta {a_4}^{1/4}(\xi-\xi_0),m\big)}{b_3\mathrm{sn}\big(\pm\eta {a_4}^{1/4}(\xi-\xi_0),m\big)+b_4\mathrm{cn}\big(\pm\eta {a_4}^{1/4}(\xi-\xi_0),m\big)}-\frac{5\alpha}{8\beta}\right)^{1/2}, \tag{26}$$

where

$$\begin{aligned} &b_1 = \sigma_1b_3+s_1b_4, \quad b_2=\sigma_1b_4-s_1b_3, \quad b_3=-s_1-\frac{s_2}{m_1}, \quad b_4=\sigma_1-\sigma_2,\\ &E = \frac{(\sigma_1-\sigma_2)^2+s_1^2+s_2^2}{2s_1s_2}, \quad m_1 = E+\sqrt{E^2-1}, \quad m^2 = \frac{m_1^2-1}{m_1^2},\\ &\eta = \frac{s_2\sqrt{m_1^2a_3^2+a_4^2}}{\sqrt{a_3^2+a_4^2}}. \end{aligned} \tag{27}$$

*Case* 1.9. $D_2 > 0$, $D_3 > 0$, $D_4 = 0$. If $a_4 > 0$, $f(w) = (w-\bar{\alpha})(w-\bar{\beta})(w-\bar{\gamma})^2$, where $\bar{\alpha} > \bar{\beta}$. Denote $\delta_3 = (\bar{\gamma}-\bar{\alpha})(\bar{\beta}-\bar{\gamma})$, $\delta_4 = \bar{\alpha}+\bar{\beta}-2\bar{\gamma}$. When $w > \bar{\alpha}$, if $\bar{\beta} < \bar{\gamma} < \bar{\alpha}$, the solution of eq. (2) is

$$u = \pm\left(a_4^{-1/4}\bar{\gamma} + \frac{2a_4^{-1/4}\delta_3}{\pm(\bar{\alpha}-\bar{\beta})\sin(a_4^{1/4}\sqrt{\delta_3}(\xi-\xi_0)) + \delta_4} - \frac{5\alpha}{8\beta}\right)^{1/2}, \tag{28}$$

and if $\bar{\gamma} > \bar{\alpha}$ or $\bar{\gamma} < \bar{\beta}$, the solution of eq. (2) is

$$u = \pm\left(a_4^{-1/4}\bar{\gamma} + \frac{2a_4^{-1/4}(-\delta_3)}{(\bar{\alpha}-\bar{\beta})\cosh(\sqrt{-\delta_3}a_4{}^{1/4}(\xi-\xi_0)) + \delta_4} - \frac{5\alpha}{8\beta}\right)^{1/2}. \tag{29}$$

If $a_4 < 0$, $f(w) = (\bar{\alpha}-w)(w-\bar{\beta})(w-\bar{\gamma})^2$, where $\bar{\alpha} > \bar{\beta}$. When $\bar{\beta} < w < \bar{\alpha}$, if $\bar{\gamma} > \bar{\alpha}$ or $\bar{\gamma} < \bar{\beta}$, the solution of eq. (1) is:

$$u = \pm\left((-a_4)^{-1/4}\bar{\gamma} + \frac{2(-a_4)^{-1/4}\delta_3}{\pm(\bar{\alpha}-\bar{\beta})\sin(\sqrt{\delta_3}(-a_4)^{1/4}(\xi-\xi_0)) - \delta_4} - \frac{5\alpha}{8\beta}\right)^{1/2}, \tag{30}$$

and if $\bar{\beta} < \bar{\gamma} < \bar{\alpha}$, the solution of eq. (2) is

$$u = \pm\left((-a_4)^{-1/4}\bar{\gamma} - \frac{2(-a_4)^{-1/4}(-\delta_3)}{(\bar{\alpha}-\bar{\beta})\cosh(\sqrt{-\delta_3}(-a_4)^{1/4}(\xi-\xi_0)) - \delta_4} - \frac{5\alpha}{8\beta}\right)^{1/2}. \tag{31}$$

*Case* 2. $c_1 = 0$.

In this case, eq. (4) becomes

$$(u')^2 = -\frac{\beta}{15(\gamma+\delta)}uf(u), \tag{32}$$

where

$$f(u) = -\frac{15(\gamma+\delta)}{\beta}c_0 - \frac{15c}{\beta}u + \frac{5\alpha}{2\beta}u^3 + u^5. \tag{33}$$

Denote

$$s = -\frac{15(\gamma+\delta)}{\beta}, \quad g = -\frac{15(\gamma+\delta)}{\beta}c_0,$$

the complete discrimination system of $F(w)$ is given by

$$
\begin{aligned}
D_2 &= -\frac{5\alpha}{2\beta}, \qquad D_3 = -\frac{375\alpha(\alpha^2+8\beta c)}{2\beta^3},\\
D_4 &= -\frac{625(45\alpha^4 c + 792\alpha^2\beta c^2 + 3456\beta^2 c^3 - 2\alpha\beta^4 g^2)}{4\beta^5},\\
D_5 &= \frac{3125}{8\beta^7}(-5400\alpha^4 c^3 - 103680\alpha^2\beta c^4 - 497664\beta^2 c^5 + 27\alpha^5\beta^2 g^2\\
&\qquad + 540\alpha^3\beta^3 c g^2 + 2880\alpha\beta^4 c^2 g^2 + 8\beta^7 g^4),\\
E_2 &= \frac{15625\alpha^2(18\alpha^3 c + 144\alpha\beta c^2 + \beta^4 g^2)}{4\beta^6}, \quad F_2 = \frac{300\alpha c}{\beta^2}.
\end{aligned} \tag{34}
$$

*Case* 2.1. $D_5 = D_4 = 0$, $D_3 > 0$, $E_2 \neq 0$, then $f(u) = (u-\alpha_1)^2(u-\beta_1)^2(u-\gamma_1)$, where $\alpha_1, \beta_1, \gamma_1$ are real and $\alpha_1 \neq \beta_1 \neq \gamma_1$, $\alpha_1 \neq 0$.

*Case* 2.1.1. $\gamma_1 = 0$. When $s > 0$, we have

$$
\pm\sqrt{s}(\xi - \xi_0) = \frac{\alpha_1 \ln\left|\frac{u}{u-\beta_1}\right| - \beta_1 \ln\left|\frac{u}{u-\alpha_1}\right|}{\alpha_1^2\beta_1 - \alpha_1\beta_1^2}. \tag{35}
$$

*Case* 2.1.2. $\beta_1 = 0$.

When $s > 0$, and $u > \max(0, \gamma_1)$, or $u < \min(0, \gamma_1)$, if $\alpha_1(\alpha_1 - \gamma_1) > 0$, we have

$$
\begin{aligned}
&\pm\alpha_1\sqrt{s}(\xi - \xi_0)\\
&= \frac{\operatorname{arccosh}\left(\frac{2\alpha_1(\alpha_1-\gamma_1) + (2\alpha_1-\gamma_1)(u-\alpha_1)}{|\gamma_1|\,(u-\alpha_1)}\right)}{\sqrt{\alpha_1(\alpha_1-\gamma_1)}} - \frac{2\sqrt{u(u-\gamma_1)}}{\gamma_1 u},
\end{aligned} \tag{36}
$$

and if $\alpha_1(\alpha_1 - \gamma_1) < 0$, we have

$$
\begin{aligned}
&\pm\alpha_1\sqrt{s}(\xi - \xi_0)\\
&= \frac{\arcsin\left(\frac{2\alpha_1(\alpha_1-\gamma_1) + (2\alpha_1-\gamma_1)(u-\alpha_1)}{|\gamma_1|\,(u-\alpha_1)}\right)}{\sqrt{-\alpha_1(\alpha_1-\gamma_1)}} - \frac{2\sqrt{u(u-\gamma_1)}}{\gamma_1 u}.
\end{aligned} \tag{37}
$$

When $s < 0$, and $\min(0, \gamma_1) < u < \max(0, \gamma_1)$, if $\alpha_1(\alpha_1 - \gamma_1) > 0$, we have

$$\pm \alpha_1\sqrt{-s}(\xi - \xi_0) = \frac{\arcsin\left(-\frac{2\alpha_1(\alpha_1-\gamma_1)+(2\alpha_1-\gamma_1)(u-\alpha_1)}{|\gamma_1|\,(u-\alpha_1)}\right)}{\sqrt{\alpha_1(\alpha_1-\gamma_1)}} + \frac{2\sqrt{-u(u-\gamma_1)}}{\gamma_1 u}, \tag{38}$$

and if $\alpha_1(\alpha_1 - \gamma_1) < 0$, we have

$$\pm \alpha_1\sqrt{-s}(\xi - \xi_0) = \frac{\operatorname{arccosh}\left(-\frac{2\alpha_1(\alpha_1-\gamma_1)+(2\alpha_1-\gamma_1)(u-\alpha_1)}{|\gamma_1|\,(u-\alpha_1)}\right)}{\sqrt{-\alpha_1(\alpha_1-\gamma_1)}} + \frac{2\sqrt{-u(u-\gamma_1)}}{\gamma_1 u}. \tag{39}$$

*Case* 2.1.3. $\gamma_1 \neq 0, \beta_1 \neq 0$.

When $s > 0$, $u > \max(0, \gamma_1)$ or $u < \min(0, \gamma_1)$, if $\alpha_1(\alpha_1 - \gamma_1) > 0$, $\beta_1(\beta_1 - \gamma_1) > 0$, we have

$$\begin{aligned} \pm\, (\alpha_1 - \beta_1)\sqrt{s}(\xi - \xi_0) &= \frac{\operatorname{arccosh}\left(\frac{2\alpha_1(\alpha_1-\gamma_1)+(2\alpha_1-\gamma_1)(u-\alpha_1)}{|\gamma_1|\,(u-\alpha_1)}\right)}{\sqrt{\alpha_1(\alpha_1-\gamma_1)}} \\ &\quad - \frac{\operatorname{arccosh}\left(\frac{2\beta_1(\beta_1-\gamma_1)+(2\beta_1-\gamma_1)(u-\beta_1)}{|\gamma_1|\,(u-\beta_1)}\right)}{\sqrt{\beta_1(\beta_1-\gamma_1)}}, \end{aligned} \tag{40}$$

if $\alpha_1(\alpha_1 - \gamma_1) < 0$, $\beta_1(\beta_1 - \gamma_1) < 0$, we have

$$\begin{aligned} \pm (\alpha_1 - \beta_1)\sqrt{s}(\xi - \xi_0) &= \frac{\arcsin\left(\frac{2\alpha_1(\alpha_1-\gamma_1)+(2\alpha_1-\gamma_1)(u-\alpha_1)}{|\gamma_1|\,(u-\alpha_1)}\right)}{\sqrt{\alpha_1}(\gamma_1-\alpha_1)} \\ &\quad - \frac{\arcsin\left(\frac{2\beta_1(\beta_1-\gamma_1)+(2\beta_1-\gamma_1)(u-\beta_1)}{|\gamma_1|\,(u-\beta_1)}\right)}{\sqrt{\beta_1(\gamma_1-\beta_1)}}, \end{aligned} \tag{41}$$

if $\alpha_1(\alpha_1-\gamma_1)>0$, $\beta_1(\beta_1-\gamma_1)<0$, we have

$$\pm(\alpha_1-\beta_1)\sqrt{s}(\xi-\xi_0) = \frac{\operatorname{arccosh}\left(\dfrac{2\alpha_1(\alpha_1-\gamma_1)+(2\alpha_1-\gamma_1)(u-\alpha_1)}{|\gamma_1|\,(u-\alpha_1)}\right)}{\sqrt{\alpha_1(\alpha_1-\gamma_1)}} - \frac{\arcsin\left(\dfrac{2\beta_1(\beta_1-\gamma_1)+(2\beta_1-\gamma_1)(u-\beta_1)}{|\gamma_1|\,(u-\beta_1)}\right)}{\sqrt{\beta_1(\gamma_1-\beta_1)}}, \tag{42}$$

and if $\alpha_1(\alpha_1-\gamma_1)<0$, $\beta_1(\beta_1-\gamma_1)>0$, we have

$$\pm(\alpha_1-\beta_1)\sqrt{s}(\xi-\xi_0) = \frac{\arcsin\left(\dfrac{2\alpha_1(\alpha_1-\gamma_1)+(2\alpha_1-\gamma_1)(u-\alpha_1)}{|\gamma_1|\,(u-\alpha_1)}\right)}{\sqrt{\alpha_1(\gamma_1-\alpha_1)}} - \frac{\operatorname{arccosh}\left(\dfrac{2\beta_1(\beta_1-\gamma_1)+(2\beta_1-\gamma_1)(u-\beta_1)}{|\gamma_1|\,(u-\beta_1)}\right)}{\sqrt{\beta_1(\beta_1-\gamma_1)}}. \tag{43}$$

When $s<0$, $\gamma_1<u<0$ or $\gamma_1>u>0$, if $\alpha_1(\alpha_1-\gamma_1)>0$, $\beta_1(\beta_1-\gamma_1)>0$, we have

$$\pm(\alpha_1-\beta_1)\sqrt{-s}(\xi-\xi_0) = \frac{\arcsin\left(\dfrac{2\alpha_1(\alpha_1-\gamma_1)+(2\alpha_1-\gamma_1)(u-\alpha_1)}{|\gamma_1|\,(u-\alpha_1)}\right)}{-\sqrt{\alpha_1(\alpha_1-\gamma_1)}} - \frac{\arcsin\left(\dfrac{2\beta_1(\beta_1-\gamma_1)+(2\beta_1-\gamma_1)(u-\beta_1)}{|\gamma_1|\,(u-\beta_1)}\right)}{-\sqrt{\beta_1(\beta_1-\gamma_1)}}, \tag{44}$$

if $\alpha_1(\alpha_1-\gamma_1)<0$, $\beta_1(\beta_1-\gamma_1)<0$, we have

$$\pm(\alpha_1-\beta_1)\sqrt{-s}(\xi-\xi_0) = \frac{\operatorname{arccosh}\left(-\dfrac{2\alpha_1(\alpha_1-\gamma_1)+(2\alpha_1-\gamma_1)(u-\alpha_1)}{|\gamma_1|\,(u-\alpha_1)}\right)}{\sqrt{-\alpha_1(\alpha_1-\gamma_1)}} - \frac{\operatorname{arccosh}\left(-\dfrac{2\beta_1(\beta_1-\gamma_1)+(2\beta_1-\gamma_1)(u-\beta_1)}{|\gamma_1|\,(u-\beta_1)}\right)}{\sqrt{-\beta_1(\beta_1-\gamma_1)}}, \tag{45}$$

if $\alpha_1(\alpha_1-\gamma_1)>0$, $\beta_1(\beta_1-\gamma_1)<0$, we have

$$\pm(\alpha_1-\beta_1)\sqrt{-s}(\xi-\xi_0) = \frac{\arcsin\left(\dfrac{2\alpha_1(\alpha_1-\gamma_1)+(2\alpha_1-\gamma_1)(u-\alpha_1)}{|\gamma_1|\,(u-\alpha_1)}\right)}{-\sqrt{\alpha_1(\alpha_1-\gamma_1)}} - \frac{\operatorname{arccosh}\left(-\dfrac{2\beta_1(\beta_1-\gamma_1)+(2\beta_1-\gamma_1)(u-\beta_1)}{|\gamma_1|\,(u-\beta_1)}\right)}{\sqrt{-\beta_1(\beta_1-\gamma_1)}}, \tag{46}$$

and if $\alpha_1(\alpha_1-\gamma_1)<0$, $\beta_1(\beta_1-\gamma_1)>0$, we have

$$\pm(\alpha_1-\beta_1)\sqrt{-s}(\xi-\xi_0) = \frac{\operatorname{arccosh}\left(\dfrac{-2\alpha_1(\alpha_1-\gamma_1)-(2\alpha_1-\gamma_1)(u-\alpha_1)}{|\gamma_1|\,(u-\alpha_1)}\right)}{\sqrt{-\alpha_1(\alpha_1-\gamma_1)}} + \frac{\arcsin\left(\dfrac{2\beta_1(\beta_1-\gamma_1)+(2\beta_1-\gamma_1)(u-\beta_1)}{|\gamma_1|\,(u-\beta_1)}\right)}{\sqrt{\beta_1(\beta_1-\gamma_1)}}. \tag{47}$$

*Case* 2.2. $D_5=D_4=D_3=0$, $D_2\neq 0$, $F_2\neq 0$, then $f(u)=(u-\alpha_2)^3(u-\beta_2)^2$, where $\alpha_2$, $\beta_2$ are real and $\alpha_2\neq\beta_2$.

*Case* 2.2.1. $\alpha_2=0$. When $s>0$, $u>\max(0,\beta_2)$ or $u<\min(0,\beta_2)$, we have

$$\pm\sqrt{s}(\xi-\xi_0)=\frac{1}{\beta_2^2}\left(\frac{\beta_2}{u}+\ln\left|\frac{u-\beta_2}{u}\right|\right). \tag{48}$$

*Case* 2.2.2. $\beta_2=0$.

When $s>0$, and $u>\max(0,\alpha_2)$ or $u<\min(0,\alpha_2)$, we have

$$\pm\sqrt{s}(\xi-\xi_0)=\frac{2(2u-\alpha_2)}{\alpha_2^2\sqrt{u(u-\alpha_2)}}. \tag{49}$$

When $s<0$, and $\max(0,\alpha_2)>u>\min(0,\alpha_2)$, we have

$$\pm\sqrt{-s}(\xi-\xi_0)=\frac{2(2u-\alpha_2)}{\alpha_2^2\sqrt{-u(u-\alpha_2)}}. \tag{50}$$

*Case* 2.2.3. $\alpha_2 \neq 0, \beta_2 \neq 0$.

When $s > 0$, if $\beta_2(\beta_2 - \alpha_2) > 0$, we have

$$\pm (\alpha_2 - \beta_2)\sqrt{s}(\xi - \xi_0) = -\frac{2\sqrt{u(u-\alpha_2)}}{\alpha_2(u-\alpha_2)} - \frac{\operatorname{arccosh}\left(\frac{2\beta_2(\beta_2-\alpha_2)+(2\beta_2-\alpha_2)(u-\beta_2)}{|\alpha_2|\,(u-\beta_2)}\right)}{\sqrt{\beta_2(\beta_2-\alpha_2)}}, \tag{51}$$

and if $\beta_2(\beta_2 - \alpha_2) < 0$, we have

$$\pm (\alpha_2 - \beta_2)\sqrt{s}(\xi - \xi_0) = -\frac{2\sqrt{u(u-\alpha_2)}}{\alpha_2(u-\alpha_2)} - \frac{\arcsin\left(\frac{2\beta_2(\beta_2-\alpha_2)+(2\beta_2-\alpha_2)(u-\beta_2)}{|\alpha_2|\,(u-\beta_2)}\right)}{\sqrt{-\beta_2(\beta_2-\alpha_2)}}. \tag{52}$$

When $s < 0$, if $\beta_2(\beta_2 - \alpha_2) > 0$, we have

$$\pm (\alpha_2 - \beta_2)\sqrt{-s}(\xi - \xi_0) = \frac{2\sqrt{-u(u-\alpha_2)}}{\alpha_2(u-\alpha_2)} - \frac{\arcsin\left(\frac{-2\beta_2(\beta_2-\alpha_2)-(2\beta_2-\alpha_2)(u-\beta_2)}{|\alpha_2|\,(u-\beta_2)}\right)}{\sqrt{\beta_2(\beta_2-\alpha_2)}}, \tag{53}$$

and if $\beta_2(\beta_2 - \alpha_2) < 0$, we have

$$\pm (\alpha_2 - \beta_2)\sqrt{-s}(\xi - \xi_0) = \frac{2\sqrt{-u(u-\alpha_2)}}{\alpha_2(u-\alpha_2)} - \frac{\operatorname{arccosh}\left(\frac{-2\beta_2(\beta_2-\alpha_2)-(2\beta_2-\alpha_2)(u-\beta_2)}{|\alpha_2|\,(u-\beta_2)}\right)}{\sqrt{-\beta_2(\beta_2-\alpha_2)}}. \tag{54}$$

*Case* 2.3. $D_5 = D_4 = D_3 = 0$, $D_2 \neq 0$, $F_2 = 0$, then $f(u) = (u - \alpha_3)^4(u - \beta_3)$, where $\alpha_3, \beta_3$ are real and $\alpha_3 \neq \beta_3$.

*Case* 2.3.1. $\alpha_3 = 0$. We have

$$\pm(\xi - \xi_0) = \frac{2}{3s\beta_3}\left(-\frac{1}{u^2} + \frac{2}{\beta_3 u}\right)\sqrt{su(u-\beta_3)}. \tag{55}$$

*Case* 2.3.2. $\beta_3 = 0$. When $s > 0$, we have

$$\pm\sqrt{s}(\xi - \xi_0) = \frac{1}{\alpha_3^2}\left(\ln\left|\frac{u}{u-\alpha_3}\right| - \frac{\alpha_3}{u-\alpha_3}\right). \tag{56}$$

*Case* 2.3.3. $\alpha_3 \neq 0, \beta_3 \neq 0$.

When $s > 0$, and $u > \max(0, \beta_3)$ or $u < \min(0, \beta_3)$, if $\alpha_3(\alpha_3 - \beta_3) > 0$, we have

$$\pm \alpha_3(\alpha_3-\beta_3)\sqrt{s}(\xi-\xi_0) = -\frac{\sqrt{u(u-\beta_3)}}{u-\alpha_3} - \frac{(2\alpha_3-\beta_3)\operatorname{arccosh}\left(\frac{2\alpha_3(\alpha_3-\beta_3)+(2\alpha_3-\beta_3)(u-\alpha_3)}{|\beta_3|\,(u-\alpha_3)}\right)}{2(\alpha_3^2-\alpha_3\beta_3)^{1/2}}, \tag{57}$$

and if $\alpha_3(\alpha_3-\beta_3)<0$, we have

$$\pm\,\alpha_3(\alpha_3-\beta_3)\sqrt{s}(\xi-\xi_0) = \frac{\sqrt{u(u-\beta_3)}}{u-\alpha_3} + \frac{(2\alpha_3-\beta_3)\arcsin\left(\frac{2\alpha_3(\alpha_3-\beta_3)+(2\alpha_3-\beta_3)(u-\alpha_3)}{|\beta_3|\,(u-\alpha_3)}\right)}{2(-\alpha_3^2+\alpha_3\beta_3)^{1/2}}. \tag{58}$$

When $s<0$, if $\alpha_3(\alpha_3-\beta_3)<0$, we have

$$\pm\,\alpha_3(\alpha_3-\beta_3)\sqrt{s}(\xi-\xi_0) = \frac{\sqrt{-u(u-\beta_3)}}{-(u-\alpha_3)} + \frac{(2\alpha_3-\beta_3)\mathrm{arccosh}\left(\frac{2\alpha_3(\alpha_3-\beta_3)+(2\alpha_3-\beta_3)(u-\alpha_3)}{-|\beta_3|\,(u-\alpha_3)}\right)}{2(-\alpha_3^2+\alpha_3\beta_3)^{1/2}}, \tag{59}$$

and if $\alpha_3(\alpha_3-\beta_3)>0$, we have

$$\pm\,\alpha_3(\alpha_3-\beta_3)\sqrt{s}(\xi-\xi_0) = \frac{\sqrt{-u(u-\beta_3)}}{u-\alpha_3} + \frac{(2\alpha_3-\beta_3)\arcsin\left(\frac{2\alpha_3(\alpha_3-\beta_3)+(2\alpha_3-\beta_3)(u-\alpha_3)}{|\beta_3|\,(u-\alpha_3)}\right)}{2(\alpha_3^2-\alpha_3\beta_3)^{1/2}}. \tag{60}$$

*Case* 2.4. $D_5=D_4=D_3=D_2=0$, then $f(u)=u^5$. When $s>0$,

$$\pm\sqrt{s}(\xi-\xi_0) = -\frac{1}{2}u^{-2}. \tag{61}$$

The expressions (35)–(61) are all possible elementary function solutions of eq. (32).

*Case* 2.5. If $D_5=D_4=0$, $D_3<0$, $E_2\neq 0$, then $f(u)=(u^2+\alpha_5u+\beta_5)^2(u-\gamma_5)$, where $\alpha_5^2-4\beta_5<0$, if $D_5=0$, $D_4>0$, then $f(u)=(u-\alpha_6)^2(u-\beta_1)(u-\beta_2)(u-\beta_3)$, if $D_5=D_4=0$, $D_3<0$, $E_2=0$, then $f(u)=(u-\alpha_7)^3((u-l_1)^2+s_1^2)$, if $D_5=0$, $D_4<0$, then $f(u)=(u-\alpha_8)^2(u-\beta_8)((u-l_1)^2+s_1^2)$, if $D_5=D_4=0$, $D_3>0$, $E_2=0$, then $f(u)=(u-\alpha_9)^3(u-\beta_9)(u-\gamma_9)$, if $D_5>0$, $D_4>0$, $D_3>0$, $D_2>0$, then $f(u)=(u-\alpha_{10})(u-\alpha_{11})(u-\alpha_{12})(u-\alpha_{13})(u-\alpha_{14})$, if $D_5<0$, then $f(u)=(u-\alpha_{15})(u-\alpha_{16})(u-\alpha_{17})((u-l_1)^2+s_1^2)$, and if $D_5<0$ and $D_4\leq 0$ or $D_5<0$ or $D_2\leq 0$, then $f(u)=(u-\alpha_{18})(u-\alpha_{19})(u-\alpha_{20})((u-l_1)^2+s_1^2)$. In this case, yields $\pm(\xi-\xi_0)=\int \mathrm{d}u/\sqrt{suf(u)}$, and we can solve it with elliptic functions. For brevity these are omitted here.

## 3. Conclusions

In this paper, by the complete discrimination system for polynomial method, we obtained the classifications of travelling wave solutions to the ZK equation with $p=2$. We obtained lots of explicit solutions eqs (13)–(31) and implicit solutions eqs (35)–(61). Explicit solutions included trigonometric periodic solutions such as eqs (13), (28), (30), rational function solution such as eqs (14), (17), (18), hyperbolic function solutions such as eqs (15), (16), (19), (29), (31), Jacobi elliptic function solutions such as eqs (20)–(27). In fact, if $p = 1$, we can classify its travelling wave solutions with nine cases. If $p \neq 1$ and $p \neq 2$, we can obtain some solutions by using Liu's trial equation method [14–19]. For brevity these are omitted here.

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