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The symmetries and conservation laws of some Gordon-type equations in Milne space-time

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Abstract. In this letter, the Lie point symmetries of a class of Gordon-type wave equations that arise in the Milne space-time are presented and analysed. Using the Lie point symmetries, it is showed how to reduce Gordon-type wave equations using the method of invariants, and to obtain exact solutions corresponding to some boundary values. The Noether point symmetries and conservation laws are obtained for the Klein–Gordon equation in one case. Finally, the existence of higher-order variational symmetries of a projection of the Klein–Gordon equation is investigated using the multiplier approach.

Keywords. Conservation laws; Milne space-time; Gordon-type equations.

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1. Introduction

A vast amount of work has been published in the literature studying differential equations (DEs) in terms of the Lie point symmetries admitted by them [1,2]. These symmetries play an important role in finding exact analytic solutions of the nonlinear DEs. Other than Lie point symmetries, Noether symmetries are also widely studied and are associated, in particular, with those DEs which possess Lagrangians. These symmetries represent physical features of DEs via the conservation laws they admit. The interesting link between symmetries and conservation laws in mathematical physics is provided in the classic work of Noether [3] showing that for every infinitesimal transformation admitted by the action integral of a system, there exists a conservation law. The Noether symmetries, which are symmetries of the Euler–Lagrange systems, have interesting applications in the study of properties of particles moving under the influence of gravitational fields.

Recently, some published results were aimed at understanding Noether symmetries of Lagrangians that arise from certain pseudo-Riemannian metrics of interest [4,5]. More recently, Noether symmetries of the Euler–Lagrange equations on the Milne metric [6] were found and a discussion of the results were given by comparing Noether symmetries on the Milne metric with those of other conventional symmetries of the same space-time [7]. Concerning the pure wave equation (homogeneous), it is *a priori* clear that it will admit a maximal Noether symmetry group on a flat manifold. In that spirit, the work of Mahadi [7] gives limited information and needs further understanding. With this example in mind, we extend the work of Mahadi [7] by studying Klein–Gordon [8] equation on the Milne metric and see how Noether symmetry structures change when classical wave equations are coupled with an inhomogeneous term. For completeness, we also investigate the existence of higher-order variational symmetries of a projection of the Klein–Gordon equation using the multiplier approach.

The plan of the paper is as follows: In §2 we derive Lie point symmetries of some Gordon-type wave equations and illustrate the reduction of a Gordon-type wave equation on a Milne manifold. In §3, we determine the Noether point symmetries of the Klein–Gordon equation and construct the associated conserved densities. Lastly, we list some higher-order symmetries and conservation laws of a projected Klein–Gordon equation in §4.

We present some of the definitions and notations below. Intrinsic to a Lie algebraic treatment of differential equations is the universal space $\mathcal{A}$ (see [2]). The space $\mathcal{A}$ is the vector space of all differential functions of all finite orders and forms an algebra. Consider an rth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \mu = 1, \ldots, \tilde{m}, \tag{1}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collections of all first-, second-, ..., rth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$ respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \qquad i = 1, \ldots, n, \tag{2}$$

where the summation convention is used whenever appropriate. A current $\Phi = (\Phi^1, \ldots, \Phi^n)$ is conserved if it satisfies

$$D_i \Phi^i = 0 \tag{3}$$

along the solutions of (1). It can be shown that every admitted conservation law arises from multipliers $Q_\mu(x, u, u_{(1)}, \ldots)$ such that

$$Q_\mu G^\mu = D_i \Phi^i \tag{4}$$

holds identically (that is, off the solution space) for some current Φ. The conserved vector may then be obtained by the homotopy operator (see [1,9,10]). Other works on symmetries and conservation laws can be found in [11–13].

DEFINITION 1

The Euler operator, for each dependent variable u^α, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \quad \alpha = 1,\ldots,m. \tag{5}$$

Note. In most literature, a variational problem consists of finding the extrema (maxima or minima) of a functional

$$\mathcal{L}[u] = \int_\Omega L(x, u_{(n)})\mathrm{d}x$$

in some class of functions $u = f(x)$ defined over Ω, where $\Omega \subset X$ is an open, connected subset with smooth boundary $\partial\Omega$ (we consider the Euclidean space with $X = R^n$). The integrand $L(x, u_{(n)})$, called the Lagrangian of the variational problem $\mathcal{L}$, is a smooth function of x, u and various derivatives of u [2].

DEFINITION 2

A Lie–Bäcklund operator is given by

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}\zeta^\alpha_{i_1\ldots i_s}\frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}}, \tag{6}$$

where $\xi^i, \eta^\alpha \in \mathcal{A}$ and the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned}\zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij},\\ \zeta^\alpha_{i_1\ldots i_s} &= D_{i_1}\ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{ji_1\ldots i_s}, \qquad s > 1.\end{aligned} \tag{7}$$

In (7), W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{8}$$

A Lie symmetry generator of (1) is a one-parameter Lie group transformation that leaves the given differential equation invariant under the transformation of all independent and dependent variables. In this paper, we shall assume that X is a Lie point operator, i.e., ξ and η are functions of x and u and are independent of derivatives of u. A generalized operator of the form $\tilde{X} = \eta^\alpha\partial/\partial u^\alpha + \cdots$ is called a canonical or evolutionary representation of X.

DEFINITION 3

If we include point-dependent gauge terms $f_1,\ldots,f_n$, the Noether symmetries X are given by

$$X(L) + LD_i(\xi^i) = D_i(f_i). \tag{9}$$

DEFINITION 4

The Noether operator associated with a Lie–Bäcklund operator X is given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1}\cdots D_{i_s}(W^\alpha)\frac{\delta}{\delta u^\alpha_{i i_1\cdots i_s}}, \qquad i = 1,\dots,n, \quad (10)$$

where the Euler–Lagrange operators with respect to derivatives of u^α are obtained from (5) by replacing u^α by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1}\cdots D_{j_s}\frac{\partial}{\partial u^\alpha_{i j_1\cdots j_s}},$$
$$i = 1,\dots,n, \qquad \alpha = 1,\dots,m. \quad (11)$$

Noether's Theorem. *For any Noether symmetry X corresponding to a given Lagrangian $L \in \mathcal{A}$, there exists a conserved current $\Phi^i = (\Phi^1,\dots,\Phi^n)$, $\Phi^i \in \mathcal{A}$, defined by*

$$\Phi^i = f_i - N^i(L), \qquad i = 1,\dots,n, \quad (12)$$

which is a conserved current of the Euler–Lagrange equations $(\delta L/\delta u^\alpha) = 0$.

2. Lie symmetries of Gordon-type equations in Milne space-time

Consider the Milne metric [6]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^2(\mathrm{d}x^2 + \mathrm{e}^{2x}(\mathrm{d}y^2 + \mathrm{d}z^2)) \quad (13)$$

which represents an empty Universe and is of interest in relativity for being a special case of a well-known Friedmann–Lemaître–Robertson–Walker metric [6,8]. The Klein–Gordon equation [8] on (13) is obtained by

$$\Box u = \frac{1}{\sqrt{|-g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|-g|}g^{ij}\frac{\partial}{\partial x^i}u\right) = k(u), \quad (14)$$

and takes the form

$$u_{xx} - t^2 u_{tt} + \mathrm{e}^{-2x}u_{yy} + \mathrm{e}^{-2x}u_{zz} - 3tu_t + 2u_x - t^2 k(u) = 0. \quad (15)$$

In order to find the Lie point symmetries of the above Gordon-type equation we restrict $k(u)$ to some special cases. These cases are assumed by keeping in mind the fact that we allow the inhomogeneous term, $k(u)$, to be taken as $\sin(u)$ and some powers of u. The criterion that yields the Lie point is given by the invariance condition [2]

$$X[u_{xx} - t^2u_{tt} + \mathrm{e}^{-2x}u_{yy} + \mathrm{e}^{-2x}u_{zz} - 3tu_t + 2u_x - t^2k(u)]|_{\mathrm{eq.}(15)=0} = 0, \quad (16)$$

where X is the prolonged symmetry generator in the jet space. Thus, the invariance of differential equations (15) leads to the Lie point symmetries possessed by (15). The procedure for finding Lie point symmetries is well known [2] and therefore will be given without derivations. It turns out, from the symmetry study, that some special polynomial cases of $k(u)$ arise. Also, in line with the literature, we consider the sine-Gordon equation.

Thus, we study the four cases for $k(u)$ in (15) given by

(i) $k(u) = \sin(u)$ (sine-Gordon)
(ii) $k(u) = u$ (Klein–Gordon)
(iii) $k(u) = u^3$
(iv) $k(u) = u^n$, $n \neq 0, 1, 3$.

Case i. Following the symmetry criterion, we find that eq. (16) in this case admits ten Lie point symmetries given by

$$
\begin{aligned}
X_1 &= \frac{\mathrm{e}^x}{t}\partial_x - \mathrm{e}^x\partial_t,\\
X_2 &= \partial_y,\\
X_3 &= \frac{\mathrm{e}^{-x}}{t}\partial_y - \mathrm{e}^x y\partial_t + \frac{\mathrm{e}^x y}{t}\partial_x,\\
X_4 &= \partial_z,\\
X_5 &= \frac{\mathrm{e}^{-x}}{t}\partial_z - \mathrm{e}^x z\partial_t + \frac{\mathrm{e}^x z}{t}\partial_x,\\
X_6 &= y\partial_z - z\partial_y,\\
X_7 &= -\partial_x + y\partial_y + z\partial_z,\\
X_8 &= \frac{2\mathrm{e}^{-x}y}{t}\partial_y + \frac{2\mathrm{e}^{-x}z}{t}\partial_z + \mathrm{e}^{-x}(-1 - \mathrm{e}^{2x}(y^2+z^2))\partial_t\\
&\quad + \frac{\mathrm{e}^{-x}(-1 + \mathrm{e}^{2x}(y^2+z^2))}{t}\partial_x,\\
X_9 &= 2y\partial_x - 2yz\partial_z + (\mathrm{e}^{-2x} - y^2 + z^2)\partial_y,\\
X_{10} &= -2yz\partial_y + 2z\partial_x + (\mathrm{e}^{-2x} + y^2 - z^2)\partial_z.
\end{aligned}
$$

Case ii. When $k(u) = u$, we have a Klein–Gordon equation. Equation (16) in this case admits 13 Lie point symmetries given by

$$
\begin{aligned}
X_1 &= u\partial_u,\\
X_2 &= \mathcal{F}_1(x,y,z,t)\partial_u,\\
X_3 &= \frac{\mathrm{e}^x}{t}\partial_x - \mathrm{e}^x\partial_t,\\
X_4 &= \partial_y,\\
X_5 &= \frac{\mathrm{e}^{-x}}{t}\partial_y - \mathrm{e}^x y\partial_t + \frac{\mathrm{e}^x y}{t}\partial_x,
\end{aligned}
$$

$$
\begin{aligned}
X_6 &= y\partial_z - z\partial_y,\\
X_7 &= -\partial_x + y\partial_y + z\partial_z,\\
X_8 &= \partial_z,\\
X_9 &= \frac{e^{-x}}{t}\partial_z - e^x z\partial_t + \frac{e^x z}{t}\partial_x,\\
X_{10} &= -2\partial_x + 2y\partial_y + 2z\partial_z + u\partial_u,\\
X_{11} &= \frac{2e^{-x}y}{t}\partial_y + \frac{2e^{-x}z}{t}\partial_z + e^{-x}(-1 - e^{2x}(y^2+z^2))\partial_t\\
&\quad + \frac{e^{-x}(-1 + e^{2x}(y^2+z^2))}{t}\partial_x,\\
X_{12} &= 2y\partial_x - 2yz\partial_z + (e^{-2x} - y^2 + z^2)\partial_y,\\
X_{13} &= -2yz\partial_y + 2z\partial_x + (e^{-2x} + y^2 - z^2)\partial_z,
\end{aligned}
$$

where

$$
\begin{aligned}
&e^{2x}t^2\mathcal{F}_1(t,x,y,z) - \mathcal{F}_{1,zz} - \mathcal{F}_{1,yy} - 2e^{2x}\mathcal{F}_{1,x} - e^{2x}\mathcal{F}_{1,xx}\\
&\quad + 3e^{2x}t\mathcal{F}_{1,t} + e^{2x}t^2\mathcal{F}_{1,tt} = 0.
\end{aligned}
$$

Case iii. When $k(u) = u^3$, the (Gordon-type) eq. (16) yields a set of 15 Lie point symmetries:

$$
\begin{aligned}
X_1 &= t\partial_t - u\partial_u,\\
X_2 &= -2e^x tu\partial_u + e^x(1+t^2)\partial_t + \frac{e^x(-1+t^2)}{t}\partial_x,\\
X_3 &= -2e^x tu\partial_u + e^x(-1+t^2)\partial_t + \frac{e^x(1+t^2)}{t}\partial_x,\\
X_4 &= \partial_y,\\
X_5 &= -2e^x tyu\partial_u + \frac{e^{-x}(-1+t^2)}{t}\partial_y + e^x(1+t^2)y\partial_t + \frac{e^x(-1+t^2)y}{t}\partial_x,\\
X_6 &= y\partial_z - z\partial_y,\\
X_7 &= -\partial_x + y\partial_y + z\partial_z,\\
X_8 &= -2e^x tyu\partial_u + \frac{e^{-x}(1+t^2)}{t}\partial_y + e^x(-1+t^2)y\partial_t + \frac{e^x(1+t^2)y}{t}\partial_x,\\
X_9 &= \partial_z,\\
X_{10} &= 2e^x tzu\partial_u - \frac{e^{-x}(1+t^2)}{t}\partial_z - e^x(-1+t^2)z\partial_t - \frac{e^x(1+t^2)z}{t}\partial_x,
\end{aligned}
$$

$$X_{11} = -2e^x tzu\partial_u + \frac{e^{-x}(-1+t^2)}{t}\partial_z + e^x(1+t^2)z\partial_t + \frac{e^x(-1+t^2)z}{t}\partial_x,$$

$$X_{12} = -2y\partial_x + 2yz\partial_z + (-e^{-2x} + y^2 - z^2)\partial_y,$$

$$\begin{aligned} X_{13} = {} & \frac{2e^{-x}(-1+t^2)y}{t}\partial_y + \frac{2e^{-x}(-1+t^2)z}{t}\partial_z \\ & - 2e^{-x}tu(1+e^{2x}(y^2+z^2))\partial_u + e^{-x}(1+t^2)(1+e^{2x}(y^2+z^2))\partial_t \\ & + \frac{e^{-x}(-1+t^2)(-1+e^{2x}(y^2+z^2))}{t}\partial_x, \end{aligned}$$

$$\begin{aligned} X_{14} = {} & \frac{2e^{-x}(1+t^2)y}{t}\partial_y + \frac{2e^{-x}(1+t^2)z}{t}\partial_z - 2e^{-x}tu(1+e^{2x}(y^2+z^2))\partial_u \\ & + e^{-x}(-1+t^2)(1+e^{2x}(y^2+z^2))\partial_t \\ & + \frac{e^{-x}(1+t^2)(-1+e^{2x}(y^2+z^2))}{t}\partial_x, \end{aligned}$$

$$X_{15} = 2yz\partial_y - 2z\partial_x + (-e^{-2x} - y^2 + z^2)\partial_z.$$

Case iv. In this case $k(u) = u^n$, $n \neq 0, 1, 3$, the general polynomial Gordon-type equation (16) admits 11 Lie point symmetries given by

$$X_1 = \frac{2u}{-3+n}\partial_u + \frac{t-nt}{-3+n}\partial_t,$$

$$X_2 = \frac{e^x}{t}\partial_x - e^x\partial_t,$$

$$X_3 = \partial_y,$$

$$X_4 = \frac{e^{-x}}{t}\partial_y - e^x y\partial_t + \frac{e^x y}{t}\partial_x,$$

$$X_5 = \partial_z,$$

$$X_6 = y\partial_z - z\partial_y,$$

$$X_7 = -\partial_x + y\partial_y + z\partial_z,$$

$$X_8 = \frac{e^{-x}}{t}\partial_z - e^x z\partial_t + \frac{e^x z}{t}\partial_x,$$

$$\begin{aligned} X_9 = {} & \frac{2e^{-x}y}{t}\partial_y + \frac{2e^{-x}z}{t}\partial_z + e^{-x}(-1-e^{2x}(y^2+z^2))\partial_t \\ & + \frac{e^{-x}(-1+e^{2x}(y^2+z^2))}{t}\partial_x, \end{aligned}$$

$$X_{10} = -2y\partial_x + 2yz\partial_z + (-e^{-2x} + y^2 - z^2)\partial_y,$$

$$X_{11} = 2yz\partial_y - 2z\partial_x + (-e^{-2x} - y^2 + z^2)\partial_z.$$

2.1 *Reduction of order of eq. (15)* – an illustration

In this section we briefly show how the order of (1+3) Klein–Gordon equation (15) can be reduced using its symmetries. In the first reduction, the equation with four independent variables is reduced to a partial DE that has two independent variables. The reduced equation may then be analysed further using another Lie symmetry reduction or an appropriate alternative method. In the second reduction (§2.1.1), we obtain exact solutions.

Since $[X_6, X_7] = 0$, where X_6 and X_7 appear as Lie symmetries in all the above cases, we may begin reducing with either X_6 or X_7. Suppose we reduce (15) by $X_6 = y\partial_z - z\partial_y$. The characteristic equations are

$$\frac{\mathrm{d}x}{0} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}y}{-z} = \frac{\mathrm{d}z}{y} = \frac{\mathrm{d}u}{0}.$$

Integrating yields $\alpha = y^2 + z^2$ and eq. (15) is reduced to

$$\frac{1}{t^2}u_{xx} - u_{tt} + \frac{2}{t^2}\mathrm{e}^{-2x}(\alpha u_{\alpha\alpha} + u_\alpha) - \frac{3}{t}u_t + \frac{2}{t^2}u_x - k(u) = 0 \tag{17}$$

with $u = u(x, t, \alpha)$.

If we then reduce eq. (17) by $X_7 = -\partial_x + y\partial_y + z\partial_z$, we obtain the scaling transformation $\bar{X} = -\partial_x + 2\alpha\partial_\alpha$. We now have the characteristic equations,

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{-1} = \frac{\mathrm{d}\alpha}{2\alpha} = \frac{\mathrm{d}u}{0}.$$

By integrating, we obtain $\beta = \ln\alpha + 2x$ and eq. (17) reduces to

$$\frac{2}{t^2}u_{\beta\beta}(2 + \mathrm{e}^{-\beta}) - u_{tt} + \frac{4}{t^2}u_\beta - \frac{3}{t}u_t - k(u) = 0 \tag{18}$$

with $u = u(t, \beta)$.

Equation (18) may be further analysed or reduced using the underlying symmetries. It turns out that the Lie point symmetries are cumbersome and involve special functions such as Bessel functions for $k = u$.

For $k = u^3$, eq. (18) admits one symmetry

$$t\partial_t - u\partial_u.$$

For $k = u^4$, the symmetries of (18) are

$$\begin{aligned} X^* &= \frac{3}{4}t\partial_t - \frac{1}{2}u\partial_u, \\ X^{**} &= 4t^2(1 + 2\mathrm{e}^\beta)\partial_\beta + t^3(1 + 4\mathrm{e}^\beta)\partial_t - 2t^2u(1 + 4\mathrm{e}^\beta)\partial_u. \end{aligned}$$

Using X^{**}, eq. (18) reduces to the ordinary differential equation (ODE)

$$\gamma^4 F_{\gamma\gamma} + \gamma F_\gamma + 4F - 8\gamma^2 F^4 = 0, \tag{19}$$

where

$$\gamma = \frac{t\mathrm{e}^{-\beta/4}}{(1+2\mathrm{e}^{\beta})^{1/4}} \qquad \text{and} \qquad F = u\mathrm{e}^{\beta/2}(1+2\mathrm{e}^{\beta})^{1/2}.$$

It turns out that eq. (19) admits the Lie point symmetry

$$G = -\frac{3}{2}\beta\partial_{\beta} + F\partial_F$$

which leads us to the first-order ODE

$$q_p = \frac{2q + 24q^{2/3}p^{4/3} - 12q^{2/3}p^{1/3}}{2p + 3q^{1/3}p^{2/3}}, \tag{20}$$

where $p = \gamma^2 F^3$ and $q = \gamma^5 F'^3$.

There are no symmetries for $k = u^n$, $n \neq 0, 1, 3, 4$ and $k = \sin u$ in eq. (18).

Also, one may consider reduction by studying the underlying conservation laws. This would require methods other than the variational one, i.e., Noether's theorem, since eq. (18) is not variational.

For the Klein–Gordon case $k(u) = u$ in (18), it can be shown, for e.g., that a conserved vector of (18) is (Φ^{β}, Φ^{t}), where

$$\begin{aligned}
\Phi^{\beta} &= 2(1+2\mathrm{e}^{\beta})\ \mathrm{BesselJ}(1,t)u_{\beta},\\
\Phi^{t} &= \frac{1}{2}\mathrm{e}^{\beta}t[(t\ \mathrm{BesselJ}(0,t) - 2\ \mathrm{BesselJ}(1,t)\\
&\quad - t\ \mathrm{BesselJ}(2,t))u - 2t\ \mathrm{BesselJ}(1,t)u_t],
\end{aligned}$$

such that $D_{\beta}\Phi^{\beta} + D_t\Phi^{t} = 0$ along the solutions of (18) and where BesselJ is the Bessel function of the first kind ((Φ^{β} is the conserved flow and Φ^{t}, the conserved density).

For $k(u) = u^3$ in (18), the components of the conserved vector are:

$$\begin{aligned}
\Phi^{\beta} &= (1+2\mathrm{e}^{\beta})t^2[u_t u_{\beta} - u u_{\beta t}],\\
\Phi^{t} &= -\frac{1}{4}t^2[\mathrm{e}^{\beta}t^2u^4 + 2\mathrm{e}^{\beta}t^2{u_t}^2 + 4u(\mathrm{e}^{\beta}tu_t - 2\mathrm{e}^{\beta}u_{\beta} - (1+2\mathrm{e}^{\beta})u_{\beta\beta})].
\end{aligned}$$

Similarly, for $k(u) = u^4$ in (18), the components of the conserved vector are

$$\begin{aligned}
\Phi^{\beta} &= -\frac{1}{5}(1+2\mathrm{e}^{\beta})t^3(40\mathrm{e}^{\beta}u^2 + 4\mathrm{e}^{\beta}t^2u^5 - 5u_{\beta}((1+4\mathrm{e}^{\beta})tu_t\\
&\quad + 4(1+2\mathrm{e}^{\beta})u_{\beta}) + 5tu(10\mathrm{e}^{\beta}u_t + 2\mathrm{e}^{\beta}tu_{tt} + (1+4\mathrm{e}^{\beta})u_{\beta t})),\\
\Phi^{t} &= -t^4(-2\mathrm{e}^{\beta}(1+4\mathrm{e}^{\beta})u^2 + \frac{1}{5}\mathrm{e}^{\beta}(1+4\mathrm{e}^{\beta})t^2u^5 + \frac{1}{2}\mathrm{e}^{\beta}tu_t((1+4\mathrm{e}^{\beta})tu_t\\
&\quad + 4(1+2\mathrm{e}^{\beta})u_{\beta}) - u(2\mathrm{e}^{\beta}(3+8\mathrm{e}^{\beta})u_{\beta}\\
&\quad + (1+2\mathrm{e}^{\beta})(2\mathrm{e}^{\beta}tu_{\beta t} + (1+4\mathrm{e}^{\beta})u_{\beta\beta}))).
\end{aligned}$$

The cases $k(u) = \sin u$ and $k(u) = u^n$, $n \neq 1, 3, 4$ in eq. (18) do not yield any conserved vectors.

2.1.1 *Exact solutions/boundary conditions.* We reduce eq. (15) by first using the symmetry $X = \partial_y$ to obtain

$$\frac{1}{t^2}u_{xx} - u_{tt} + \frac{1}{t^2}e^{-2x}u_{zz} - \frac{3}{t}u_t + \frac{2}{t^2}u_x - k(u) = 0,$$
$$u = u(x, z, t), \quad u(x, 0, t) = 0, \quad u(x, 1, t) = 1. \tag{21}$$

Reduce (21) further using the symmetry $X = (e^x/t)\partial_x - e^x\partial_t$. The characteristic equations are

$$\frac{t\,dx}{e^x} = \frac{dt}{-e^x} = \frac{dz}{0} = \frac{du}{0}.$$

Integrating yields $\tilde{t} = (1/t)e^{-x}$ and eq. (21) is reduced to

$$\frac{1}{\tilde{t}^2}u_{zz} - k(u) = 0, \tag{22}$$

where $u(z, \tilde{t})$ and boundary conditions transform to

$$u(0, \tilde{t}) = 0, \quad u(1, \tilde{t}) = 1.$$

If $k(u) = u$, as in the Klein–Gordon case, then the general solution to eq. (22) is

$$u(z, \tilde{t}) = e^{\tilde{t}z}C_1(\tilde{t}) + e^{-\tilde{t}z}C_2(\tilde{t}). \tag{23}$$

We plot the function $\tilde{t}$ over different ranges: $\{x, -50, 50\}, \{t, 1, 10\}$ (figure 1) and $\{x, -10, 10\}, \{t, 1, 10\}$ (figure 2). When $t \to \pm\infty$, $\tilde{t} \to 0$, and when $t \to 0$, $\tilde{t}$ becomes large.

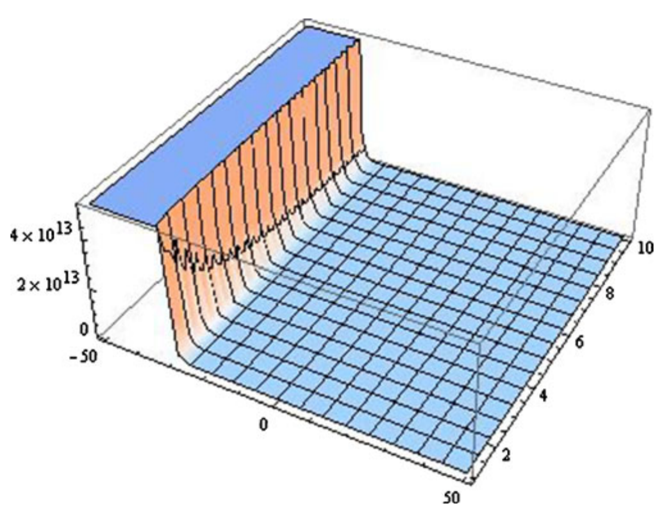

Figure 1. $\tilde{t} = \frac{1}{t}e^{-x}$.

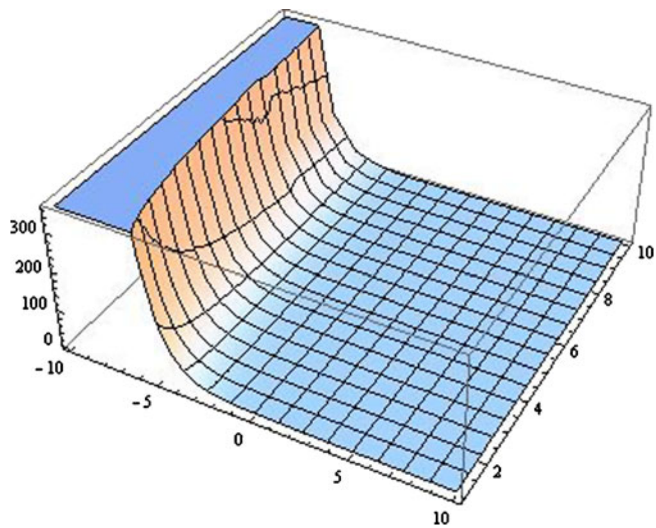

Figure 2. $\tilde{t} = \frac{1}{t}e^{-x}$.

Keeping this in mind, we plot $u(z, \tilde{t})$ from (23) in figure 3: $\{\tilde{t}, 0, 10\}$, $\{z, -5, 5\}$ and figure 4: $\{\tilde{t}, 0, 10\}$, $\{z, -5, 5\}$, choosing particular C_1s and C_2s.

If we impose the boundary conditions on eq. (23), the solution of $u(z, \tilde{t})$ may be expressed in terms of the hyperbolic sine function, namely,

$$u(z, \tilde{t}) = \frac{1}{2\sinh(\tilde{t})}e^{\tilde{t}z} - \frac{1}{2\sinh(\tilde{t})}e^{-\tilde{t}z}.$$

The graph of this solution, for the range $\{\tilde{t}, 0, 10\}$, $\{z, -5, 5\}$, is given in figure 5.

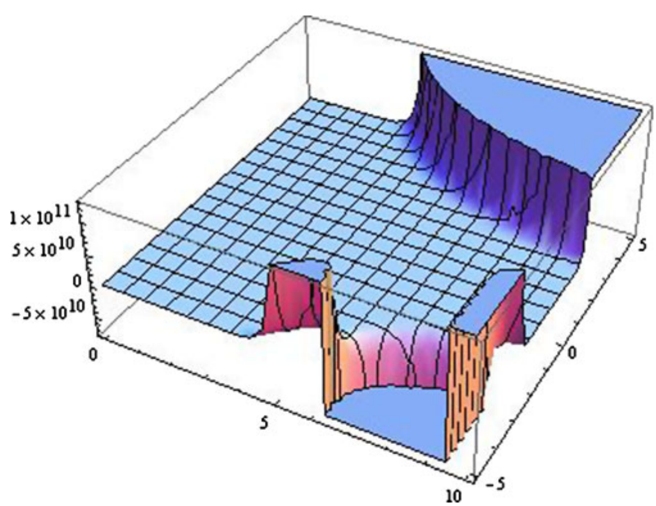

Figure 3. $C_1 = 100\tilde{t}^2$, $C_2 = -2\sin(\tilde{t})$.

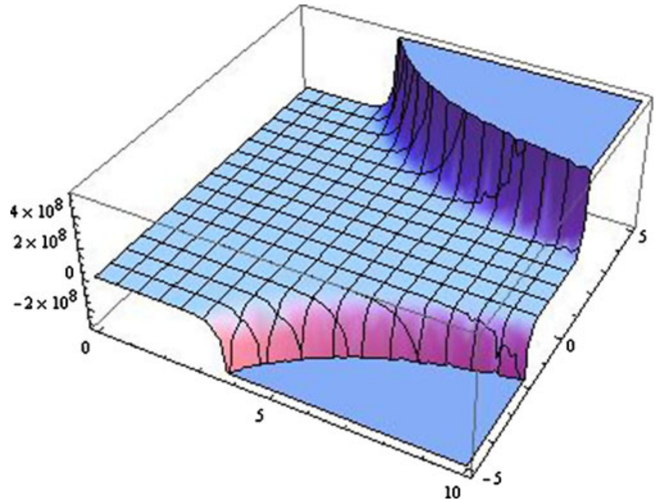

Figure 4. $C_1 = 100,\ C_2 = -2$.

Remark. For the sine-Gordon case, with $k(u) = \sin(u)$ in eq. (22), we obtain

$$u(z, \tilde{t}) = \pm 2 \text{ Jacobi Amplitude} \times \left[\frac{1}{2}\sqrt{-(2\tilde{t}^2 - D_1(\tilde{t}))(z + D_2(\tilde{t}))^2}, -\frac{4\tilde{t}^2}{-2\tilde{t}^2 + D_1(\tilde{t})}\right],$$

where Jacobi amplitude $[v, m]$ refers to the amplitude $am(v \mid m)$ for Jacobi elliptic functions.

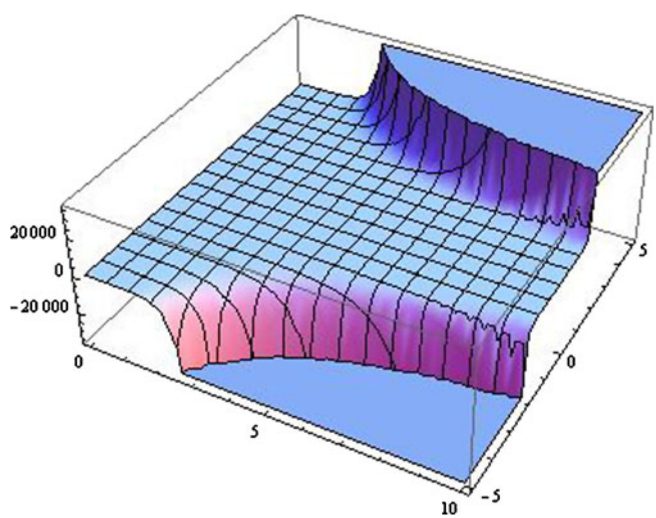

Figure 5. $u(z, \tilde{t}) = (1/2 \sinh(\tilde{t}))e^{\tilde{t}z} - (1/2 \sinh(\tilde{t}))e^{-\tilde{t}z}$.

3. Noether symmetries of Klein–Gordon equation for the Milne space-time

Consider the wave eq. (15) with $k(u) = u$ (Klein–Gordon), which has the Lagrangian,

$$L = \frac{1}{2}t^3 e^{2x} u^2 + \frac{1}{2}t e^{2x} u_x^2 + \frac{1}{2}t u_y^2 + \frac{1}{2}t u_z^2 - \frac{1}{2}t^3 e^{2x} u_t^2. \tag{24}$$

We assume that

$$\begin{aligned} X &= \xi(t,x,y,z,u)\partial_x + \tau(t,x,y,z,u)\partial_t + \eta(t,x,y,z,u)\partial_y \\ &\quad + \gamma(t,x,y,z,u)\partial_z + \phi(t,x,y,z,u)\partial_u \end{aligned}$$

is a Noether point operator that satisfies (9) with gauge vector f_i $(i = 1,2,3,4)$, dependent on (t,x,y,z,u). Then the Noether symmetry criterion (9) for the Lagrangian given by eq. (24) takes the form,

$$XL + L[D_t\tau + D_x\xi + D_y\eta + D_z\gamma] = D_t f_1 + D_x f_2 + D_y f_3 + D_z f_4.$$

Separation by derivatives of u yields the following overdetermined system:

$$\begin{aligned}
u_t^3 &: \tau_u = 0, \\
u_x^3 &: \xi_u = 0, \\
u_y^3 &: \eta_u = 0, \\
u_z^3 &: \gamma_u = 0, \\
u_t^2 &: -e^{2x}t^3\xi - \frac{3}{2}e^{2x}t^2\tau - \frac{1}{2}e^{2x}t^3\gamma_z - \frac{1}{2}e^{2x}t^3\eta_y - \frac{1}{2}e^{2x}t^3\xi_x \\
&\quad + \frac{1}{2}e^{2x}t^3\tau_t - e^{2x}t^3\phi_u = 0, \\
u_x^2 &: e^{2x}t\xi + \frac{1}{2}e^{2x}\tau + \frac{1}{2}e^{2x}t\gamma_z + \frac{1}{2}e^{2x}t\eta_y - \frac{1}{2}e^{2x}t\xi_x + \frac{1}{2}e^{2x}t\tau_t \\
&\quad + e^{2x}t\phi_u = 0, \\
u_y^2 &: \frac{1}{2}\tau + \frac{1}{2}t\gamma_z - \frac{1}{2}t\eta_y + \frac{1}{2}t\xi_x + \frac{1}{2}t\tau_t + t\phi_u = 0, \\
u_z^2 &: \frac{1}{2}\tau - \frac{1}{2}t\gamma_z + \frac{1}{2}t\eta_y + \frac{1}{2}t\xi_x + \frac{1}{2}t\tau_t + t\phi_u = 0, \\
u_t u_z &: e^{2x}t^3\gamma_t - t\tau_z = 0, \\
u_x u_z &: -e^{2x}t\gamma_x - t\xi_z = 0, \\
u_y u_z &: -t\gamma_y - t\eta_z = 0, \\
u_t u_y &: e^{2x}t^3\eta_t - t\tau_y = 0, \\
u_y u_x &: -e^{2x}t\eta_x - t\xi_y = 0,
\end{aligned}$$

$$\begin{aligned}
u_t u_x &: \mathrm{e}^{2x} t^3 \xi_t - \mathrm{e}^{2x} t \tau_x = 0,\\
u_t &: -f_{1,u} - \mathrm{e}^{2x} t^3 \phi_t = 0,\\
u_x &: -f_{2,u} + \mathrm{e}^{2x} t \phi_x = 0,\\
u_y &: -f_{3,u} + t \phi_y = 0,\\
u_z &: -f_{4,u} + t \phi_z = 0,\\
1 &: \mathrm{e}^{2x} t^3 u^2 \xi + \frac{3}{2} \mathrm{e}^{2x} t^2 u^2 \tau + \mathrm{e}^{2x} t^3 u \phi + \frac{1}{2} t^3 \mathrm{e}^{2x} u^2 \left(\tau_t + \xi_x + \eta_y + \gamma_z\right)\\
&\quad - \left(f_{1,t} + f_{2,x} + f_{3,y} + f_{4,z}\right) = 0. \qquad (25)
\end{aligned}$$

Solving the above system, one finds that the Noether point symmetries of the Klein–Gordon equation for the Milne metric are given by

$$\begin{aligned}
X_1 &= \mathrm{e}^{2x} t^3 \left(\frac{\mathrm{e}^x}{t}\partial_x - \mathrm{e}^x \partial_t\right), \quad f_i = 0,\\
X_2 &= \mathrm{e}^{2x} t^3 \partial_y, \quad f_i = 0,\\
X_3 &= \mathrm{e}^{2x} t^3 \left(\frac{\mathrm{e}^{-x}}{t}\partial_y - \mathrm{e}^x y \partial_t + \frac{\mathrm{e}^x y}{t}\partial_x\right), \qquad f_i = 0,\\
X_4 &= \mathrm{e}^{2x} t^3 \partial_z, \quad f_i = 0,\\
X_5 &= \mathrm{e}^{2x} t^3 \left(\frac{\mathrm{e}^{-x}}{t}\partial_z - \mathrm{e}^x z \partial_t + \frac{\mathrm{e}^x z}{t}\partial_x\right), \qquad f_i = 0,\\
X_6 &= \mathrm{e}^{2x} t^3 (y\partial_z - z\partial_y), \qquad f_i = 0,\\
X_7 &= \mathrm{e}^{2x} t^3 (-\partial_x + y\partial_y + z\partial_z), \quad f_i = 0,\\
X_8 &= \mathrm{e}^{2x} t^3 \Bigg(\frac{2\mathrm{e}^{-x} y}{t}\partial_y + \frac{2\mathrm{e}^{-x} z}{t}\partial_z + \mathrm{e}^{-x}(-1 - \mathrm{e}^{2x}(y^2 + z^2))\partial_t\\
&\qquad + \frac{\mathrm{e}^{-x}(-1 + \mathrm{e}^{2x}(y^2 + z^2))}{t}\partial_x\Bigg), \qquad f_i = 0,\\
X_9 &= \mathrm{e}^{2x} t^3 (2y\partial_x - 2yz\partial_z + (\mathrm{e}^{-2x} - y^2 + z^2)\partial_y), \qquad f_i = 0,\\
X_{10} &= \mathrm{e}^{2x} t^3 (-2yz\partial_y + 2z\partial_x + (\mathrm{e}^{-2x} + y^2 - z^2)\partial_z), \qquad f_i = 0.
\end{aligned}$$

We may then obtain the corresponding conserved flows for each X_i ($i = 1, \ldots, 10$). For example, the symmetry X_6 yields the conserved flow,

$$\begin{aligned}
\Phi_6^t &= -\frac{1}{2}\mathrm{e}^{2x} t^3 (y u_z u_t - z u_y u_t + u(-y u_{tz} + z u_{ty}))\\
\Phi_6^x &= \frac{1}{2}\mathrm{e}^{2x} t (y u_z u_x - z u_y u_x + u(-y u_{xz} + z u_{xy}))\\
\Phi_6^y &= \frac{1}{2} t (\mathrm{e}^{2x} t^2 z u^2 + u_y (y u_z - z u_y) - u(u_z + z u_{zz} + y u_{yz} - 3\mathrm{e}^{2x} t z u_t\\
&\quad - \mathrm{e}^{2x} t^2 z u_{tt} + 2\mathrm{e}^{2x} z u_x + \mathrm{e}^{2x} z u_{xx}))\\
\Phi_6^z &= -\frac{1}{2} t (\mathrm{e}^{2x} t^2 y u^2 + u_z(-y u_z + z u_y) - u(u_y + z u_{yz}\\
&\quad + y(u_{yy} + \mathrm{e}^{2x}(-3t u_t - t^2 u_{tt} + 2u_x + u_{xx})))).
\end{aligned}$$

4. Klein–Gordon equations and higher-order variational symmetries and conservation laws in Milne space-time

Consider the Klein–Gordon wave equation in Milne space-time with dependent variable u as a function of x, t and y only, i.e., we have removed the spatial variable z from the original wave equation (15) – the calculations are extremely cumbersome producing no final outcomes. We consider the multiplier method for eq. (15), by choosing $k(u) = u$. That is, since the Euler–Lagrange operator annihilates total divergences, we get

$$\frac{\delta}{\delta u}\left[\mathcal{Q}\left(\frac{1}{t^2}u_{xx} - u_{tt} + \frac{1}{t^2}\mathrm{e}^{-2x}u_{yy} - \frac{3}{t}u_t + \frac{2}{t^2}u_x - u\right)\right] = 0, \tag{26}$$

where $\mathcal{Q} = \mathcal{Q}(x, y, t, u_x, u_x, u_{xx}, u_{xy}, u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy})$. Although not pursued here, the calculations may include derivatives of u with respect to t. Then

$$\begin{aligned}
&\mathcal{Q}\left[\left(\frac{1}{t^2}u_{xx} - u_{tt} + \frac{1}{t^2}\mathrm{e}^{-2x}u_{yy} - \frac{3}{t}u_t + \frac{2}{t^2}u_x - u\right)\right] \\
&\quad = D_t\Phi^t + D_x\Phi^x + D_y\Phi^y,
\end{aligned}$$

where (Φ^x, Φ^y, Φ^t) is the conserved flow (Φ^t being the conserved density). We obtain the set of multipliers $\mathcal{Q}_i$, namely,

$$\begin{aligned}
\mathcal{Q}_1 &= t^3\mathrm{e}^{2x}\left(-\frac{1}{3}u_{xxx} - \frac{1}{6}yu_y + yu_{xxy} + \frac{1}{2}y^2u_{yy}\right. \\
&\qquad \left. + \frac{1}{3}y^3u_{yyy} - y^2u_{xyy} - \frac{1}{2}u_{xx} + \frac{1}{12}u\right), \\
\mathcal{Q}_2 &= t^3\mathrm{e}^{2x}(-2yu_{xyy} + u_{xxy} + yu_{yy} + y^2u_{yyy}), \\
\mathcal{Q}_3 &= \frac{1}{2}t^3\mathrm{e}^{-2x}u_{yyy} + t^3u_{xxy} \\
&\qquad + t^3\mathrm{e}^{2x}\left(2y^3u_{xyy} - y^3u_{yy} - \frac{1}{2}y(u) - 3y^2u_{xxy}\right. \\
&\qquad \left. + \frac{1}{2}y^2u_y + 3yu_{xx} + 2yu_{xxx} - \frac{1}{2}y^4u_{yyy}\right), \\
\mathcal{Q}_4 &= \frac{1}{2}t^3\mathrm{e}^{2x}(2u_{xyy} - u_{yy} - 2yu_{yyy}), \\
\mathcal{Q}_5 &= t^3\mathrm{e}^{2x}u_{yyy}, \\
\mathcal{Q}_7 &= t^3\mathrm{e}^{2x}\left(u_x + \frac{1}{2}u - yu_y\right), \\
\mathcal{Q}_8 &= t^3e^{2x}u_y.
\end{aligned}$$

These multipliers yield a set of eight conserved flows, for example, the corresponding components of the conserved vector for $\mathcal{Q}_4$ are

$$\begin{aligned}
\Phi_4^x &= \frac{1}{12}t(2e^{2x}t^2{u_y}^2 + 2{u_{yy}}^2 - 6e^{2x}tu_{yy}u_t - 2e^{2x}t^2u_{yy}u_{tt}\\
&\quad + e^{2x}u_{yy}u_x - 6e^{2x}yu_{yyy}u_x + 6e^{2x}u_xu_{xyy} + 2e^{2x}u_{yy}u_{xx}\\
&\quad - 2u_y(u_{yyy} + e^{2x}(-3tu_{ty} - t^2u_{tty} + 2u_{xy} + u_{xxy}))\\
&\quad + u(-4e^{2x}t^2u_{yy} + 2u_{yyyy} + e^{2x}(-6tu_{tyy} - 2t^2u_{ttyy}\\
&\quad + 7u_{xyy} + 6yu_{xyyy} - 4u_{xxyy}))),\\
\Phi_4^t &= \frac{1}{4}e^{2x}t^3(u_{yy}u_t + 2yu_{yyy}u_t - uu_{tyy} - 2yuu_{tyyy} - 2u_tu_{xyy} + 2uu_{xtyy}),\\
\Phi_4^y &= \frac{1}{12}t(-6e^{2x}t^2y{u_y}^2 - 2(3y{u_{yy}}^2 + u_{yyy}u_x\\
&\quad + u_{yy}(-9e^{2x}tyu_t - 3e^{2x}t^2yu_{tt} + 6e^{2x}yu_x - 2u_{xy} + 3e^{2x}yu_{xx})\\
&\quad + e^{2x}(-3tu_{ty}u_x - t^2u_{tty}u_x + 6tu_tu_{xy} + 2t^2u_{tt}u_{xy}\\
&\quad - 2u_xu_{xy} - 2u_{xy}u_{xx} + u_xu_{xxy}))\\
&\quad + u_y(3e^{2x}tu_t - 18e^{2x}tyu_{ty} + e^{2x}t^2u_{tt}\\
&\quad - 6e^{2x}t^2yu_{tty} - 2e^{2x}u_x + 4e^{2x}t^2u_x + 12e^{2x}yu_{xy} + 4u_{xyy}\\
&\quad + 6e^{2x}tu_{xt} + 2e^{2x}t^2u_{xtt} - 5e^{2x}u_{xx} + 6e^{2x}yu_{xxy} - 2e^{2x}u_{xxx})\\
&\quad + u(2e^{2x}t^2u_y + 12e^{2x}t^2yu_{yy} + 3e^{2x}tu_{ty}\\
&\quad + 18e^{2x}tyu_{tyy} + e^{2x}t^2u_{tty} + 6e^{2x}t^2yu_{ttyy} - 2e^{2x}u_{xy}\\
&\quad - 8e^{2x}t^2u_{xy} - 12e^{2x}yu_{xyy}\\
&\quad - 2u_{xyyy} - 12e^{2x}tu_{xty} - 4e^{2x}t^2u_{xtty} + 7e^{2x}u_{xxy}\\
&\quad - 6e^{2x}yu_{xxyy} + 4e^{2x}u_{xxxy})).
\end{aligned}$$

5. Concluding remarks

This paper investigates a class of wave and Gordon-type equations in Milne space-time. In particular, we conducted a Lie and Noether symmetry analysis of a Klein–Gordon equation on this manifold. We have given some symmetry reductions to show how the (1+3)-dimensional wave equation can be reduced to an ordinary differential equation using the method of invariants, and obtained some exact solutions. A conserved density of the Klein–Gordon equation is constructed. Finally, some higher-order symmetries for the projected equation and associated conservation laws are presented. It is hoped that an analysis of the nonlinear Klein–Gordon equation in a genuinely curved space-time will provide interesting insight from the point of view of conserved quantities.

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