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Absolute parametric instability in a nonuniform plane plasma waveguide

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Abstract. The paper reports an analysis of the effect of spatial plasma nonuniformity on absolute parametric instability (API) of electrostatic waves in magnetized plane waveguides subjected to an intense high-frequency (HF) electric field using the separation method. In this case the effect of strong static magnetic field is considered. The problem of strong magnetic field is solved in 1D nonuniform plane plasma waveguide. The equation describing the spatial part of the electric potential is obtained. Also, the growth rates and conditions of the parametric instability for periodic and aperiodic cases are obtained. It is found that the spatial nonuniformity of the plasma exerts a stabilizing effect on the API. It is shown that the growth rates of periodic and aperiodic API in nonuniform plasma are less compared to that of uniform plasma.

Keywords. Absolute parametric instability; separation method; anisotropic plane; plasma waveguide; nonuniform plasma.

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1. Introduction

The parametric interaction of an external HF electric field with an electrostatic surface wave in anisotropic nonuniform plasma has been previously investigated using a special method based on the separation of variables [1]. The method makes it possible to separate the problem into two parts. The 'dynamical' (temporal) part describes the parametric excitation of waves and the corresponding equations within the renormalization of natural (eigen) frequencies coincides with equations for parametrically unstable waves in a uniform plasma [2,3]. Natural frequencies of surface waves and spatial distribution of the self-consistent electric field amplitude are determined from the solution of a boundary-value problem ('spatial' part) by taking into account specific spatial distribution of plasma density. The proposed approach ('separation method') is significantly simpler than the method previously used in the theory of parametric resonance in a nonuniform plasma (e.g., ref. [4] and references therein). Therefore, it is of special interest to apply the separation method to solve different problems involving parametric excitation of electrostatic waves in bounded nonuniform plasma.

It is known that (e.g. [5]) the spatial nonuniformity of plasma density may lead to the localization of a parametrically unstable waves in a finite region of a plasma volume. This suggests that instability has assumed an absolute character. From an experimental point of view, it is quite important to know whether a given parametric instability is absolute or convective. This is so essential because the nature of instability determines the mechanism of their saturation. The convective instability reaches saturation at a comparatively low level, due to convection of energy of the unstable waves away from the resonance region. The absolute instability saturates at a higher level of energy under various nonlinear effects. From this point of view, the absolute parametric instability (API) plays a crucial role in the transfer of energy from the electromagnetic radiation to the plasma and may have important consequences for experiments on RF plasma heating in tokamaks and for laser fusion [6–21].

In ref. [1] the problem of parametric excitation of natural modes of semi-infinite plasma (surface waves) was analysed as an initial value problem. In other words, surface waves are excited due to an initial perturbation at the boundary and a dispersion equation determines the complex frequency ω as a function of the real wave number k. It is of practical interest to use the separation method described in [1] for solving eigenvalues problems when the wave number k is found as a function of the real frequency ω. This means that one has to treat forced oscillations in a plasma by an external source (generator) with a fixed frequency ω_g (for more details see e.g. [22], where the initial value problem for the problem of surface wave transformation at the plasma resonance in a transition layer is treated).

Demchenko *et al* [23] have reported the analysis of the effect of spatial plasma nonuniformity on parametric instability of electrostatic waves in magnetized cylindrical waveguides subjected to an intense HF electric field.

A method is expounded in this paper which permits reducing the API excited by a monochromatic pumping field of arbitrary amplitude in nonuniform magnetoactive plasma to the parametric excitation of spatial oscillations in uniform isotropic plasma. Below, we shall discuss the parametric excitation of low-frequency waves whose dispersion is completely determined by a HF field, in a strong magnetic field, i.e., here, we shall apply the method of ref. [1] to investigate the API in a 1D nonuniform plasma waveguide subjected to an intense HF electric field as an eigenvalue problem.

Using the separation method, we investigate the API in bounded nonuniform plasma under the pump field and static magnetic field. Both the pump field $\vec{E}_{\mathrm{P}} = \vec{E}_0 \sin(\omega_0 t)$ and the static magnetic field $\vec{B}_0$ are directed along the z-axis. Assuming the intensity of the magnetic field to be strong enough ($\omega_{c_\alpha} \gg \omega_{\mathrm{P}_\alpha}$), the motion of plasma particles are considered to be confined along z-axis only. We are going to study API in 1D nonuniform bounded plasma.

2. Separation method in API in a 1D nonuniform bounded plasma

Let us suppose that the plane waveguide is filled by nonuniform plasma ($n_{\alpha_0} = n_{\alpha_0}(x); \alpha = e, i$). A uniform strong static magnetic field $\vec{B}_0$ ($\omega_{c_\alpha} \gg \omega_{\mathrm{P}_\alpha}$) and a HF electric field $\vec{E}_{\mathrm{P}} = \vec{E}_0 \sin(\omega_0 t)$ are directed along the z-axis. We choose the electric field of an ordinary wave as an HF pump field. The equilibrium particle velocity $\vec{u}_\alpha(0,\ 0,\ u_\alpha)$ is determined by the following expression:

$$\vec{u}_\alpha = -\frac{e_\alpha \vec{E}_0}{m_\alpha \omega_0} \cos(\omega_0 t). \tag{1}$$

Represent the perturbations of velocity $\delta\vec{V}_\alpha(0,\ 0,\ \delta V_\alpha)$, density and electrical potential Φ in the form $(\delta\vec{V}_\alpha, \delta n_\alpha, \Phi) \sim \exp(ik_z z)$.

For our case the initial system of equations consists of the hydrodynamical equations in conjunction with the Poisson's equation:

$$\frac{\partial \vec{V}_\alpha}{\partial t} + (\vec{V}_\alpha \cdot \nabla)(\vec{V}_\alpha) = \frac{e_\alpha}{m_\alpha}\left(\vec{E}_{\mathrm{P}} + \frac{1}{c}[\vec{V}_\alpha \times \vec{B}_0] - \nabla\Phi\right) \tag{2}$$

$$\frac{\partial n_\alpha}{\partial t} + \mathrm{div}(n_\alpha \vec{V}_\alpha) = 0, \tag{3}$$

$$\Delta\Phi = -4\pi \sum_\alpha e_\alpha n_\alpha, \tag{4}$$

where n_α and $\vec{V}_\alpha$ are the density and velocity of particles of species α and Φ is the potential self-consistent electric field. Suppose that $k_z \equiv k$; $(\partial\delta V_\alpha/\partial t) \sim \omega\delta V_\alpha \sim \omega_{\mathrm{P}_\alpha}\delta V_\alpha, k u_\alpha \sim \omega_{\mathrm{P}_\alpha}$. It means that particles are 'frozen' and cannot move across the magnetic field $\delta V_{\alpha_x} = \delta V_{\alpha_y} = 0$. From linearized eq. (2) we find

$$ikn_{\alpha_0}\frac{\partial\delta V_\alpha}{\partial t} = k^2 \frac{e_\alpha}{m_\alpha} n_{\alpha_0}\Phi, \quad \delta V_{\alpha_z} = \delta V_\alpha. \tag{5}$$

The continuity eq. (3) reduces to

$$-\left[\frac{\partial\delta n_\alpha}{\partial t} + iku_\alpha\delta n_\alpha\right] = ikn_{\alpha_0}\delta V_\alpha. \tag{6}$$

Introducing new variables $\nu_\alpha = e_\alpha\delta n_\alpha \mathrm{e}^{iA_\alpha}$, $A_\alpha = -a_\alpha \sin(\omega_0 t)$, $a_e \gg a_i$, $a_\alpha \equiv \left(e_\alpha k E_0/m_\alpha\omega_0^2\right) \approx a_e$, the set of eqs (5) and (6) can be rewritten as follows:

$$\frac{\partial^2\nu_\alpha}{\partial t^2} = -\frac{e_\alpha^2}{m_\alpha}\mathrm{e}^{-iA_\alpha}\hat{L}_2\Phi, \tag{7}$$

where

$$\hat{L}_2\Phi = -k^2 n_{\alpha_0}(x).$$

The Poisson's equation takes the form

$$\frac{\partial^2\Phi}{\partial x^2} - k^2\Phi = -4\sum_\alpha \nu_\alpha \mathrm{e}^{iA_\alpha(t)}. \tag{8}$$

Assuming $\nu_\alpha(x,t) = \nu_{\alpha_1}(t)\nu_{\alpha_2}(x)$, $\Phi(x,t) = \Phi_1(t)\Phi_2(x)$ and separating variables in eqs (7) and (8), we have

$$\frac{d^2\nu_{\alpha_1}}{\partial t^2} = -\frac{m_e p_\alpha^2}{e^2} e^{iA_\alpha} \sum_{\beta=e,i} \nu_{\beta_1} \frac{e_\beta^2}{m_\beta} e^{-iA_\beta}, \tag{9}$$

$$\frac{\partial^2\Phi_2}{\partial x^2} - k^2\varepsilon(x,p)\Phi_2 = 0, \tag{10}$$

where $\varepsilon(x,p) = 1 - \omega_{\mathrm{P}_e}^2/p^2$ and p is the separation constant. The set of eq. (9) describe 'temporal' (dynamical) part of the problem. Comparing the derived equations with the system describing volumetric oscillations in a uniform plasma [8,9], we find that the presence of plasma nonuniformity results in the renormalization of natural plasma frequencies $\omega_{\mathrm{P}_e}^2 \to p^2$, $\omega_{\mathrm{P}_i}^2 \to (m_e/m_i)p^2$. This fact enables us to use the method developed in [3] to solve the system of equations with periodical coefficients (9). Equation (10) corresponds to the 'spatial' (stationary) part of the problem. If the profile of plasma density and boundary conditions are specified, solution of eq. (10) gives us the needed value of the constant p. The distinguishing feature of eq. (10) is that the amplitude of HF electric field is not part of it.

3. Solution of the spatial eq. (10)

We shall consider API in nonuniform plasma in which the density distribution is determined by the relation $n = n_0(1 - x^2/L^2)$ [24] where L is the characteristic scale of nonuniformity. In this case, eq. (10) takes the form

$$\frac{\partial^2\Phi_2}{\partial x^2} - k^2\Phi_2 + \frac{k^2}{p^2}\left(\frac{4\pi e^2 n_0}{m}\right)\left(1 - \frac{x^2}{L^2}\right)\Phi_2 = 0. \tag{11}$$

Equation (10) yields

$$\frac{\partial^2\Phi_2}{\partial x^2} + (A - Bx^2)\Phi_2 = 0, \tag{12}$$

where

$$A = -k^2\varepsilon_0, \quad B = \frac{\omega_{\mathrm{P}_0}^2 k^2}{p^2 L^2}, \quad \varepsilon_0 = 1 - \frac{4\pi e^2 n_0}{m_e p^2}.$$

The solution of eq. (12), which describes trapped oscillations, is possible for $A < 0(\varepsilon_0 < 0)$ in the region $-\sqrt{|A|/B} < x < \sqrt{|A|/B}$. Then eq. (12) takes the form

$$\frac{\partial^2\Phi_2}{\partial \xi^2} + \left(\frac{A}{\sqrt{B}} - \xi^2\right)\Phi_2 = 0, \quad \frac{|A|}{\sqrt{B}} = \frac{k|\varepsilon_0|pL}{\omega_{\mathrm{P}_0}}, \tag{13}$$

where $\xi = (k\omega_{\mathrm{P}_0}/Lp)^{1/2}x$. Making the substitution $\Phi_2 = \psi(\xi)\exp(-\xi^2/2)$ and introducing the notation

$$2n + 1 = \frac{|A|}{\sqrt{B}} = \frac{k|\varepsilon_0|pL}{\omega_{\mathrm{P}_0}} \tag{14}$$

we obtain the equation

$$\psi''_{\xi\xi} - 2\xi\psi'_{\xi} + 2n\psi = 0 \tag{15}$$

for the function $\psi(\xi)$. The solutions of this equation are Hermite polynomials [25]:

$$\psi \sim H_n(\xi) = (-1)^n \, \mathrm{e}^{\xi^2} \frac{\mathrm{d}^n \mathrm{e}^{-\xi^2}}{\mathrm{d}\xi^n} \tag{16}$$

satisfying the localizability condition (the width of the region of localizability of the oscillations is assumed to be significantly less than the width of the plasma layer) only for integral positive values of the number n (including zero). This fact permits considering eq. (14) as an analog of the quantization rule, which serves to determine the possible values of the quantity p. Thus, the solution of eq. (13) takes the form

$$\Phi_2 = c_n \mathrm{e}^{-\xi^2/2} H_n(\xi). \tag{17}$$

From eq. (14), we get

$$p^2 + \left(\left[\frac{2n+1}{kL}\right]\omega_{\mathrm{P}_0}\right)p - \omega_{\mathrm{P}_0}^2 = 0. \tag{18}$$

Thus, from eq. (18), we get

$$p = \left(\frac{\omega_{\mathrm{P}_0}}{2}\right)\left\{\left(\left[\frac{2n+1}{kL}\right]^2 + 4\right)^{1/2} - \left(\frac{2n+1}{kL}\right)\right\}. \tag{19}$$

At $L \to \infty$, $p \to \omega_{\mathrm{P}_0}$ (plasma waves in uniform plasma). Equation (18) takes the form

$$p^2 = \omega_{\mathrm{P}_0}^2 (1-\delta)\,,$$

$$\delta = \frac{1}{2}\frac{2n+1}{kL}\left\{\left[4 + \left(\frac{2n+1}{kL}\right)^2\right]^{1/2} - \frac{2n+1}{kL}\right\}. \tag{20}$$

This equation is the same as in a uniform plasma case [24]; i.e., the nonuniform plasma has no effect on the space part of the problem.

4. Solution of the 'temporal' (time-dependent) equations

Following the procedure, developed in [2,3], from eq. (9) we can derive dispersion equation of low-frequency oscillations ($\omega \approx (m_e/m_i)p$).

Under the parametric resonance condition ($n\omega_0 \approx s$, n-integer), we get

$$\omega^4 - \frac{\Delta_n^2 p^2}{4}\omega^2 - \frac{m_e}{2m_i} p^4 \Delta_n J_n^2(a) = 0, \tag{21}$$

where $\Delta_n = (p/n\omega_0)^2 - 1$, and we suppose here that the resonance 'mismatch' Δ_n satisfies the inequalities $(m_e/m_i) \ll |\Delta_n| \ll 1$. For convenience a_e is written as a throughout. From eq. (21) we find the frequencies of parametrically excited plasma oscillations

$$\omega^2 = \frac{\Delta_n^2 p^2}{8}\left(1 \pm \left[1 + \frac{32}{\Delta_n^3} J_n^2(a) \frac{m_e}{m_i}\right]^{1/2}\right), \tag{22}$$

where $J_n(a)$ is the Bessel function. Expression (22) yields an unstable solution in two cases.

4.1 *Periodic instability* ($\Delta_n < 0$)

In this case $\gamma_{\rm per} = {\rm Im}\,\omega > 0$, i.e., small perturbations in a plasma grow exponentially in time, if the following condition is satisfied:

$$0 > \Delta_n > -2\left(4J_n^2(a)\,\frac{m_e}{m_i}\right)^{1/3}. \tag{23}$$

Condition (23) means that when the *nth* harmonic of the external HF field approaches p, the surface oscillations grow, provided

$$n\omega_0 = p\left[1 + \left(4J_n^2(a)\,\frac{m_e}{m_i}\right)^{1/3}\right].$$

The growth rate of instability is determined by the expression

$$\gamma_{\rm per} = p\frac{|\Delta_n|}{4}\left[-1 + \left(\frac{32J_n^2(a)}{|\Delta_n|^3}\frac{m_e}{m_i}\right)^{1/2}\right]^{1/2}. \tag{24}$$

The maximum value of the growth rate $\gamma_{\rm per}$ is reached at

$$(n\omega_0)_{\max} = p\left[1 + \left(\frac{1}{4}J_n^2\,(a)\,\frac{m_e}{m_i}\right)^{1/3}\right]. \tag{25}$$

Substituting (25) into (24) we find

$$\gamma_{\rm per}^{\max} = p\left[\frac{\sqrt{27}}{32}J_n^2\,(a)\,\frac{m_e}{m_i}\right]^{1/3}. \tag{26}$$

4.2 *Aperiodic instability* ($\Delta_n > 0$)

In this case eq. (22) describes the growth of oscillations when the minus sign is taken. We have then the following expression for the growth rate:

$$\gamma_{\rm aper} = \frac{\Delta_n}{2\sqrt{2}}p\left[\left(1 + \frac{32J_n^2(a)}{\Delta_n^3}\frac{m_e}{m_i}\right)^{1/2} - 1\right]^{1/2}. \tag{27}$$

The maximum of the growth rate

$$\gamma_{\rm aper}^{\max} = p\left[\frac{1}{2}J_n^2(a)\,\frac{m_e}{m_i}\right]^{1/3} \tag{28}$$

is attained under the condition

$$(n\omega_0)_{\rm aper}^{\max} = p\left[1 - \left(\frac{1}{2}J_n^2(a)\,\frac{m_e}{m_i}\right)^{1/3}\right]. \tag{29}$$

The main feature of eqs (24)–(26) is in the existence of a separation constant p which enables us to account for the plasma nonuniformity.

At $\delta \ll 1$, expressions (26) and (28) become

$$\gamma_{\text{per}}^{\max} \cong \left(1 - \frac{\delta}{2}\right) \gamma_{\text{per}}^{\max U}, \quad \gamma_{\text{aper}}^{\max} \cong \left(1 - \frac{\delta}{2}\right) \gamma_{\text{aper}}^{\max U}, \tag{30}$$

where $\gamma_{\text{per}}^{\max U}$ and $\gamma_{\text{aper}}^{\max U}$ are values of growth rates of the periodical and the aperiodical API at vanishing density gradients.

From eq. (24) we conclude that, the threshold value of the HF field amplitude in the case of periodic instability is determined by the relation

$$32 J_n^2(a_{\text{thr}}) \frac{m_e}{m_i} = |\Delta_n|^3, \qquad |\Delta_n| = -\frac{p^2}{(n\omega_0)^2} + 1. \tag{31}$$

At small amplitudes of the pumping wave $a \ll 1$, $n = 1$, from eq. (31) we get

$$a_{\text{thr}}^2 = \frac{m_i}{8 m_e} \left[|\Delta_0| + \delta \frac{\omega_{\text{P}_0}^2}{(n\omega_0)^2} \right]^3, \qquad |\Delta_0| = \frac{\omega_{\text{P}_0}^2}{(n\omega_0)^2} - 1. \tag{32}$$

It follows from expressions (30) and (32) that nonuniformity of the plasma density results in a decrease of the growth rate of absolutely unstable oscillations and an increase in the threshold value of the pump wave amplitude compared to the case of a uniform plasma waveguide.

It should be noted that our approach is significantly simpler than the method ordinarily employed in the theory of parametric excitation of waves in nonuniform plasma. Therefore, it is of practical interest to apply the method to solve different problems in parametric resonance in nonuniform plasma taking into account finite plasma temperature and nonuniformities of the HF electric field and static magnetic field.

5. Results and conclusions

We study in this paper the effect of 1D plasma nonuniformity on absolute parametric instability (API) of electrostatic waves in a magnetized plane waveguide subjected to an intense HF electric field by using the separation method.

It follows from eqs (20), (26), (28), (31) and (32) that taking into account the nonuniformity of plasma density results in the decrease of the maximum values of the oscillation build up increments and an increase in the threshold value of the electric field amplitude of the pumping wave compared to the case of uniform plasma. These results are consistent with the results of refs [5,24]. Equation (20) is the same equation in uniform plasma case [5,24], i.e., the nonuniform plasma has no effect on the space part of the problem. The main feature of eq. (9) enables us to account for the plasma nonuniformity.

From expressions (26) and (29), we conclude that the growth rates of periodic and aperiodic API decrease more in nonuniform plasma than in uniform plasma (Demchenko *et al* [24]).

It should be noted that our approach is significantly simpler than the method ordinarily employed in the theory of parametric excitation of waves in nonuniform plasma.

Therefore, it is of practical interest to apply the method to solve different problems in parametric resonance in nonuniform plasma taking into account relativistic electron plasma and nonuniformities of the HF electric field and static magnetic field.

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