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Synchronizability on complex networks via pinning control

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Abstract. It is proved that the maximum eigenvalue sequence of the principal submatrices of coupling matrix is decreasing. The method of calculating the number of pinning nodes is given based on this theory. The findings reveal the relationship between the decreasing speed of maximum eigenvalue sequence of the principal submatrices for coupling matrix and the synchronizability on complex networks via pinning control. We discuss the synchronizability on some networks, such as scale-free networks and small-world networks. Numerical simulations show that different pinning strategies have different pinning synchronizability on the same complex network, and the synchronizability with pinning control is consistent with one without pinning control in various complex networks.

Keywords. Complex network; the pinning synchronization; synchronizability.

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1. Introduction

In the real world, many systems, such as Internet, World Wide Web, food webs, biological neural networks, electric power grids, social networks, etc., can be described by models of complex networks. So far, complex networks have been intensively investigated across many fields of science and engineering [1–5]. The synchronization of all dynamical nodes in a network is one of the most interesting and significant phenomenon, which has been widely studied in complex dynamical networks [6–12]. As we know, the real-world complex networks usually have a large number of nodes. Therefore, it is difficult to realize synchronization by adding controllers to all nodes. It is important to achieve synchronization by pinning part of the nodes, which is known as pinning synchronization control. Some significant results have been obtained about the pinning synchronization on complex networks [13–16]. Synchronizability (synchronization ability) on a complex network is the ease by which synchronization can be achieved. However, there is no unique definition for the synchronization ability on complex networks. The synchronizability has been studied in some literatures [17–21]. Nishikawa *et al* [17] found that the synchronizability on complex networks with a homogeneous distribution of connectivity is better than heterogeneous one. Hong *et al* [18] investigated the effects of various factors such as degree, characteristic path length, heterogeneity and betweenness centrality on synchronization, and found a consistent trend between the synchronization and the betweenness centrality. Comellas and Gago [19] studied the relevant factors influencing the synchronization of complex networks by using spectral graph theory. Ma and Wang [20] studied the effects of connectivity, coupling strength, average distance, heterogeneity, clusters and weight distribution on the synchronizability on complex networks by introducing a weight matrix extensively and deeply. Jalili [21] studied the robustness of synchronizability against random deletion of nodes in dynamical scale-free networks, and found that as the network size decreases, the robustness of its synchronizability against random removal of nodes declines. However, all the findings above on synchronizability were based mainly on the topological structures of complex networks, and used the eigenvalue ratio λ_N/λ_2 for the coupling matrix A ($A = -\mathbf{G}$, and $\mathbf{G}$ is defined in the following text) as a measure of synchronizability without pinning control.

In this paper, we investigate the synchronizability on complex networks from the perspective of pinning control, and use the decreasing speed of maximum eigenvalue sequence of the principal submatrices for coupling matrix as the synchronizability indicator. The effects of different pinning strategies in some main complex networks on synchronizability are analysed, which include nearest-neighbour coupled networks, star-coupled networks, globally coupled networks, scale-free networks and small-world networks. We find that different pinning strategies have different synchronizabilities on the same complex network, and synchronizability with pinning control is similar to the one without pinning control on complex networks. For example, synchronizability of the high-degree pinning is stronger than one of the random pinning and the low-degree pinning strategies on small-world and scale-free networks. Two different complex networks, small-world and scale-free networks, are more easily synchronized than the nearest-neighbour coupled networks under the conditions without pinning control and with pinning control.

The rest of the paper is organized as follows. Some necessary preliminaries, hypothesis and lemmas are presented in §2. Section 3 discusses the pinning synchronization scheme, and gives the decreasing law of maximum eigenvalue sequence of the principal submatrices for coupling matrix. In §4, the synchronizability on complex networks with pinning control is analysed, and numerical simulations are given. Finally, conclusion is given in §5.

2. Preliminaries

Consider a complex dynamical network of N identical coupled nodes, with each node being an n-dimensional dynamical system [14].

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}(\mathbf{x}_i(t)) + \sigma \sum_{j=1}^{N} g_{ij} \boldsymbol{\Gamma} \mathbf{x}_j(t), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $\mathbf{x}_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbf{R}^n$ is the state vector of the ith node, $\mathbf{f}$: $\mathbf{R}^n \to \mathbf{R}^n$ is a smooth nonlinear vector field, σ is the overall strength of the coupling, $\Gamma \in \mathbf{R}^{n\times n}$ is the inner coupling matrix (Γ describes the individual coupling between connected nodes of the network), $\mathbf{G} = (g_{ij})_{N\times N}$ is the coupling matrix representing the topological structure of the network, where g_{ij} are defined as follows: if there is a link from node i to node j ($j \neq i$), then $g_{ij} = 1$; otherwise, $g_{ij} = 0$ ($-\mathbf{G}$ is a Laplacian matrix). In addition, assume that $\mathbf{G}$ is a diffusive matrix, satisfying

$$g_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{N} g_{ij} = -\sum_{\substack{j=1\\ j\neq i}}^{N} g_{ji}, \quad i = 1, 2, \ldots, N. \tag{2}$$

Assume that $\mathbf{s}(t) \in \mathbf{R}^n$ is a solution of an isolated node, satisfying

$$\dot{\mathbf{s}}(t) = \mathbf{f}(\mathbf{s}(t)), \tag{3}$$

$\mathbf{s}(t)$ can be an equilibrium point, a periodic orbit, an aperiodic orbit, even a chaotic orbit in the phase space.

If

$$\lim_{t\to\infty} \|\mathbf{x}_i(t) - \mathbf{s}(t)\|_2 = 0, \quad i = 1, 2, \ldots, N, \tag{4}$$

the coupled network (1) is said to achieve synchronization.

Define the error vector as

$$\mathbf{e}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t), \quad i = 1, 2, \ldots, N. \tag{5}$$

Next, we present some hypothesis and lemmas for later use.

Hypothesis 1. There exists a nonnegative constant ω and a positive definite matrix $\Gamma \in \mathbf{R}^{n\times n}$, such that $\mathbf{f}$ satisfies the following inequality:

$$(\mathbf{x} - \mathbf{y})^T[\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})] \leq \omega(\mathbf{x} - \mathbf{y})^T\Gamma(\mathbf{x} - \mathbf{y}), \tag{6}$$

for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$.

Lemma 1 [22]. Let eigenvalues of the matrix $\mathbf{A}$ have $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let eigenvalues of the matrix $\mathbf{B}$ have $\mu_1, \mu_2, \ldots, \mu_m$, then $n{*}m$ eigenvalues of the matrix $\mathbf{A} \otimes \mathbf{B}$ are $\lambda_i\mu_j$ ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$).

Lemma 2 [23]. The following linear matrix inequality (LMI)

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^H & \mathbf{C} \end{pmatrix} < 0 \tag{7}$$

is equivalent to one of the following conditions:

(1) $\mathbf{A} < 0$ and $\mathbf{C} - \mathbf{B}^H\mathbf{A}^{-1}\mathbf{B} < 0$,
(2) $\mathbf{C} < 0$ and $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^H < 0$,

where $\mathbf{A} = \mathbf{A}^H$ ($\mathbf{A}^H$ is the conjugate transpose matrix for $\mathbf{A}$), $\mathbf{C} = \mathbf{C}^H$, '$<$' is partial order and matrix $\mathbf{X} < 0$ represents that $\mathbf{X}$ is a negative definite matrix.

Lemma 3 (Cauchy Interlace Theorem) [24]. Let $\mathbf{A}$ be a Hermitian matrix of order n, and let $\mathbf{B}$ be a principal submatrix of $\mathbf{A}$ of order $n-1$. If $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n-1} \le \lambda_n$ lists the eigenvalues of $\mathbf{A}$ and $\mu_1 \le \mu_2 \le \cdots \le \mu_{n-2} \le \mu_{n-1}$ the eigenvalues of $\mathbf{B}$, then $\lambda_1 \le \mu_1 \le \lambda_2 \le \cdots \le \mu_{n-1} \le \lambda_n$.

3. The pinning synchronization scheme

Suppose that coupling matrix $\mathbf{G}$ is irreducible (there are no isolated nodes on a complex network). The general pinning synchronization scheme can be described by

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}(\mathbf{x}_i(t)) + \sigma \sum_{j=1}^{N} g_{ij} \boldsymbol{\Gamma} \mathbf{x}_j(t) + \mathbf{u}_i(t), \qquad i = 1, 2, \ldots, l, \tag{8}$$

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}(\mathbf{x}_i(t)) + \sigma \sum_{j=1}^{N} g_{ij} \boldsymbol{\Gamma} \mathbf{x}_j(t), \qquad i = l+1, l+2, \ldots, N, \tag{9}$$

where

$$\mathbf{u}_i(t) = -\sigma d_i \boldsymbol{\Gamma}(\mathbf{x}_i(t) - \mathbf{s}(t)) \in \mathbf{R}^n, \qquad i = 1, 2, \ldots, l, \tag{10}$$

are n-dimensional controllers with the control gains $d_i > 0$ and $\boldsymbol{\Gamma}$ is a positive definite matrix.

From eqs (8)–(10) and (3), we can obtain the following error equations:

$$\begin{aligned}\dot{\mathbf{e}}_i(t) &= \mathbf{f}(\mathbf{x}_i(t)) - \mathbf{f}(\mathbf{s}(t)) \\ &\quad + \sigma \sum_{j=1}^{N} g_{ij} \boldsymbol{\Gamma} \mathbf{x}_j(t) - \sigma d_i \boldsymbol{\Gamma}(\mathbf{x}_i(t) - \mathbf{s}(t)), \qquad i=1,2,\ldots,l,\end{aligned} \tag{11}$$

$$\begin{aligned}\dot{\mathbf{e}}_i(t) &= \mathbf{f}(\mathbf{x}_i(t)) - \mathbf{f}(\mathbf{s}(t)) \\ &\quad + \sigma \sum_{j=1}^{N} g_{ij} \boldsymbol{\Gamma} \mathbf{x}_j(t), \quad i = l+1, l+2, \ldots, N.\end{aligned} \tag{12}$$

In the following, we give a synchronization criterion of the pinning synchronization scheme in the form of Lemma 4.

Lemma 4. Suppose that Hypothesis 1 holds. The complex network via pinning control eqs (8), (9) and (10) (the pinning synchronization scheme) synchronizes globally if $\omega \mathbf{I}_N + \sigma(\mathbf{G} - \mathbf{D})$ is a negative definite matrix, where $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_l, 0, \ldots, 0)$ and $\mathbf{I}_N$ is a unit matrix of order N.

Proof. We assume the candidate Lyapunov function:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} \mathbf{e}_i(t)^T \mathbf{e}_i(t). \tag{13}$$

By differentiating V along the trajectories, we obtain:

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{N} \mathbf{e}_i(t)^T \dot{\mathbf{e}}_i(t) \\
&= \sum_{i=1}^{N} \mathbf{e}_i(t)^T\,[\mathbf{f}(\mathbf{x}_i(t)) - \mathbf{f}(\mathbf{s}(t))] + \sigma \sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij}\mathbf{e}_i(t)^T \boldsymbol{\Gamma} \mathbf{e}_j(t) \\
&\quad - \sigma \sum_{i=1}^{l} d_i \mathbf{e}_i(t)^T \boldsymbol{\Gamma} \mathbf{e}_i(t) \le \sum_{i=1}^{N} \omega \mathbf{e}_i(t)^T \boldsymbol{\Gamma} \mathbf{e}_i(t) \\
&\quad + \sigma \sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij}\mathbf{e}_i(t)^T \boldsymbol{\Gamma} \mathbf{e}_j(t) - \sigma \sum_{i=1}^{l} d_i \mathbf{e}_i(t)^T \boldsymbol{\Gamma} \mathbf{e}_i(t) \\
&= \mathbf{e}_i(t)^T\,[(\omega \mathbf{I}_N + \sigma(\mathbf{G} - \mathbf{D})) \otimes \boldsymbol{\Gamma}]\,\mathbf{e}_i(t),
\end{aligned} \tag{14}$$

where $\otimes$ is the Kronecker product, $\mathbf{e}(t) = (\mathbf{e}_1(t), \mathbf{e}_2(t), \ldots, \mathbf{e}_N(t))^T$. Because $\boldsymbol{\Gamma}$ is a positive definite matrix, according to Lemma 1, $\omega \mathbf{I}_N + \sigma(\mathbf{G} - \mathbf{D})$ is a negative definite matrix if and only if $(\omega \mathbf{I}_N + \sigma(\mathbf{G} - \mathbf{D})) \otimes \Gamma$ is a negative definite matrix. From Lyapunov stability theory, the proof is completed.

Now, we prove the decreasing law of maximum eigenvalue sequence of the principal submatrices for the coupling matrix.

Suppose that

$$\omega \mathbf{I}_N + \sigma(\mathbf{G} - \mathbf{D}) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{pmatrix}. \tag{15}$$

$\mathbf{A} = \omega \mathbf{I}_l + \sigma(\tilde{\mathbf{G}} - \tilde{\mathbf{D}})$, $\mathbf{I}_l$ is a unit matrix of order l, $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{D}}$ are principal submatrices of order for $\mathbf{G}$ and $\mathbf{D}$, respectively. $\mathbf{C} = \omega \mathbf{I}_{N-l} + \sigma \mathbf{G}\,[l+1, l+1]$. The constant ω is related to $\mathbf{f}$. If σ is fixed, appropriate values of d_i can be selected to make $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^H$ a negative definite matrix. Therefore, according to Lemma 2, $\omega \mathbf{I}_N + \sigma(\mathbf{G} - \mathbf{D})$ is a negative definite matrix if and only if $\mathbf{C}$ is a negative definite matrix. Now we can study the pinning synchronization in complex networks using negative definiteness for low order matrix $\mathbf{C}$.

Next, to calculate the number of pinning nodes and analyse the synchronizability on complex networks, we prove Theorem 1, which reveal the decreasing law of maximum eigenvalue sequence of the principal submatrices for the coupling matrix.

Theorem 1. *Suppose that* **G** *is a coupling matrix in a complex network.* **G**$[l, l]$ *is a principal submatrix of order* $N - l + 1$ *for* **G**, *and has the following form*:

$$\mathbf{G}\,[l, l] = \begin{pmatrix} g_{ll} & g_{ll+1} & \cdots & g_{lN} \\ g_{l+1l} & g_{l+1l+1} & \cdots & g_{l+1N} \\ \cdots & \cdots & \cdots & \\ g_{Nl} & g_{Nl+1} & \cdots & g_{NN} \end{pmatrix}, \tag{16}$$

then $0 = \lambda_{\max}(\mathbf{G}\,[1, 1]) > \lambda_{\max}\,(\mathbf{G}\,[2, 2]) \geq \cdots \geq \lambda_{\max}\,(\mathbf{G}\,[N, N])$, *where* $\lambda_{\max}(\mathbf{G}[l, l])$ *is the maximum eigenvalue of* $\mathbf{G}[l, l]$, $(l = 1, 2,\ldots, N)$.

Proof. Suppose that $N - l + 1$ eigenvalues of the matrix $\mathbf{G}[l, l]$ are $\lambda_i(\mathbf{G}[l, l])$, $N - l$ eigenvalues of matrix $\mathbf{G}[l + 1, l + 1]$ are $\lambda_j(\mathbf{G}\,[l + 1, l + 1])$ $(i = 1, 2, \ldots, N - l + 1;$ $j = 1, 2, \ldots, N - l)$ and $\lambda_1(\mathbf{G}\,[l, l]) \leq \lambda_2\,(\mathbf{G}\,[l, l]) \leq \cdots \leq \lambda_{N-l+1}(\mathbf{G}\,[l, l])$, $\lambda_1(\mathbf{G}\,[l + 1, l + 1]) \leq \lambda_2(\mathbf{G}\,[l + 1, l + 1]) \leq \cdots \leq \lambda_{N-l}(\mathbf{G}\,[l + 1, l + 1])$. According to Lemma 3,

$$\begin{aligned} \lambda_1(\mathbf{G}[l, l]) \leq \lambda_1(\mathbf{G}\,[l + 1, l + 1]) \cdots \leq \lambda_{N-l}(\mathbf{G}\,[l + 1, l + 1]) \\ \leq \lambda_{N-l+1}(\mathbf{G}\,[l, l]) \end{aligned}$$

holds. Therefore,

$$\lambda_{\max}(\mathbf{G}[l + 1, l + 1]) \leq \lambda_{\max}(\mathbf{G}[l, l]).$$

The **G**[1,1] has unique zero eigenvalue, all the other eigenvalues are negative. Because all eigenvalues of **G**[2,2] are negative,

$$\lambda_{\max}(\mathbf{G}[1, 1]) > \lambda_{\max}(\mathbf{G}[2, 2])$$

holds. After all,

$$0 = \lambda_{\max}(\mathbf{G}[1, 1]) > \lambda_{\max}(\mathbf{G}[2, 2]) \geq \cdots \geq \lambda_{\max}(\mathbf{G}[N, N]).$$

The proof is completed.

Figures 1–6 show examples of numerical simulations of the decreasing law of maximum eigenvalue sequence of the principal submatrices for coupling matrix in some main complex networks.

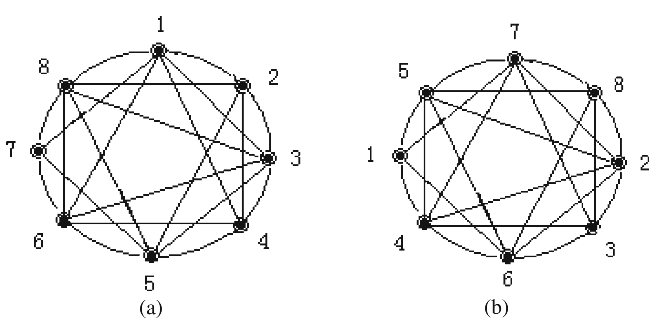

Figure 1. (**a**) The sequential numbering and (**b**) the random numbering.

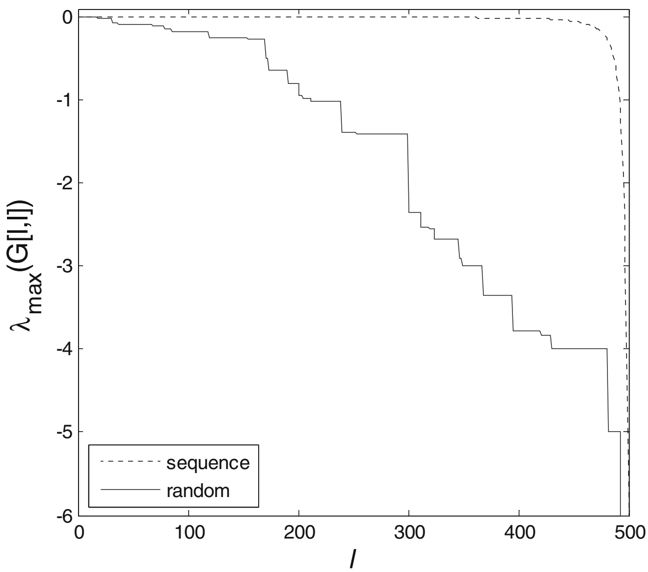

Figure 2. The relation between maximum eigenvalue sequence of the principal submatrices and the number of pinning nodes in a nearest-neighbour coupled network in two strategies.

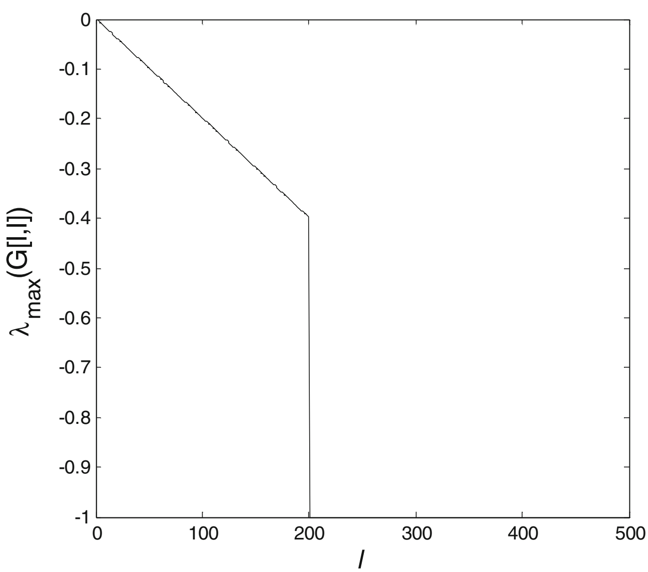

Figure 3. The relation between maximum eigenvalue sequence of the principal submatrices and the number of pinning nodes in a star-coupled network.

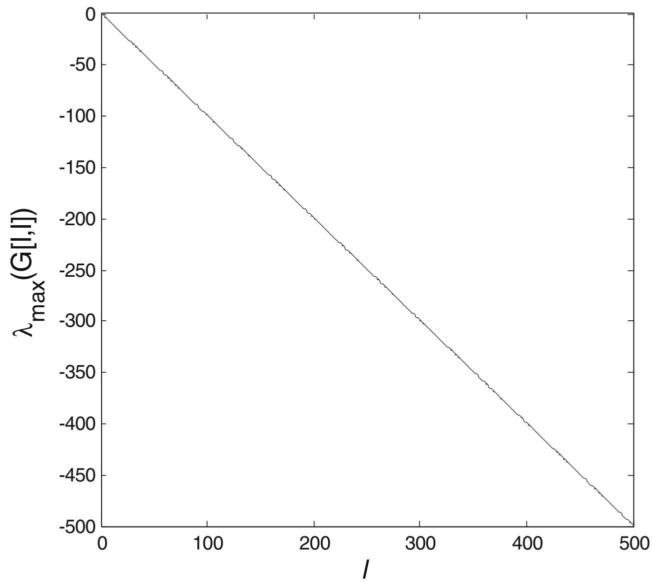

Figure 4. The relation between maximum eigenvalue sequence of the principal submatrices and the number of pinning nodes in a globally coupled network.

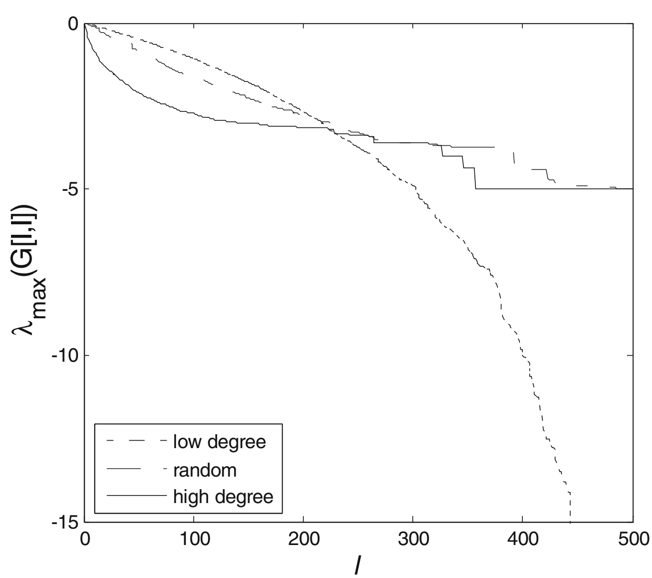

Figure 5. The relation between maximum eigenvalue sequence of the principal submatrices and the number of pinning nodes in a scale-free network in three strategies.

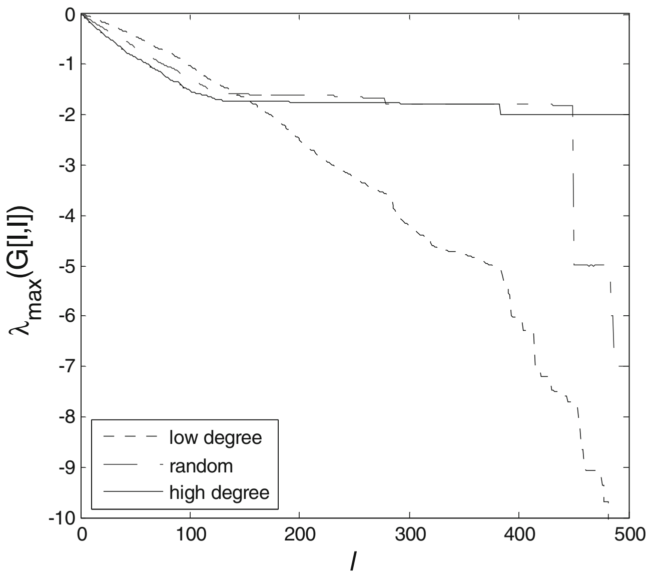

Figure 6. The relation between maximum eigenvalue sequence of the principal submatrices and the number of pinning nodes in a small-world network in three strategies.

As we can see from the proof above, if $l \geq 1$, all the eigenvalues of $\mathbf{G}[l+1, l+1]$ are negative. Hence, we can always select an appropriate value of σ to make $\mathbf{C} = \omega \mathbf{I}_{N-l} + \sigma \mathbf{G}\,[l+1, l+1]$ a negative definite matrix. In other words, if $l \geq 1$, the pinning synchronization scheme synchronizes.

Now, the relationship among the overall strength of the coupling (σ), the number of pinning nodes (l) and $\lambda_{\max}(\mathbf{G}[l+1, l+1])$ is given.

Suppose that ω is known. When

$$\sigma \lambda_{\max}(\mathbf{G}[l+1, l+1]) < -\omega,$$

or

$$\lambda_{\max}(\mathbf{G}\,[l+1, l+1]) < -\omega/\sigma,$$

$\mathbf{C}$ is negative, i.e. the pinning synchronization scheme synchronizes with l pinning nodes. Therefore, if σ is fixed, the number of pinning nodes can be calculated, or if l is fixed, σ can be calculated.

4. Synchronizability analysis and numerical simulations

Now, we propose a descriptive notion about pinning synchronizability. Synchronizability of the pinning scheme is the degree of strength of synchronization on complex networks with pinning control. Under the same condition, the smaller the number of nodes needed in the synchronization, the stronger is the synchronizability obtained in the same network. According to $\lambda_{\max}(\mathbf{G}\,[l+1, l+1]) < -\omega/\sigma$, the lesser the value $\lambda_{\max}(\mathbf{G}[l+1, l+1])$

is, the smaller is the number of pinning nodes (l) required. From Theorem 1, for any coupling matrix $\mathbf{G}$:

$$\lambda_{\max}(\mathbf{G}[1,1]) > \lambda_{\max}(\mathbf{G}[2,2]) \geq \cdots \geq \lambda_{\max}(\mathbf{G}[N,N])$$

holds. Therefore, the faster the decreasing speed of maximum eigenvalue sequence of the principal submatrices for coupling matrix is, the stronger the synchronizability is in a specific pinning strategy. Figures 1–6 show the relationship between $\lambda_{\max}(\mathbf{G}[l,l])$ and the number of pinning nodes in some main complex networks in various pinning strategies. Obviously, synchronizability of the pinning scheme can be seen in different pinning strategies.

Generally, three pinning strategies are used: the high-degree pinning strategy, the random pinning strategy and the low-degree pinning strategy. Assume that a coupling matrix is constructed by using numbering sequence of nodes from small to big in complex networks, and a numbering sequence of nodes is the same as the sequence of pinning nodes. To the specific complex network, different numbering sequences of nodes correspond to different coupling matrices and maximum eigenvalue sequence of the principal submatrices. If random pinning strategy is used to pin nodes, nodes are numbered randomly. A coupling matrix is constructed according to the random numbering sequence of nodes, and then the number of pinning nodes is calculated using the method above. If the high-degree pinning strategy is used, nodes are numbered from nodes of high degree to nodes of low degree, and then a coupling matrix is constructed to calculate the number of pinning nodes. Similarly, we can use other pinning strategies.

4.1 *Synchronizability on regular networks*

4.1.1 *Nearest-neighbour coupled networks.* It is more difficult for a nearest-neighbour coupled network to achieve synchronization. We use the sequential pinning strategy and the random pinning strategy to investigate their synchronizability. In the sequential pinning strategy, a coupling matrix is constructed by numbering the sequence of nodes. The sequential numbering and the random numbering are shown in figures 1a and 1b ($N = 8$), respectively.

Here is a nearest-neighbour coupled network with 500 nodes, and degrees of each node are 6. Figure 2 gives the relationship between the maximum eigenvalue sequence of the principal submatrices and the number of pinning nodes in the sequential pinning strategy and the random pinning strategy. It is found that synchronizability of the sequential pinning strategy is very weak, i.e., even if the number of pinning nodes (l) is larger, the pinning effect is very poor.

4.1.2 *Star-coupled networks.* A star-coupled network is a network with N nodes, which has a centre node C_0 with $N-1$ degrees, and other nodes with 1 degree. If $l < C_0$, $\lambda_{\max}(\mathbf{G}[l,l]) > -1$, otherwise ($l \geq C_0$) $\lambda_{\max}(\mathbf{G}[l,l]) = -1$.

Figure 3 shows the relation between maximum eigenvalue sequence of the principal submatrices and the number of pinning nodes in a star-coupled network, which has 500 nodes, and $C_0 = 200$. Obviously, synchronizability on star-coupled networks is weaker.

4.1.3 *Globally coupled networks.* Each node is connected to each other in a globally coupled network, and it is the easiest to achieve synchronization in these networks.

Because a coupling matrix is not related to numbering sequence of nodes in a globally coupled network, synchronizability on globally coupled networks is the same for various node sequences. It is easy to calculate $\lambda_{\max}(\mathbf{G}[l, l]) = l-1$.

Figure 4 shows the case of pinning synchronization in a globally coupled network with 500 nodes. Obviously, the synchronizability is the strongest on globally coupled networks.

4.2 *Synchronizability on scale-free networks*

Now, the relationship between maximum eigenvalue sequence of the principal submatrices and the number of pinning nodes in a scale-free network is shown in three strategies: (1) the high-degree pinning strategy, (2) the random pinning strategy and (3) the low-degree pinning strategy.

A scale-free network with $N = 500$ is constructed using the Barabási–Albert model with $m_0 = 5$ starting nodes [25]. From figure 5, it can be found that the synchronizability of the high-degree pinning strategy is the strongest when the number of pinning nodes is less than 50% of the nodes, and the random pinning strategy is the next.

4.3 *Synchronizability on small-world networks*

We use the mean degree 8 and the remove probability 0.06 to construct a small-world network with 500 nodes [1]. The evolution trend in a small-world network is similar to the one in the scale-free networks in figure 6.

Remark

(1) From the simulations, different pinning strategies have different synchronizability except globally coupled networks. Hence, it is important to select an appropriate pinning strategy.
(2) We can find that synchronizability with pinning control is consistent with one without pinning control on various complex networks. For example, synchronizability on globally coupled networks is much stronger than the one in nearest-neighbour coupled networks.

4.4 *Simulation of the pinning synchronization*

Next, let us verify the pinning synchronization scheme. The simulation example is on a scale-free network by using the random pinning strategy, and dynamical system on each node is the Lorenz system [26]. A scale-free network is constructed by using the parameters in the above example. We calculate the number of pinning nodes using Hypothesis 1. In fact, other systems such as Lü system [27], Chen system [28] and Chua's circuit [29], also satisfy Hypothesis 1.

The ith node Lorenz system is written as follows:

$$\begin{cases} \dot{x}_{i1} = a\,(x_{i2} - x_{i1}) \\ \dot{x}_{i2} = cx_{i1} - x_{i1}x_{i3} - x_{i2}\,, \quad 1 \le i \le 500, \\ \dot{x}_{i3} = x_{i1}x_{i2} - bx_{i3} \end{cases} \tag{17}$$

where $a = 10$, $c = 30$, $b = 8/3$, then the Lorenz system leads to chaos.

It is known that $|x_{i1}| \leq 29$, $|x_{i2}| \leq 29$, $-1 \leq x_{i3} \leq 57$, $|s_1| \leq 29$, $|s_2| \leq 29$, $-1 \leq s_3 \leq 57$ [30]. Let $\mathbf{\Gamma} = \mathrm{diag}(1, 1, 1)$.

$$\begin{aligned}\mathbf{e}_i^T\,[\mathbf{f}(\mathbf{x}_i) - \mathbf{f}(\mathbf{s})] &= -ae_{i1}^2 - e_{i2}^2 - be_{i3}^2 + (a + c - x_{i3})e_{i1}e_{i2} + x_{i2}e_{i1}e_{i3} \\ &\leq -ae_{i1}^2 - e_{i2}^2 - be_{i3}^2 + (a + c + 1)\,|e_{i1}e_{i2}| + 29\,|e_{i1}e_{i3}| \\ &\leq \left(-a + \frac{39\rho}{2} + \frac{29\eta}{2}\right)e_{i1}^2 + \left(-1 + \frac{39}{2\rho}\right)e_{i2}^2 + \left(-b + \frac{29}{2\eta}\right)e_{i3}^2.\end{aligned}$$

Choosing $\rho = 0.9810$, $\eta = 0.6730$, then

$$\mathbf{e}_i^T\,[\mathbf{f}\,(\mathbf{x}_i) - \mathbf{f}\,(\mathbf{s})] \leq 18.888\mathbf{e}_i^T\,\mathbf{\Gamma}\mathbf{e}_i. \tag{18}$$

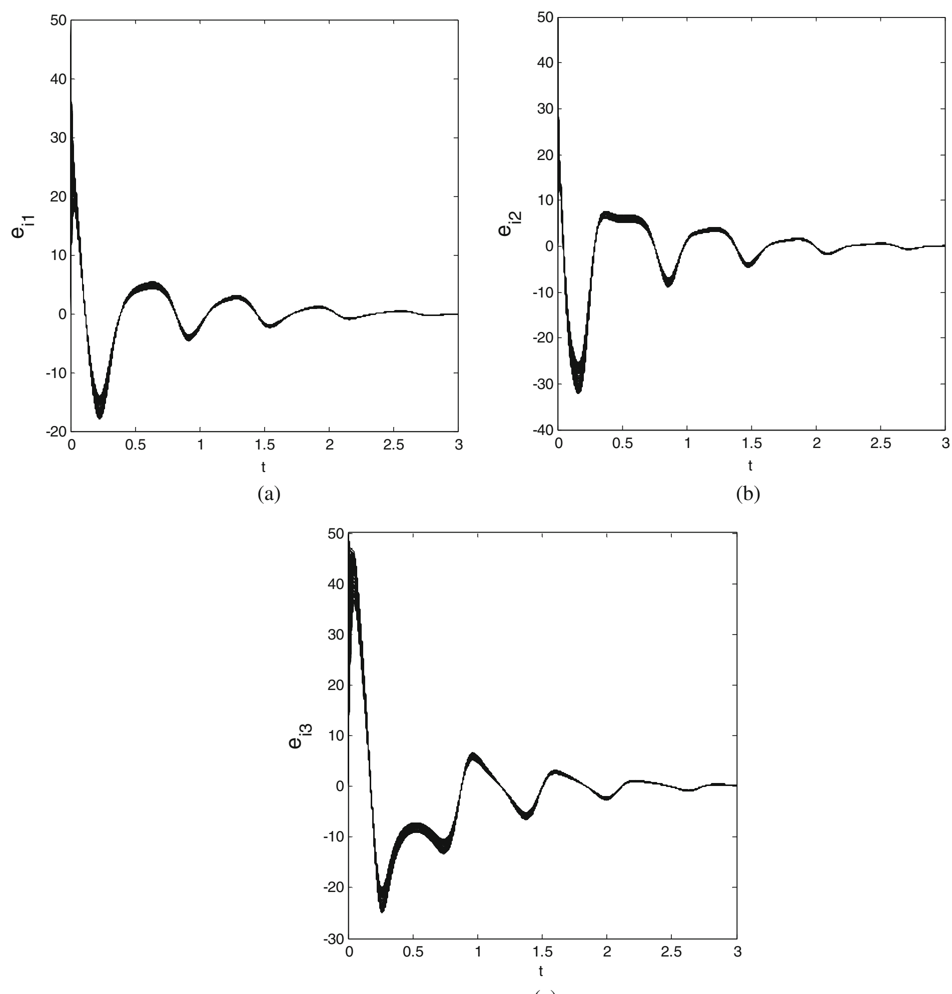

Figure 7. Evolution trends of e_{i1}, e_{i2}, e_{i3} of error components, $1 \leq i \leq 500$.

Let $\omega = 18.888$, $\sigma = 20$. In order to make $\lambda_{\max}(\mathbf{G}[l+1, l+1]) < -\omega/\sigma = -18.888/20$, we get the first $l = 57$, i.e., if the number of pinning nodes is 57, the scheme synchronizes.

By using Runge–Kutta, we choose initial values: $s(0) = (4, 5, 6)$, $x_{i1} = 4 + 0.1 \times i$, $x_{i2} = 5 + 0.1 \times i$, $x_{i3} = 6 + 0.1 \times i$; $d_i = 2, 1 \leq i \leq l$. Figures 7a–7c show evolution trends of errors for a scale-free network.

5. Conclusions

In this paper, synchronization on complex networks is investigated from the perspective of pinning control. The pinning synchronization scheme is reviewed, and asymptotic stability of synchronization solution is discussed. We find the decreasing law of maximum eigenvalue sequence of the principal submatrices for coupling matrix, and reveal the relationship between maximum eigenvalue sequence of the principal submatriecs and the number of pinning nodes. The decreasing speed of the maximum eigenvalue sequence of the principal submatrices for coupling matrix is used to study the synchronizability on some main complex networks. The results obtained are that different pinning strategies have different synchronizabilities on the same complex network and the synchronizability with pinning control is consistent with the one without pinning control in various complex networks. For networks above, the synchronizability of scale-free and small-world networks has close relationship with specific pinning strategy. The synchronizability of the high-degree pinning strategy is strongest in the three pinning strategies when the number of pinning nodes is less than 50% of the nodes, and in general, the number of pinning nodes account for a very small part of the total nodes when the pinning scheme is used. Hence, the high-degree pinning strategy is better.

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