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Adaptive control and synchronization of a fractional-order chaotic system

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Abstract. In this paper, the chaotic dynamics of a three-dimensional fractional-order chaotic system is investigated. The lowest order for exhibiting chaos in the fractional-order system is obtained. Adaptive schemes are proposed for control and synchronization of the fractional-order chaotic system based on the stability theory of fractional-order dynamic systems. The presented schemes, which contain only a single-state variable, are simple and flexible. Numerical simulations are used to demonstrate the feasibility of the presented methods.

Keywords. Fractional order; adaptive scheme; control; synchronization.

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1. Introduction

The concept of fractional-order calculus has been known since the contribution of Leibniz and L'Hospital in 1695 [1], but its applications to engineering, physics and mathematical biology are just recent topics of interest [2–5]. It was found that, with the help of fractional calculus, most of the systems in interdisciplinary fields can be depicted delicately [6–9]. As a matter of fact, all the physical phenomena in nature exist in the form of fractional order [10], and integer-order (traditional) differential equation is just a special case of the fractional one. The importance of fractional-order models is that they can yield a more explicit description and offer a deeper insight into the physical processes underlying a long-range memory behaviour. So, the fractional description is closer to reality.

One of the important problems in this field is to find minimum effective dimension for remaining chaotic in a fractional-order dynamical system. Another challenge that concerns the fractional field is the stability theory and the method for chaos control and synchronization. There are many significant differences between the integer-order differential systems and the corresponding fractional-order differential systems. Most of the conclusions for the integer-order dynamic system cannot be simply extended to the fractional order.

Many dynamic systems, such as Chua, Lorenz, Lü, Chen and Rössler systems of fractional-order demonstrate chaotic behaviour [11–15]. Recently, some investigations are devoted to achieve chaos stabilization and synchronization in fractional-order chaotic or hyperchaotic systems [16–22]. It should be pointed out that most of these are investigated through numerical simulations that are based on the stability criteria of the linear fractional-order dynamics systems, Laplace transform theory or the fractional Routh–Hurwitz conditions, and these controllers are complicated.

In this study, we concentrate on the fractional version of a new chaotic system. The minimum effective dimension of the fractional-order chaotic system to remain chaotic is investigated. Adaptive feedback methods are introduced for achieving control and synchronization of this fractional-order system based on the stability theory of fractional-order dynamic systems. The proposed controller, which only contains a single-state variable, to our knowledge, is the simplest scheme for control and synchronization of fractional-order chaotic system. Moreover, the control scheme is flexible, and is suitable both for design and implementation. Finally, numerical simulation results illustrate the effectiveness of the obtained control strategies.

2. Derivative and stability theorem on fractional-order system

2.1 *Fractional calculus*

There are many definitions of fractional derivatives. One of the definitions is introduced by Caputo [23], which is often used in real applications:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\tau)^{-\alpha+n-1} f^{(n)}(\tau)\, \mathrm{d}\tau, \tag{1}$$

where $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α, $\Gamma(\cdot)$ is the gamma function and D^{α} is generally called the 'α-order Caputo differential operator'.

2.2 *Stability theorem*

In order to investigate the chaotic behaviour, control and synchronization of the new fractional-order dynamic system, we introduce an indispensable stability theorem in the first instance.

Theorem 1 [24]. *Consider a commensurate fractional-order system*

$$D^{\alpha} X = F(X) \tag{2}$$

with $0 < \alpha \leq 1$, $X \in R^{m}$. *The equilibria of system* (2) *can be obtained by calculating* $F(X) = 0$. *These equilibrium points are locally asymptotically stable if all the eigenvalues* λ *of the Jacobian matrix* $J = \partial F(X)/\partial X$ *evaluated at the equilibrium points satisfy* $|\arg(\lambda)| > \alpha\pi/2$.

3. Fractional-order chaotic system

3.1 *New chaotic system*

Since the first chaotic model was found by Lorenz in 1963 [25], researchers have laid themselves out to construct new chaotic systems and analyse their dynamical behaviours. Many chaotic systems, such as Chen system [26], Liu system [27], Lü system [28], Qi system [29] have been proposed by developing Lorenz system. However, for these chaotic systems presented, the Lyapunov exponent spectra vary gradually and range from stable equilibrium points, periodic orbits to chaotic oscillations with the change in system parameters.

Recently, Li proposed a new chaotic system based on the construction pattern of Chen and Liu chaotic systems [30], which reads

$$\begin{cases} \dot{x}_1 = a\,(x_2 - x_1), \\ \dot{x}_2 = (c-a)x_1 + cx_2 - dx_1x_3, \\ \dot{x}_3 = -bx_3 + ex_2^2, \end{cases} \tag{3}$$

where x_1, x_2, x_3 are the state variables, a, b, c, d, e are the positive constant parameters.

It is found that when the parameters d and e vary, the Lyapunov exponent spectrum keeps invariable, and the signal amplitude can be controlled by adjusting d and e. Therefore, it can be concluded that this chaotic system has a more complicated dynamics compared to Lorenz, Chen, Liu, Lü and Qi systems. When $a = 38$, $b = 3$, $c = 30$, $d = 1$, $e = 1$, the corresponding 3D attractor, Lyapunov exponent spectrum and signal amplitude curve vs. d are shown in figure 1. For details about the properties of this new chaotic system, we refer the reader to ref. [30].

3.2 *Fractional-order chaotic system*

Here, we consider the fractional-order chaotic system:

$$\begin{cases} D^{\alpha}x_1 = a(x_2 - x_1), \\ D^{\alpha}x_2 = (c-a)x_1 + cx_2 - dx_1x_3, \\ D^{\alpha}x_3 = -bx_3 + ex_2^2, \end{cases} \tag{4}$$

where $\alpha \in (0, 1]$ $(i = 1, 2, 3)$ is the fractional order.

When $2c - a \geq 0$, one can easily get three equilibria of system (4): $P_0(0,0,0)$, $P_1(\sqrt{b(2c-a)/de}, \sqrt{b(2c-a)/de}, (2c-a)/d)$, $P_2(-\sqrt{b(2c-a)/de}, -\sqrt{b(2c-a)/de}, (2c-a)/d)$.

With the aid of Theorem 1, a necessary condition for the fractional-order systems (4) to remain chaotic is keeping at least one eigenvalue λ^* in the unstable region. This means

$$\alpha > \frac{2}{\pi} a \tan\left(\frac{|\mathrm{Im}(\lambda^*)|}{\mathrm{Re}(\lambda^*)}\right). \tag{5}$$

When $a = 38$, $b = 3$, $c = 30$, $d = 1$, $e = 1$, the equilibria of system (4) and the corresponding eigenvalues are

$$P_0(0,0,0): \quad \lambda_1 = -33.189, \quad \lambda_2 = 25.189, \quad \lambda_3 = -3$$

$$P_{1,2}(\pm 8.124, \pm 8.124, 22): \quad \lambda_1 = -17.9176, \quad \lambda_2 = 3.4588 + 16.3702i,$$
$$\lambda_3 = 3.4588 - 16.3702i.$$

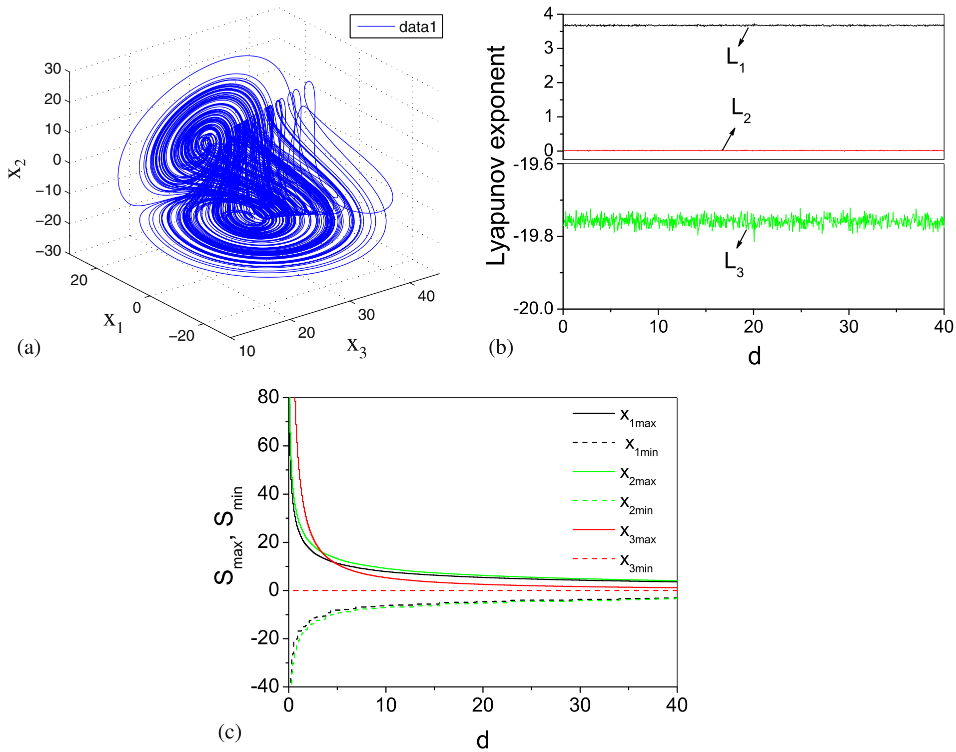

Figure 1. **(a)** Chaotic attractor, **(b)** Lyapunov exponent spectrum vs. d, **(c)** signal amplitude curve vs. d of system (1).

Therefore, when $a = 38$, $b = 3$, $c = 30$, $d = 1$, $e = 1$, all the three equilibria are unstable. According to the necessary condition (5), it follows that $\alpha > 0.8674$. It means that when $\alpha > 0.8674$, system (4) with commensurate fractional-order and the above parameters exhibits a chaotic behaviour, which is useful for further numerical simulation.

For the purpose of numerical simulations, let $a = 38$, $b = 3$, $c = 30$, $d = 1$, $e = 1$ and $\alpha = 0.92$, to ensure chaotic motion. The 3D phase diagram and time series are shown in figure 2 which reveals chaotic behaviour.

4. Adaptive control of the fractional-order chaotic system

In this section, before proposing an adaptive-feedback control method for fractional-order system, we first introduce a stability theory of fractional-order dynamic systems.

4.1 *Stability theory for fractional-order dynamic system*

Theorem 2. *Consider the following fractional-order dynamic system*:

$$D^{\alpha} X = F(X) = A(X)X \tag{6}$$

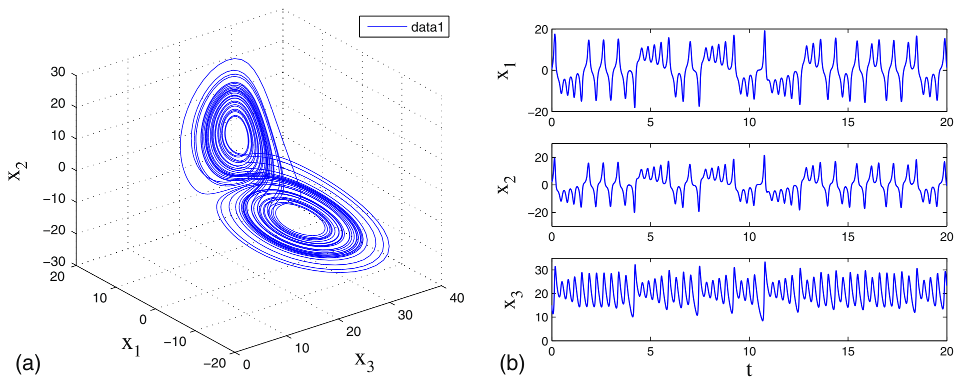

Figure 2. Chaotic attractor and time series of fractional-order system (4): **(a)** Chaotic attractor, **(b)** time series.

with $X \in R^m$, $A(X) \in R^{m\times m}$, $\alpha \in (0, 1]$. *If there exists* $B^T = B > 0$, *such that*

$$\Pi = X^T \cdot B \cdot D^\alpha X \le 0 \tag{7}$$

then system (6) *is asymptotically stable.*

Proof. When $\alpha = 1$, this is the case of stability for integer-order dynamic system, the conclusion is evident.

When $0 < \alpha < 1$, let λ^* be one of the eigenvalues of $A(X)$, $\xi \in R^m$ is the corresponding nonzero eigenvector. Then one obtain

$$A(X)\xi = \lambda^* \xi \tag{8}$$

and

$$\xi^H A(X)^H = \overline{\lambda^*}\xi^H. \tag{9}$$

Further, we have

$$\xi^H B A(X)\xi = \lambda^* \xi^H B\xi, \tag{10}$$

$$\xi^H A(X)^H B\xi = \overline{\lambda^*}\xi^H B\xi. \tag{11}$$

From eqs (10) and (11), it holds that

$$\xi^H\big(BA(X) + A(X)^H B\big)\xi = \big(\lambda^* + \overline{\lambda^*}\big)\,\xi^H B\xi. \tag{12}$$

Since $\Pi = X^T B A(X) X = 0.5X^T\big(BA(X) + A(X)^H B\big)X \le 0$, we have the following inequality:

$$\lambda^* + \overline{\lambda^*} \le 0. \tag{13}$$

Therefore, the condition $|\arg(\lambda^*)| \ge \pi/2 > \alpha\pi/2$ is satisfied and according to Theorem 1, system (6) is asymptotically stable.

4.2 *Adaptive control scheme for the fractional-order chaotic system*

The controlled fractional-order dynamic system (4) is described in a compact form

$$D^{\alpha}X = F(X) + U_i, \quad \alpha \in (0, 1], \tag{14}$$

where U_i is the controller added to the ith term of the system equation.

The controller is designed as

$$U_i = -\kappa(x_i - x_{i^*}), \tag{15}$$

where κ is the control strength, whose amplitude is adjusted according to the on-line adaptive law

$$D^{\beta}\kappa = \mu(x_i - x_{i^*})^2, \tag{16}$$

where $X = (x_1, x_2, \ldots, x_m)^T$, $F(X) = (f_1(X), f_2(X), \ldots, f_m(X))^T$, $\mu > 0$, $\beta \in (0, 1]$, $X_* = (x_{1^*}, x_{2^*}, \ldots, x_{m^*})$ is the equilibrium of the uncontrolled system.

Assumption 1. Function $f_i(X)$ is smooth in equilibrium point X_*, and there is a positive constant ρ, such that

$$\|f_i(X) - f_i(X_*)\| = \|f_i(X)\| \leq \rho\|X - X_*\|_{\infty}, \tag{17}$$

where $\|X - X_*\|_{\infty}$ is the ∞-norm of $X - X_*$, i.e., $\|X - X_*\|_{\infty} = \max_j \|x_j - x_{*j}\|$, $j = 1, 2, \ldots, m$.

In implementing the adaptive control scheme, the control strength κ is regarded as a variable (resembling the state variable $x_i, i = 1, 2, \ldots, m$). So, for systems (14) and (16), we define an augmented error vector as below:

$$E^T = (X - X_*, \kappa - \kappa_0) = (x_1 - x_{1^*}, x_2 - x_{2^*}, \ldots, x_m - x_{m^*}, \kappa - \kappa_0)$$

with $\kappa_0 \geq m\rho$. For the augmented systems (14) and (16), the candidate function Π can be selected as

$$\Pi = E^T \cdot B \cdot D^{\gamma} E \leq 0,$$

where $D^{\gamma} = (D^{\alpha}, D^{\alpha}, \ldots, D^{\alpha}, D^{\beta})$, and the operator B is selected as

$$B = \begin{bmatrix} 1 & 0 \cdots 0 & 0 \\ 0 & 1 \cdots 0 & 0 \\ \vdots & & \\ 0 & 0 \cdots 1 & 0 \\ 0 & 0 \cdots 0 & 1/\mu \end{bmatrix}.$$

Then, we have

$$\Pi = (X - X_*)^T D^{\alpha}(X - X_*) + \frac{(\kappa - \kappa_0)}{\mu} D^{\beta}(\kappa - \kappa_0).$$

Considering the Lipschitz condition (17) yields

$$\begin{aligned} \Pi &= (X - X_*)^T [F(X) - \kappa(x_i - x_{i^*})] + (\kappa - \kappa_0)(x_i - x_{i^*})^2 \\ &= (X - X_*)^T F(X) - \kappa(x_i - x_{i^*})^2 + (\kappa - \kappa_0)(x_i - x_{i^*})^2 \\ &= (X - X_*)^T F(X) - \kappa_0(x_i - x_{i^*})^2 \\ &\leq (m\rho - \kappa_0)\|X - X_*\|_{\infty}^2 \\ &\leq 0. \end{aligned} \tag{18}$$

Therefore, according to Theorem 2, systems (14) and (16) are asymptotically stable, that is, X tends to X_* and κ tends to κ_0 as $t \to \infty$.

Remark 1. The proposed scheme has the simplest expression for the controller and the adaptive updated law. And this scheme is flexible which can be added to any term of the system equation. What is more, β can have any value from 0 to 1.

4.3 *Numerical simulation*

The controller is added to the second term of eq. (4), which is described as

$$\begin{cases} D^{\alpha}x_1 = a(x_2 - x_1), \\ D^{\alpha}x_2 = (c-a)x_1 + cx_2 - dx_1x_3 - \kappa(x_2 - x_{2^*}), \\ D^{\alpha}x_3 = -bx_3 + ex_2^2, \\ D^{\beta}\kappa = \mu(x_2 - x_{2^*})^2 \,. \end{cases} \tag{19}$$

In the numerical simulation, the parameters are $a = 38$, $b = 3$, $c = 30$, $d = 1$, $e = 1$ and $\alpha = 0.92$, $\beta = 0.3$, $\mu = 5$, respectively; the initial conditions for system (19) are set to $x_1(0) = 0.5$, $x_2(0) = 0.2$, $x_3(0) = 15$, $\mu(0) = 0.1$; x_{2^*} is selected as 0, 8.124, -8.124,

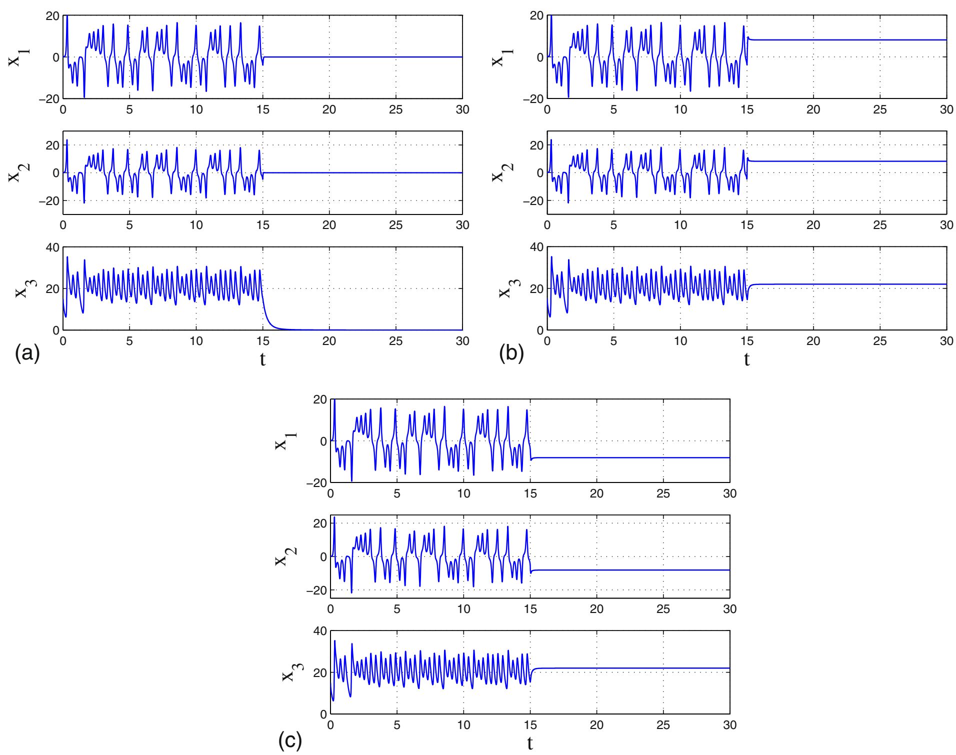

Figure 3. The trajectories of the controlled system eq. (19) stabilizing equilibrium point: **(a)** P_0, **(b)** P_1, **(c)** P_2.

respectively; the control is active when $t \geq 15$ s. The time response of the states of controlled system is illustrated in figure 3. As can be seen from the figure, the state of system (19) reaches the equilibrium point.

5. Adaptive synchronization of the fractional-order chaotic system

5.1 *Adaptive synchronization scheme for the fractional-order chaotic system*

In order to observe the synchronization between the two identical fractional-order chaotic systems, we assume that the controlled drive system is given as (14). The response system is expressed by

$$D^{\alpha} Y = F(Y), \quad \alpha \in (0, 1]. \tag{20}$$

The aim is to choose a suitable scheme such that the states of the drive and response systems are synchronized, i.e., $\lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \|X - Y\| = 0$.

By subtracting (14) from (20), we have

$$D^{\alpha} e = F(X) - F(Y) + U_i. \tag{21}$$

Choose the adaptive-feedback control scheme as

$$U_i = -\kappa(x_i - y_i), \tag{22}$$

$$D^{\beta}\kappa = \mu(x_i - y_i)^2. \tag{23}$$

Considering a candidate function

$$\Pi = e^T D^{\alpha} e + \frac{(\kappa - \kappa_0)}{\mu} D^{\beta} (\kappa - \kappa_0), \tag{24}$$

yielding

$$\begin{aligned} \Pi &= e^T [F(X) - F(Y) - \kappa e_i] + (\kappa - \kappa_0) e_i^2 \\ &\leq m\rho \|e\|^2 - \kappa e_i^2 + (\kappa - \kappa_0) e_i^2 \\ &= (m\rho - \kappa_0) \|e\|_{\infty}^2 \\ &\leq 0. \end{aligned}$$

Therefore, according to Theorem 2, systems (21) and (23) are asymptotically stable, that is, X tends to Y and κ tends to κ_0 as $t \to \infty$.

5.2 *Numerical simulation*

The controller is added to the second term of eq. (4), which is described as

$$\begin{cases} D^{\alpha} x_1 = a(x_2 - x_1), \\ D^{\alpha} x_2 = (c - a)x_1 + cx_2 - dx_1x_3 - \kappa(x_2 - y_2), \\ D^{\alpha} x_3 = -bx_3 + ex_2^2, \\ D^{\beta}\kappa = \mu(x_2 - y_2)^2. \end{cases} \tag{25}$$

Let $a = 38$, $b = 3$, $c = 30$, $d = 1$, $e = 1$ and $\alpha_1 = 0.92$, $\alpha_2 = 0.88$, $\alpha_3 = 0.95$, $\mu = 5$, respectively; but β is set to 0.8; the initial conditions for the drive system and response

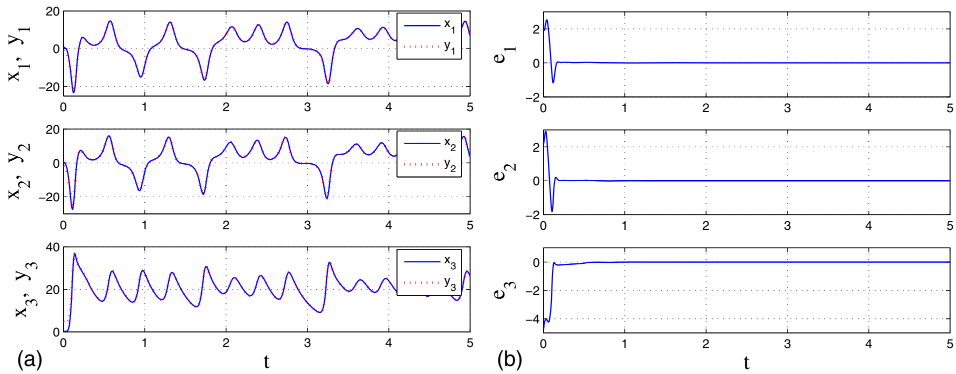

Figure 4. Synchronization result: **(a)** time response of the states, **(b)** synchronization error.

system are set to $x_1(0) = 1$, $x_2(0) = 0.2$, $x_3(0) = 0.3$, $y_1(0) = -1$, $y_2(0) = -2$, $y_3(0) = 5$. The curves of time response of the states and synchronization error are shown in figure 4, indicating that the chaotic systems (14) and (20) are synchronized.

6. Conclusion

Chaos is undesirable in many applications, and therefore it is necessary to control the chaotic behaviours. Synchronization of chaotic systems is regarded to have both theory and utility value in secure communication since the pioneering work carried out by Pecora and Carroll.

The control and synchronization of classical integer-order chaotic system have been deeply and fully investigated in recent decades. Compared to integer-order systems, fractional calculus can provide an excellent instrument for describing memory and hereditary properties of various materials and processes. Recently, study on control and synchronization of fractional-order dynamic systems has become an active research field. However, since the stability theory and method of fractional calculus are far from fully developed, this issue still remains challenging and open.

In this paper, we investigated the dynamic behaviours of a new fractional-order chaotic system. The necessary condition for the fractional-order chaotic system to remain chaotic is derived. Adaptive-feedback methods are introduced for achieving control and synchronization of this fractional-order system. The proposed controller, which only contains a single-state variable, to our knowledge, is the simplest scheme for control and synchronization of fractional-order chaotic system. Moreover, the control scheme is flexible, its design is simple and suitable for implementation. Simulation results were presented to illustrate the effectiveness of the obtained control strategies.

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