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Symmetries and casimir of an extended classical long wave system

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Abstract. In this paper, we derive Lie point, generalized, master and time-dependent symmetries of a dispersionless equation, which is an extension of a classical long wave system. This equation also admits an infinite-dimensional Lie algebraic structure of Virasoro-type, as in the dispersive integrable systems. We discuss the construction of a sequence of negative ranking symmetries through the property of uniformity in rank. More interestingly, we obtain the conserved quantities directly from the casimir of Poisson pencil.

Keywords. Dispersionless equations; symmetries; casimir; conserved quantities.

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1. Introduction

In recent years, dispersionless equations attracted a lot of attention, because they naturally appear in many physical problems [1–5]. Hence, it is very important to investigate various integrability properties of this class of dispersionless equations. Among the dispersionless equations, Benney system [1] is a ubiquitous one. This equation has strong physical reasons and rich mathematical structures. Also, it is of recent interest to obtain many integrable systems from Benney system under various reductions and the study of these reduced systems also assumes significance [6,7]. It is well known that Benney system [1]

$$u_t = uu_x + gh_x - u_y \int_0^h u_x \,\mathrm{d}y, \qquad h_t = \left(\int_0^h u \,\mathrm{d}y\right)_x, \tag{1.1}$$

describes long waves on the surface of two-dimensional shallow incompressible fluid, where $u = u(x, y, t)$ is the horizontal component of velocity, $h = h(x, t)$ is the free surface over the flat bottom $\{y = 0\}$, x and y are the horizontal and vertical coordinates, respectively, $0 \le y \le h$, t is the (minus) time coordinate and g is the gravitational constant. The subscripts of u and h with respect to x, y and t denote partial derivatives.

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For moments of u, $A_n(x,t) = \int_0^h u^n \mathrm{d}y, \forall\, n \in Z_{\geq 0}$, the system (1.1) implies an autonomous evolution equation,

$$A_{n,t} = A_{n+1,x} + gnA_{n-1}A_{0,x}, \qquad \forall\, n \in Z_{\geq 0}. \tag{1.2}$$

The above infinite-component system is irreducible, i.e., there exist no finite-component subsystems of (1.2). To overcome this difficulty, Kupershmidt [8] has extended the equations of motion of the Benney system from which one can obtain a finite subsystem of (1.2) given by

$$\begin{aligned} A_{n,t} &= \frac{1}{2}v^n A_{1,x} + \frac{1}{2}(nA_n v_x + (A_n v)_x) + gnA_{n-1}A_{0,x}, \qquad \forall\, n \in Z_{\geq 0}, \\ v_t &= vv_x + gA_{0,x}, \end{aligned} \tag{1.3}$$

where $v = v(x,t) = u(x,y,t)\,|_h = u(x,h(x,t),t)$, is the velocity on the surface.

Consider a 3-by-3 subsystem of (1.3) for $n = 0$ and $n = 1$ and introducing the variable transform $E = (A_1 - hv)^{1/2}$, we get [8]

$$v_t = vv_x + gh_x, \quad h_t = (hv)_x + EE_x, \quad E_t = \frac{1}{2}(vE)_x. \tag{1.4}$$

Note that, eq. (1.4) is the genuine extension of classical long wave system. By exhibiting the bi-Hamiltonian structure for (1.4), Kupershmidt [8] has proved that (1.4) is integrable.

The paper is organized as follows. In §2, we study Lie point symmetry for (1.4) and its various similarity reductions. In §3, we derive various symmetries of (1.4), including generalized, master and time-dependant symmetries. Our analysis reveals a rich mathematical structure analogous to soliton equations for this systems as well. In §4, by exploiting uniformity in rank approach, we present negative ranking generalized symmetries of (1.4). In §5, we construct casimir of Poisson pencil for (1.4), a closed form formula to derive all conserved densities.

2. Lie point symmetries

Application of Lie symmetry analysis to integro-differential equations or infinite systems of differential equations encounters difficulties in obtaining the overdetermined linear equations for infinitesimal generators of symmetry group [9,10]. Therefore, one needs to consider the local form of a system for our analysis. In this paper, we consider a 3-by-3 subsystem (1.4) which is a reduction of the extended equations of moments chain of Benney system (1.3). Now, the reduced system is expressed in terms of local variables. This system shares many common properties with Benney system [8]. Using this approach [9–12], we obtain a four-dimensional Lie algebra of symmetry vector fields for (1.4),

$$\begin{aligned} V_1 &= \partial_t, \\ V_2 &= \partial_x, \\ V_3 &= -x\partial_x - 2t\partial_t + v\partial_v + 2h\partial_h + \frac{3E}{2}\partial_E, \\ V_4 &= -10t\partial_x + 12\partial_v - \left(\frac{2v}{g}\right)\partial_h + \left(\frac{v^2 - 4gh}{Eg}\right)\partial_E. \end{aligned} \tag{2.1}$$

The non-vanishing commutators of (2.1) are

$$[V_1, V_3] = -2V_2, \quad [V_1, V_4] = -10V_2, \quad [V_2, V_3] = -V_2, \quad [V_3, V_4] = -V_4.$$

Next, we consider the similarity reductions of (1.4) under various choices of vector fields (2.1). Let us first consider the vector field V_3 and the corresponding characteristic equations can be written as

$$\frac{\mathrm{d}x}{-x} = \frac{\mathrm{d}t}{-2t} = \frac{\mathrm{d}v}{v} = \frac{\mathrm{d}h}{2h} = \frac{\mathrm{d}E}{3E/2}. \tag{2.2}$$

From (2.2), we get the similarity variable and similarity functions

$$\zeta = \frac{x}{\sqrt{t}}, \quad \tilde{V}(\zeta) = v\sqrt{t}, \quad \tilde{H}(\zeta) = ht, \quad \tilde{E}(\zeta) = Et^{3/4}. \tag{2.3}$$

Using the above transformations in (1.4), we get the similarity reduced equations

$$\begin{aligned}
&\zeta \tilde{V}_\zeta + 2g\tilde{H}_\zeta + 2\tilde{V}\tilde{V}_\zeta + \tilde{V} = 0,\\
&2\tilde{E}\tilde{E}_\zeta + 2(\tilde{V}\tilde{H})_\zeta + 2\tilde{H} + \zeta\tilde{H}_\zeta = 0,\\
&3\tilde{E} + 2\zeta\tilde{E}_\zeta + 2(\tilde{V}\tilde{E})_\zeta = 0.
\end{aligned} \tag{2.4}$$

Next, we consider the linear combination of vector fields V_1 and V_4,

$$V = V_1 + V_4 = \partial_t - 10t\partial_x + 12\partial_v - \frac{2v}{g}\partial_h + \left(\frac{v^2 - 4gh}{Eg}\right)\partial_E. \tag{2.5}$$

The corresponding similarity variable and similarity functions are given by

$$\begin{aligned}
&\zeta = x + 5t^2, \quad \tilde{V}(\zeta) = v - 12t, \quad \tilde{H}(\zeta) = gh + 12t^2 + 2\tilde{V}(\zeta)t,\\
&\tilde{E}(\zeta) = \frac{gE^2}{2} - 64t^3 - 16t^2\tilde{V}(\zeta) - t\tilde{V}(\zeta)^2 + 4t\tilde{H}(\zeta),
\end{aligned} \tag{2.6}$$

and the similarity reduced equations are

$$\begin{aligned}
&\tilde{V}\tilde{V}_\zeta + \tilde{H}_\zeta - 12 = 0,\\
&2\tilde{V} + (\tilde{V}\tilde{H})_\zeta + \tilde{E}_\zeta = 0,\\
&\tilde{V}^2 - 4\tilde{H} - \frac{1}{2}\tilde{E}_\zeta\tilde{V} - \tilde{V}_\zeta\tilde{E} = 0.
\end{aligned} \tag{2.7}$$

It is important to note that (2.4) and (2.7) are not having the Painlevé property [13].

3. Generalized, master and time-dependant symmetries

The existence of infinitely many commuting generalized symmetries and bi-Hamiltonian structures are important properties for integrable evolution equations [12,14,15]. The study of these along with master symmetries for dispersionless equations assumes significance. In this section, we briefly recall some basic facts. Using them, we derive new results concerning the generalized, master and time-dependent symmetries.

Consider an evolution equation $u_t = K(u)$, where $K(u)$ is a nonlinear function which involves u and its derivatives. $\sigma(u)$ is called a generalized symmetry if it satisfies the linearized equation [11,12,14–16],

$$D_t\sigma = K'(u)[\sigma], \tag{3.1}$$

where K' is the Fréchet derivative of K defined by

$$K'(u)[\sigma] = \frac{\partial}{\partial\epsilon}K(u+\epsilon\sigma)|_{\epsilon=0} \tag{3.2}$$

and D_t is the total derivative with respect to t. An operator valued function Φ is called a recursion operator if

$$\Phi'[K] - K'\Phi + \Phi K' = 0. \tag{3.3}$$

For the given evolution equation if Φ is constructed then infinitely many generalized symmetries follow from the recurrence relation,

$$K_{m+n} = \Phi^n K_m. \tag{3.4}$$

The bi-Hamiltonian structure for (1.4) has been derived by Kupershmidt [8] with Hamiltonian operators,

$$B_1 = \begin{pmatrix} 0 & \partial & 0 \\ \partial & 0 & 0 \\ 0 & 0 & \frac{1}{2}\partial \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 2g\partial & \partial v & 0 \\ v\partial & h\partial + \partial h & E\partial \\ 0 & \partial E & 0 \end{pmatrix}. \tag{3.5}$$

Now, the recursion operator [8,14–16] is defined in terms of B_1 and B_2 as

$$\Phi = B_2 B_1^{-1}. \tag{3.6}$$

From eqs (3.5) and (3.6), we get

$$\Phi = \begin{pmatrix} \partial v\partial^{-1} & 2g & 0 \\ h + \partial h\partial^{-1} & v & 2E \\ \partial E\partial^{-1} & 0 & 0 \end{pmatrix}, \tag{3.7}$$

where

$$\partial = \frac{\partial}{\partial x} \quad \text{and} \quad \partial\partial^{-1} = \partial^{-1}\partial = 1.$$

Using eq. (3.7), the first few generalized symmetries of (1.4) are listed as follows:

$$K_0 = \begin{pmatrix} v_x \\ h_x \\ E_x \end{pmatrix}, \quad K_1 = \begin{pmatrix} vv_x + gh_x \\ (hv)_x + EE_x \\ \frac{1}{2}(vE)_x \end{pmatrix},$$

$$K_2 = \begin{pmatrix} 2gEE_x + 3g(vh)_x + \frac{3}{2}v^2 v_x \\ 3ghh_x + \frac{3}{2}(v^2 h)_x + (E^2 v)_x \\ g(hE)_x + \frac{1}{2}(v^2 E)_x \end{pmatrix}. \tag{3.8}$$

A vector field $T(u)$ is called a master symmetry [16] of $u_t = K(u)$ if

$$[K, [K, T]] = 0,$$

where the commutator of two vector fields is defined by

$$[A(u), B(u)] = A'[B] - B'[A].$$

One can observe that the sequence of master symmetries T_n can be obtained by the action of a recursion operator on seed symmetry T_0 given by

$$T_0 = \begin{pmatrix} xv_x + v \\ xh_x + 2h \\ xE_x + \frac{3}{2}E \end{pmatrix}. \tag{3.9}$$

The higher-order master symmetries can be obtained from the repeated action of the recursion operator by the following rule:

$$T_n = \Phi^n T_0, \qquad n \in Z_{\geq 0}.$$

We give the first nontrivial master symmetry,

$$T_1 = \begin{pmatrix} 2xvv_x + v^2 + 2xgh_x + 4gh \\ 2x(hv)_x + 4hv + 2xEE_x + 3E^2 \\ (xvE)_x \end{pmatrix}.$$

From the above generalized and master symmetries, we found that the following algebraic structure holds:

$$[K, T_0] = 2K,$$

$$\Phi'[T_0] + \Phi[T_0'] - T_0'[\Phi] = \Phi, \tag{3.10}$$

$$[K_n, K_m] = 0,$$

$$[K_m,\ T_n] = (1+m)K_{n+m},$$

$$[T_m,\ T_n] = (m-n)T_{n+m}, \tag{3.11}$$

for $n, m \in Z_{\geq 0}$. From eq. (3.11), it is clear that system (1.4) constitute an infinite-dimensional symmetry Lie algebra of the Virasoro-type [17]. In addition, it is possible to define a time-dependent symmetry of (1.4) as

$$\Sigma_n = (1+n)tK_n + \Phi^n T_0. \tag{3.12}$$

4. Negative ranking symmetries

Having obtained Lie point and various symmetries of (1.4), in this section, we attempt to explore further for the existence of other types of symmetries. It is well known that to generate higher-order generalized symmetries one can use the recurrence relation (3.4). In principle, the inverse of the recursion operator should generate non-local/negative ranking generalized symmetries. But it is known that getting these negative ranking symmetries using the recursion operator is quite difficult. On the other hand, getting negative ranking generalized symmetries using uniform rank approach is straightforward. As a first step, one needs to consider a set of monomials with uniform rank property [18,19]. Here, we would like to emphasize that this investigation lead us to find a new sequence of negative ranking generalized symmetries, which are expressed by local quantities. This may be a property for dispersionless equations. It is worth recalling that in general, for soliton systems negative ranking generalized symmetries used to be non-local. To determine the rank for this class of generalized symmetries one needs to know the weights associated with the independent and dependent variables. From the vector field V_3, we arrive at the following scaling transformation of (1.4):

$$(x, t, v, h, E) \to (\lambda^{-1}x, \lambda^{-2}t, \lambda^{1}v, \lambda^{2}h, \lambda^{3/2}E). \tag{4.1}$$

Assigning weights (denoted by ω) to the variables based on the exponents in λ and setting $\omega(\partial_x) = 1$ (or equivalently, $\omega(x) = \omega(\partial_x^{-1}) = -1$) gives $\omega(v) = 1,\ \omega(h) = 2,\ \omega(E) = \frac{3}{2}$ and $\omega(t) = -2$ (or $\omega(\partial_t) = 2$). The rank of a monomial [18] is defined as the total weight of the monomial and uniform rank of an equation is defined as having uniform rank in each term of the equation.

Having these results, now we apply the uniform rank condition for finding generalized symmetries. If K_j is the generalized symmetry of (1.4), then the rank of the symmetry for each component can be written as

$$(r^{(1)}, r^{(2)}, r^{(3)})^T \sim (K_j^{(1)}, K_j^{(2)}, K_j^{(3)})^T = K_j,$$
$$r^{(i)} \in \mathbb{Q}, \qquad i = 1, 2, 3, \ldots, \qquad j = 0, 2, 3, \ldots,$$

where $K_j^{(i)}$ is the ith symmetry component and $r_j^{(i)}$ is the uniformity in rank of $K_j^{(i)}$.

Using the weights of various quantities defined earlier, one can easily obtain the ranks of the sequence of generalized symmetries of (1.4). We list below the ranks corresponding to the generalized symmetries:

$$\left(2, 3, \frac{5}{2}\right)^T \sim K_0,$$

$$\left(3, 4, \frac{7}{2}\right)^T \sim K_1,$$

$$\vdots \qquad \vdots$$

$$\left(n+2, n+3, n+\frac{5}{2}\right)^T \sim K_n, \qquad \forall\, n \in Z_{\geq 0}. \tag{4.2}$$

Clearly, the ranks of the generalized symmetries form strictly increasing sequence componentwise having positive values. However, it is interesting to obtain the symmetries with negative ranks. The essential idea behind the construction of negative ranking symmetries is to look for a sequence of symmetries, of which the ranks form decreasing sequence having negative values.

Based on the uniformity in rank, we consider the sets of elements in the field variables and their derivatives. Due to the presence of rational rank for the generalized symmetries in (4.2), we need to include rational form of monomials in order to derive negative ranking generalized symmetry. Hence, consider all monomials component-wise with rank $\frac{-1}{2}$, $\frac{1}{2}$ and 0 as the rank of the starting negative ranking generalized symmetry. Thus, we define the following sets of monomials:

$$\mathcal{R}_{-1/2} = \left\{\frac{h}{vE}, \frac{E_x}{h_x}, \frac{E_x}{E^2}, \frac{E}{h}, \frac{v}{E}, \frac{hE}{vh_x}, \frac{E}{v_x}, \frac{vE}{h_x}, \partial^{-1}\frac{E}{v}, \partial^{-1}\frac{Ev}{h},\right.$$
$$\left.\partial^{-1}\frac{E_x}{h}, \partial^{-1}\frac{E_x}{v^2}, \cdots\right\},$$

$$\mathcal{R}_{1/2} = \left\{\frac{E}{v}, \frac{E_x}{v^2}, \frac{E_x v}{E^2}, \frac{h}{E}, \frac{v^2}{E}, \frac{v_x}{E}, \frac{hv}{E_x}, \frac{E^2}{E_x}, \frac{v^3}{E_x}, \frac{Ev}{h}, \partial^{-1}E,\right.$$
$$\left.\partial^{-1}\frac{E_x}{v}, \partial^{-1}\frac{vE_x}{h}, \cdots\right\},$$

$$\mathcal{R}_0 = \left\{1, \frac{v^2}{h}, \frac{E^2}{hv}, \frac{h}{v^2}, \frac{E_x}{Ev}, \frac{Ev}{E_x}, \frac{E_x v}{hE}, \frac{hE}{E_x v}, \frac{v^3}{E^2}, \frac{E_x v^2}{E^3}, \frac{h_x}{E^2},\right.$$
$$\left.\frac{E^3}{E_x v^2}, \frac{E^2}{h_x}, \frac{vv_x}{E^2}, \frac{E_x h}{E^3}, \partial^{-1}v, \partial^{-1}\frac{h}{v}, \partial^{-1}\frac{E^2}{h}, \partial^{-1}\frac{E_x}{Ev}, \cdots\right\}, \tag{4.3}$$

where the subscripts of the sets indicate the ranks of elements in the concerned sets.

Let us construct $\tilde{S}_0 = (\tilde{S}_0^{(1)}, \tilde{S}_0^{(2)}, \tilde{S}_0^{(3)})^T$, where $\tilde{S}_0^{(1)}$, $\tilde{S}_0^{(2)}$ and $\tilde{S}_0^{(3)}$ are the linear combination of monomials from the sets $\mathcal{R}_{-1/2}$, $\mathcal{R}_{1/2}$ and $\mathcal{R}_0$ respectively. Substituting $\tilde{S}_0$ in the symmetry condition (3.1) and $K(u)$ from (1.4), we obtain a system of equations for the coefficients of constants. Solving them consistently, we finally get the generalized symmetry

$$\tilde{S}_0 = \begin{pmatrix} \tilde{S}_0^{(1)} \\ \tilde{S}_0^{(2)} \\ \tilde{S}_0^{(3)} \end{pmatrix} = \begin{pmatrix} -\dfrac{E_x}{E^2} \\ \dfrac{E_x v}{2gE^2} - \dfrac{v_x}{2gE} \\ \dfrac{E_x h}{E^3} - \dfrac{E_x v^2}{4gE^3} - \dfrac{h_x}{2E^2} + \dfrac{v v_x}{4E^2 g} \end{pmatrix}. \tag{4.4}$$

Repeating the same procedure for the sets $\mathcal{R}_{-3/2}$, $\mathcal{R}_{-1/2}$ and $\mathcal{R}_{-1}$, we get

$$\tilde{S}_1 = \begin{pmatrix} \tilde{S}_1^{(1)} \\ \tilde{S}_1^{(2)} \\ \tilde{S}_1^{(3)} \end{pmatrix} = \begin{pmatrix} f_x \\ -\dfrac{E_x}{2gE^2} - \dfrac{1}{2g}(vf)_x \\ \dfrac{vE_x}{2gE^3} - \dfrac{v_x}{4gE^2} + \dfrac{v}{4gE}(vf)_x - \dfrac{h}{2E} f_x - \dfrac{1}{2E}(hf)_x \end{pmatrix}, \tag{4.5}$$

where

$$f = \frac{1}{8gE^3}(v^2 - 4gh).$$

From the discussion above, it is clear that by considering the sets $\mathcal{R}_{-n-(1/2)}$, $\mathcal{R}_{-n+(1/2)}$ and $\mathcal{R}_{-n}$, for $n = 0, 1, 2, \ldots$, we can derive a sequence of generalized symmetries $\{\tilde{S}_n\}_0^\infty$, which are in negative ranks. It is surprising to note that eq. (1.4) admits negative ranking generalized symmetries which are not involving non-local variables. But this is not the case for soliton equations, where non-local terms appear especially in negative ranking generalized symmetry.

5. Casimir of the Poisson pencil

In this section, our objective is to derive the conserved densities for eq. (1.4) through casimir. Dubrovin and Novikov [20,21], introduced Poisson brackets for the study of hydrodynamic equations:

$$\left\{u^i(x), u^j(y)\right\} = g^{ij}[u(x)]\delta'(x-y) + \Gamma_k^{ij}[u(x)]u_x^k(x)\delta(x-y) \tag{5.1}$$

with the non-degeneracy condition $\det(g^{ij}) \neq 0$. This equality defines a skew symmetrical Poisson bracket on the functionals,

$$\{I, J\} = \int \mathrm{d}x \frac{\delta I}{\delta u^i(x)} \widehat{A^{ij}} \frac{\delta J}{\delta u^j(x)}, \tag{5.2}$$

where

$$\widehat{A^{ij}} = g^{ij}[u(x)]\frac{\mathrm{d}}{\mathrm{d}x} + \Gamma_k^{ij}[u(x)]u_x^k(x) \tag{5.3}$$

is the so-called Hamiltonian operator (or) a Dubrovin–Novikov Hamiltonian operator and we write $g^{ij}(u)$ as an arbitrary contravarient flat pseudo-Riemannian metric. Also, $\Gamma_k^{ij}[u(x)] = -g^{is}\Gamma_{sk}^{j}$, where Γ_{sk}^{j} is the Christoffel symbol of the Riemannian connection defined by the metric $g^{ij}(u)$. Moreover, for a bi-Hamiltonian structure, the two Hamiltonian structures must be compatible, i.e., $\{.,.\} = \{.,.\}_1 - \lambda\{.,.\}_2$ must be the Hamiltonian structure for all values of λ. Here, $\{.,.\}_1$ and $\{.,.\}_2$ are two Poisson brackets corresponding to B_1 and B_2.

From the Dubrovin–Novikov theorem [20], for two non-degenerate Hamiltonian operators $\widehat{A_1^{ij}}$ and $\widehat{A_2^{ij}}$ generated by the respective flat contravarient metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$, the compatibility condition implies that for arbitrary λ,

(1) $g^{ij}(u) = g_1^{ij}(u) + \lambda g_2^{ij}(u)$ is a metric of flat pencil.
(2) The metric connection for this metric has the form $\Gamma_k^{ij} = \Gamma_{1k}^{ij} + \lambda\Gamma_{2k}^{ij}$.

The bracket $\{.,.\}_\lambda = \{.,.\}_1 - \lambda\{.,.\}_2$ defines a linear pencil of Poisson brackets, used to construct the casimir of Poisson pencil. Using eqs (3.5) and (5.1) we get

$$\begin{aligned}
\{v(x), v(y)\}_\lambda &= 2g\delta'(x-y),\\
\{v(x), h(y)\}_\lambda &= (v(x)-\lambda)\delta'(x-y) + v_x(x)\delta(x-y),\\
\{v(x), E(y)\}_\lambda &= 0,\\
\{h(x), v(y)\}_\lambda &= (v(x)-\lambda)\delta'(x-y),\\
\{h(x), h(y)\}_\lambda &= 2h(x)\delta'(x-y) + h_x(x)\delta(x-y),\\
\{h(x), E(y)\}_\lambda &= E(x)\delta'(x-y),\\
\{E(x), h(y)\}_\lambda &= E(x)\delta'(x-y) + E_x(x)\delta(x-y),\\
\{E(x), E(y)\}_\lambda &= -\frac{\lambda}{2}\delta'(x-y).
\end{aligned} \tag{5.4}$$

If $\mathcal{C}(v, h, E, \lambda)$ is a casimir, then by using eq. (5.4), we find

$$\begin{aligned}
\{v(x), \mathcal{C}\}_\lambda &= 2g\partial_x\mathcal{C}_v + (v-\lambda)\partial_x\mathcal{C}_h + v_x\mathcal{C}_h,\\
\{h(x), \mathcal{C}\}_\lambda &= (v-\lambda)\partial_x\mathcal{C}_v + 2h\partial_x\mathcal{C}_h + h_x\mathcal{C}_h + E\partial_x\mathcal{C}_E,\\
\{E(x), \mathcal{C}\}_\lambda &= E\partial_x\mathcal{C}_h + E_x\mathcal{C}_h - \frac{\lambda}{2}\partial_x\mathcal{C}_E.
\end{aligned} \tag{5.5}$$

It is immediate from the definition of casmir, the right-hand side of eq. (5.5) must vanish

$$2g\partial_x\mathcal{C}_v + (v-\lambda)\partial_x\mathcal{C}_h + v_x\mathcal{C}_h = 0, \tag{5.6}$$

$$(v-\lambda)\partial_x\mathcal{C}_v + 2h\partial_x\mathcal{C}_h + h_x\mathcal{C}_h + E\partial_x\mathcal{C}_E = 0, \tag{5.7}$$

$$E\partial_x\mathcal{C}_h + E_x\mathcal{C}_h - \frac{\lambda}{2}\partial_x\mathcal{C}_E = 0. \tag{5.8}$$

Consider eq. (5.7) and rewriting we get

$$(v-\lambda)\left(\mathcal{C}_{vv}v_x+\mathcal{C}_{vh}h_x+\mathcal{C}_{vE}E_x\right)+2h\left(\mathcal{C}_{vh}v_x+\mathcal{C}_{hh}h_x+\mathcal{C}_{hE}E_x\right)$$
$$+E(\mathcal{C}_{vE}v_x+\mathcal{C}_{hE}h_x+\mathcal{C}_{EE}E_x)+h_x\mathcal{C}_h=0. \quad (5.9)$$

Collecting the coefficients of v_x, h_x and E_x in the above equation and equating them to zero, we obtain

$$(v-\lambda)\mathcal{C}_{vv}+2h\mathcal{C}_{vh}+E\mathcal{C}_{vE}=0, \quad (5.10)$$

$$(v-\lambda)\mathcal{C}_{vh}+2h\mathcal{C}_{hh}+E\mathcal{C}_{hE}+\mathcal{C}_h=0, \quad (5.11)$$

$$(v-\lambda)\mathcal{C}_{vE}+2h\mathcal{C}_{hE}+E\mathcal{C}_{EE}=0. \quad (5.12)$$

Solving the above equations consistently, we finally arrive at

$$\mathcal{C}(v,h,E,\lambda)=\frac{2ag-b\lambda}{2g}v+cE+b\sqrt{\frac{4gE^2-\lambda[(\lambda-v)^2-4gh]}{g\lambda}}, \quad (5.13)$$

which is the casimir of the Poisson pencil with arbitrary integral constants a, b and c.

Expanding $\mathcal{C}(v,h,E,\lambda)$ by power series in λ, we obtain

$$\begin{aligned}\mathcal{C}(v,h,E,\lambda)=&-\frac{1}{2}(bv-2g\alpha)\,g^{-1}\lambda+av+cE\\ &-\alpha(v+2gh\lambda^{-1}+2g(E^2+vh)\lambda^{-2}\\ &+2g(vE^2+v^2h+gh^2)\lambda^{-3}\\ &+2g(v^2E^2+2gE^2h+v^3h+3vgh^2)\lambda^{-4}\\ &+2g(gE^4+v^3E^2+6gE^2vh+v^4h\\ &\quad+6v^2gh^2+2g^2h^3)\lambda^{-5}+\cdots),\end{aligned} \quad (5.14)$$

where

$$\alpha=b\sqrt{-\frac{1}{g}}.$$

Each coefficient of $\lambda^{-1}, \lambda^{-2}, \ldots$, in eq. (5.14) gives conserved densities for the system (1.4). We list the first few of them:

$$\begin{aligned}\mathcal{H}_0&=h,\\ \mathcal{H}_1&=E^2+vh,\\ \mathcal{H}_2&=vE^2+v^2h+gh^2,\\ \mathcal{H}_3&=v^2E^2+2gE^2h+v^3h+3vgh^2,\\ \mathcal{H}_4&=gE^4+v^3E^2+6gE^2vh+v^4h+6v^2gh^2+2g^2h^3,\\ \mathcal{H}_5&=3gE^4v+v^4E^2+12gE^2v^2h+v^5h+10v^3gh^2\\ &\quad+10vg^2h^3+6g^2E^2h^2.\end{aligned} \quad (5.15)$$

All the higher conserved densities can be easily derived from the casimir (5.15). In [8], Kupershmidt has derived conserved densities using a different approach.

6. Conclusions

In this paper, we have obtained a four-dimensional Lie group (2.1), admitted by the extended classical long wave system (1.4) [8]. Similarity reductions (2.4) and (2.7) of (1.4) under various choices of vector fields do not possess the Painlevé property [13]. System (1.4) also exhibits Virasoro Lie algebraic structure (3.11), through the construction of generalized, master and time-dependent symmetries. Furthermore, negative ranking generalized symmetries (4.4) and (4.5) of (1.4) do not contain non-local field variable whereas in soliton system they do have. These symmetries have been found by using the property of uniformity in rank [18,19]. Using Dubrovin and Novikov Poisson bracket [20,21], a simple and closed form of expression for casimir is obtained. From the casimir, we have derived conserved quantities of (1.4) straightforwardly. Similar analysis for higher component subsystem of (1.3) will be interesting. We shall report the results in our future publication.

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