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On the sharp front-type solution of the Nagumo equation with nonlinear diffusion and convection

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Abstract. This paper is concerned with the Nagumo equation with nonlinear degenerate diffusion and convection which arises in several problems of population dynamics, chemical reactions and others. A sharp front-type solution with a minimum speed to this model equation is analysed using different methods. One of the methods is to solve the travelling wave equations and compute an exact solution which describes the sharp travelling wavefront. The second method is to solve numerically an initial-moving boundary-value problem for the partial differential equation and obtain an approximation for this sharp front-type solution.

Keywords. Nagumo equation; nonlinear degenerate diffusion; nonlinear convection; sharp wavefront solution; minimum wave speed.

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1. Introduction

The Nagumo equation with linear or nonlinear diffusion and convection has extensively been applied to population dynamics, ecology, neurophysiology, chemical reactions and flame propagation [1–3]. In particular, the case where the equation involves degenerate nonlinear diffusion is of considerable interest [4–8]. In this case, a travelling wavefront solution of sharp type is known to exist for exactly one value of the wave speed. Such wavefronts, for instance, represent collective motion of populations in particular collective cell spreading, invasion in ecology and concentration in chemical reactions. This paper concerns these wavefronts in the Nagumo equation with general nonlinear degenerate diffusion and convection which reads as follows:

$$\frac{\partial u}{\partial t} + \beta u^n \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\alpha u^n \frac{\partial u}{\partial x}\right) + \gamma u(1-u^m)(u^m-\delta), \tag{1}$$

where $\alpha, \beta, \gamma > 0$, and $m, n > 0$ are constants and $0 < \delta < 1$. The unknown (density or concentration) $u(x, t)$ is non-negative, x is a space coordinate, and t denotes time. This eq. (1), with one or more terms omitted, is found in many other areas of application and is connected to the porous media equation.

Here we present a computation of the travelling wavefront solution of the sharp type for eq. (1) using different methods. This paper is organized as follows. In §2, an exact solution to eq. (1) is computed in the form of sharp wavefront with a minimum speed by solving the travelling wave equations. In §3, an initial-moving boundary-value problem for the partial differential equation is solved numerically and an approximation to this sharp travelling wavefront solution is calculated. Section 4 provides the conclusion.

2. Travelling wave problem: Exact solution

In such model equation (1), a travelling wavefront of sharp type [9] $u(x, t) = \theta(x - ct) = \theta(\zeta)$ connecting two equilibrium states $\theta = 1$ and $\theta = 0$ is known to exist for exactly one value of the wave speed c, say c^*. This sharp travelling wavefront is the solution of the ordinary differential equation

$$-c\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} + \beta\theta^n\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} = \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\alpha\theta^n\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\right) + \gamma\theta(1-\theta^m)(\theta^m - \delta) \tag{2}$$

with the boundary conditions which are the equilibrium states. To find the form of this sharp travelling wavefront, we first rewrite (2) as follows:

$$\begin{aligned}-c\theta^{n-1}\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} &+ \beta\theta^n\theta^{n-1}\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} \\ &= \alpha\theta^n\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\theta^{n-1}\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\right) + \alpha\left(\theta^{n-1}\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\right)^2 + \gamma\theta^n(1-\theta^m)(\theta^m-\delta).\end{aligned} \tag{3}$$

Next, we make the following relevant nonlinear transformation:

$$\theta^{2m-1}\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} = A(1-\theta^m)(\theta^m - \delta), \tag{4}$$

where we set $n = 2m$. Substituting eq. (4) into eq. (3) and taking all the coefficients of $\theta^i\,(i = 0, m, 2m, 3m, 4m)$ to be zero, yield

$$\begin{aligned}&c\delta A = \alpha\delta^2A^2,\\ &-c(1+\delta)A = -\alpha\delta(1+\delta)mA^2 - 2\alpha\delta(1+\delta)A^2\\ &cA = \alpha(1+4\delta+\delta^2)mA^2 + \alpha(1+4\delta+\delta^2)A^2 + \beta\delta A - \gamma\delta\\ &0 = -3\alpha(1+\delta)mA^2 - 2\alpha\delta(1+\delta)A^2 - \beta(1+\delta)A + \gamma(1+\delta)\\ &0 = 2\alpha mA^2 + \alpha A^2 + \beta A - \gamma.\end{aligned}$$

These equations are solved to give

$$A = \frac{-\beta \pm \sqrt{\beta^2 + 4(2m+1)\alpha\gamma}}{2(2m+1)\alpha}, \tag{5}$$

and

$$c = \frac{-\beta\delta \pm \sqrt{\beta^2\delta^2 + 4(2m+1)\alpha\gamma\delta^2}}{2(2m+1)}, \tag{6}$$

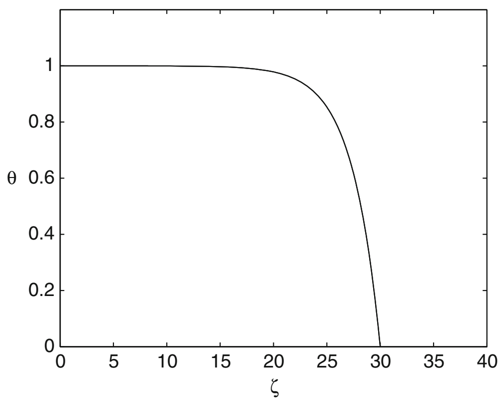

Figure 1. Graph of the resulting solution in eq. (7).

which gives the value of the minimum speed c^*. Solving eq. (4), we then obtain the following exact implicit solution:

$$\ln\left|\frac{\theta^m - 1}{(1-(\theta^m/\delta))^\delta}\right| = -\frac{(1-\delta)mc^*}{\alpha\gamma}(\zeta - \zeta^*) \tag{7}$$

to eq. (3) in the form of sharp front in travelling wave coordinates. This sharp front with the minimum speed c^* given by eq. (6), moves to the right or to the left according to $c > 0$ or $c < 0$, respectively.

In figure 1, we show a graph of the solution in eq. (7) with the minimum speed $c^* = 0.3838$, given by eq. (6) for $\alpha = \beta = \gamma = 1$, $\delta = 0.5$ and $m = 1$.

3. Time-dependent problem: Numerical solution

In this section, we numerically compute the sharp travelling wavefront solution with the minimum speed of eq. (1) for a special case by approximating the time-dependent solutions. For this purpose, we introduce a moving boundary problem, that is, a moving right-hand boundary condition at $x = x_*(t)$, and a fixed left-hand boundary condition for the partial differential equation

$$\frac{\partial u}{\partial t} + \beta u^2 \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\alpha u^2 \frac{\partial u}{\partial x}\right) + \gamma u(1-u)(u-\delta),$$
$$0 < x < x_*(t), \quad t > 0. \tag{8}$$

At the fixed left-hand boundary, we choose the Dirichlet boundary condition $u = 1$. At the moving right-hand boundary, there is the boundary condition $u = 0$. The equation of moving boundary can be derived to take the following form:

$$\frac{\mathrm{d}x_*}{\mathrm{d}t} = -u\frac{\partial u}{\partial x}. \tag{9}$$

The initial condition is chosen to have compact support in the following form:

$$u = \begin{cases} 1, & \text{if } x < 0 \\ 0, & \text{if } x > 0 \end{cases}, \qquad t = 0, \tag{10}$$

$$x_*(0) = x_0. \tag{11}$$

Equation (8) with eq. (9) and the above-mentioned boundary conditions which define the moving boundary problem may be solved numerically as in ref. [10], by mapping the equation with a suitable choice of new space coordinates onto a fixed spatial domain and using finite differences to approximate the solutions of the transformed equations. For this moving boundary problem, we introduce the transformation

$$y = x - x_*(t) \le 0. \tag{12}$$

With this, eq. (8) becomes

$$\frac{\partial u}{\partial t} + \beta u^2 \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\alpha u^2 \frac{\partial u}{\partial y}\right) + \frac{\mathrm{d}x_*}{\mathrm{d}t}\frac{\partial u}{\partial y} + \gamma u\,(1-u)\,(u-\delta),$$
$$y \le 0, t > 0, \tag{13}$$

which can be rewritten in the following form:

$$\frac{\partial u}{\partial t} = \frac{\partial J}{\partial y} + \gamma u(1-u)(u-\delta), \quad y \le 0, t > 0, \tag{14}$$

where J is the flux given by

$$J = \alpha u^2 \frac{\partial u}{\partial y} - \beta \frac{u^3}{3} + \frac{\mathrm{d}x_*}{\mathrm{d}t} u, \qquad x_* = x_*(t). \tag{15}$$

To solve eq. (14) with eq. (15), we discretize the space derivatives and integrate the resulting ordinary differential equations in time along the constant $y = y_i$ lines. Furthermore, we use the upwind difference method, approximate the differential equation for the moving boundary $x_*(t)$, eq. (9) by

$$\frac{\mathrm{d}x_*}{\mathrm{d}t} = \frac{u_N^2}{\mathrm{d}y}, \tag{16}$$

and again replace dy with dx, to obtain

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{J_i - J_{i-1}}{\mathrm{d}x} + \gamma u_i (1-u_i)\,(u_i - \delta)\,, \qquad i = 1, 2, ..., N,$$

$$J_i = \alpha\left(\frac{u_{i+1}+u_i}{2}\right)^2\left(\frac{u_{i+1}-u_i}{\mathrm{d}x}\right) - \frac{\beta}{3}u_{i+1}^3 + \frac{u_N^2}{\mathrm{d}x}u_{i+1}, \quad i = 0, 1, ..., N, \tag{17}$$

for $i = 0$, $u_{i=0} = 1$.

Then, we solve eq. (17) by using an adaptive step Runge–Kutta scheme of fourth order [11], with initial conditions as in eq. (10) and boundary conditions $u_{i=0} = 1$ and $u_{N+1} = 0$.

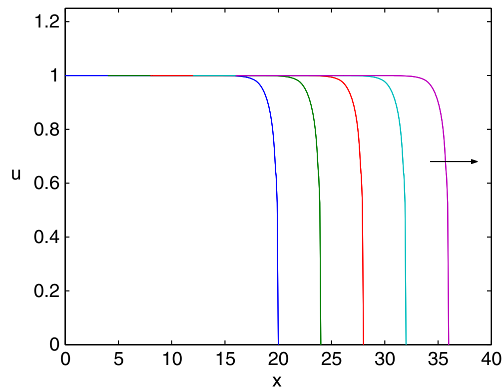

Figure 2. Numerically computed wavefront solution u as function of space x at different times. Clearly, this wavefront with the minimum speed is of sharp type.

In figure 2, we show the resulting numerical solution by sketching the profiles of $u\,(x, t)$ at different times. This figure clearly shows that already for intermediate time, we obtain the profile of the sharp wavefront-type solution. The wave speed is approximately found to be $c = 0.3764$. We have computed this speed by calculating the position of the moving boundary $x_*(t)$ and the position of a selected level point ($u = 0.5$) on the front solution profile $x_f(t)$. With this, we have found that the velocity of the moving boundary for large time is $c_{x_*} = c_{x_f} \simeq 0.3764$, where $c_{x_f}(t)$ is the velocity of $x_f(t)$. In general, this example shows that the initial conditions of the compact support evolve to the sharp front propagating with the minimum speed $c = 0.3764$, which is close to the speed c^* given by eq. (6) in §2 for the same values of model equation parameters.

4. Conclusion

In this paper, we have studied the travelling wavefront solution of the sharp type for the Nagumo equation with nonlinear degenerate diffusion and convection which is of widespread use in different areas, population dynamics, chemical reactions, flame propagation, etc. Specifically, we have computed this sharp wavefront solution to this model equation using two different methods. One method is to solve the travelling wave equation and obtain an exact solution to the equation describing such sharp travelling wavefront with a minimum speed. The other method is to solve an initial-moving boundary-value problem for the partial differential equation and find numerically an approximation to this sharp travelling wavefront solution.

Finally, we note that this study carried out in particular, the computation of an exact solution and the numerical approximation describing the sharp type front with a minimum speed in such equation gains more importance particularly, in determining the wave speed which is, for instance, the asymptotic rate of spread of populations.

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