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Double compactons in the Olver–Rosenau equation

AIYONG CHEN* and SHUANGQUAN WEN

School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin, Guangxi, 541004, People's Republic of China
*Corresponding author. E-mail: aiyongchen@163.com

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Abstract. It is showed that the fully nonlinear evolution equations of Olver and Rosenau can be reduced to Hamiltonian form by transformation of variables. The resulting Hamiltonian equations are treated by the dynamical systems theory and a phase-space analysis of their singular points is presented. The results of this study demonstrate that the equations can support double compactons. The new Olver–Rosenau compactons are different from the well-known Rosenau–Hyman compacton and Cooper–Shepard–Sodano compacton, because they are induced by a singular elliptic instead of singular straight line on phase-space.

Keywords. $K(m, n)$ equation; Rosenau–Hyman compacton; Olver–Rosenau compacton.

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1. Introduction

The study of nonlinear wave equations and their solitary wave solutions are of great importance in many areas of physics. Classically, the solitary wave solutions of nonlinear evolution equations are determined by analytic formulae (typically a sech^2 function or variants thereof) and serve as prototypical solutions that model physical localized waves. In the case of integrable systems, the solitary waves interact cleanly, and are known as solitons. For many examples, localized initial data ultimately break up into a finite collection of solitary wave solutions; this fact has been proved analytically for certain integrable equations such as the Korteweg–de Vries equation, and is observed numerically in many others. The appearance of non-analytic solitary wave solutions to new classes of nonlinear wave equations, including peakons [1–5], which have a corner at their crest, cuspons [2], having a cusped crest, and compactons [6–10], which have compact support, has vastly increased the menagerie of solutions appearing in model equations, both integrable and non-integrable. The distinguishing feature of the systems admitting non-analytic solitary wave solutions is that, contrary to the classical nonlinear

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wave equations, they all include a nonlinear dispersion term, meaning that the highest order derivatives (characterizing the dispersion relation) do not occur linearly in the system, but are typically multiplied by a function of the dependent variable.

There are two important nonlinearly dispersive equations. One is the well-known Camassa–Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

which was proposed by Camassa and Holm [1] as a model equation for unidirectional nonlinear dispersive waves in shallow water. This equation has attracted a lot of attention over the past decade due to its interesting mathematical properties. The Camassa–Holm equation has been found to have peakons, cuspons and composite wave solutions [2]. The other is the $K(m, n)$ equation

$$u_t \pm (u^m)_x + (u^n)_{xxx} = 0, \tag{1.2}$$

which was discovered by Rosenau and Hyman [6]. The $K(2, 2)$ equation supports compacton solutions

$$\begin{cases} u(x,t) = \dfrac{4c}{3}\cos^2\left(\dfrac{x-ct}{4}\right), & |x-ct| \le 2\pi, \\ u(x,t) = 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

The compactons represent travelling solitary wave solutions with compact support. That is, they vanish identically outside a finite range. The compactons are also robust within their range of existence.

Unlike the ordinary KdV equation, the generalized KdV equation (1.2) considered by Rosenau and Hyman was not derivable from a first-order Lagrangian, except for $n = 1$, and did not possess the usual conservation laws of energy and mass that the KdV equation possessed. It is presumed that the above equations are not completely integrable, but instead possess only a finite number of conservation laws (for more details see [9]). Therefore, Cooper *et al* [9] proposed a generalization of the KdV equation based on the first-order Lagrangian

$$L(l, p) = \int \left[\frac{1}{2}\phi_x\phi_t + \frac{(\phi_x)^2}{l(l-1)} - \alpha\,(\phi_x)^p\,(\phi_{xx})^2\right] \mathrm{d}x, \tag{1.4}$$

which leads to the CSS equation

$$u_t + u^{l-2}u_x - p\left[u^{p-1}\,(u_x)^2\right]_x + 2\alpha\left[u^p u_x\right]_{xx} = 0. \tag{1.5}$$

The Cooper–Shepard–Sodano (CSS) equation also has compacton, which is similar to Rosenau–Hyman compacton. Recently, Mihaila *et al* [11,12] studied the numerical stability of single compactons of the $K(m, n)$ equation and the CSS equation and their pairwise interactions by using Padé approximant method.

In 1997, Rosenau [7] studied the nonanalytic solitary waves of the following integrable nonlinear wave equation:

$$(u \pm u_{xx})_t = au_x + \frac{1}{2}\left[\left(u^2 \pm u_x^2\right)(u \pm u_{xx})\right]_x. \tag{1.6}$$

The equation is obtained by Olver and Rosenau [13] through a reshuffling procedure of the Hamiltonian operators underlying the bi-Hamiltonian structure of mKdV equations. For convenience, we call it the Olver–Rosenau equation. In this paper, we find that, in comparison with the $K(m, n)$ equation and the CSS equation, the Olver–Rosenau equation has double compactons for suitable parameters.

2. Hamiltonian system and Newton equation

For the Olver–Rosenau equation, change of variable from $u(\xi)$ to $\varphi(\xi)$ and subsequent integration over ξ leads to an ordinary differential equation

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2} = -\frac{2g + 2(a+c)\varphi + \varphi^3 + \varphi(\mathrm{d}\varphi/\mathrm{d}\xi)^2}{2c + \varphi^2 + (\mathrm{d}\varphi/\mathrm{d}\xi)^2}. \tag{2.1}$$

The right side of eq. (2.1) does not permit one to regard these equations as Hamiltonian systems and gives rise to an awkward analytical constraint for the application of dynamical systems theory. This problem can, however, be resolved by considering evolution of φ in a zero-energy hypersurface in the phase-space belonging to some potential function [14,15].

The potential representation for the analysis of travelling wave solutions of nonlinear dispersive evolution equations was introduced by Eichmann *et al* [14]. This representation is defined as

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right)^2 = -F(\varphi). \tag{2.2}$$

The left side of eq. (2.2) is identified with the non-relativistic kinetic energy and right side with the negative value of the potential energy $F(\varphi)$ because ξ and φ in eq. (2.1) can be regarded as time and space coordinates, respectively. Here $\mathrm{d}\varphi/\mathrm{d}\xi$ is the velocity of a particle moving along the φ-axis. Clearly, for the potential representation in eq. (2.2), the evolution of φ or u proceeds on the zero-energy hypersurface in the phase-space belonging to $F(\varphi)$. The function $F(\varphi)$ for the Olver–Rosenau equation is given by

$$F(\varphi) = \varphi^2 + 2c \mp 2\sqrt{-a}(\varphi - \sqrt{-2c}). \tag{2.3}$$

Using eq. (2.2) in eq. (2.1) we get

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2} = \mp\left(-\sqrt{-a} \pm \varphi\right). \tag{2.4}$$

Equation (2.4) can be written in the Newtonian form as

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2} = -\frac{\mathrm{d}V(\varphi)}{\mathrm{d}\varphi} \tag{2.5}$$

with the potential

$$V(\varphi) = \pm\left(-\sqrt{-a}\varphi \pm \frac{1}{2}\varphi^2\right). \tag{2.6}$$

The second-order differential equation (2.4) is equivalent to the Hamiltonian system

$$\begin{cases} \dfrac{d\varphi}{d\xi} = \dfrac{\partial H}{\partial \psi}, \\ \dfrac{d\psi}{d\xi} = -\dfrac{\partial H}{\partial \varphi}, \end{cases} \tag{2.7}$$

with the Hamiltonian

$$H(\varphi, \psi) = \frac{1}{2}\psi^2 + V(\varphi). \tag{2.8}$$

System (2.7) is a linear system with very simple dynamical behaviour. In fact, the second-order differential equation (2.1) is also equivalent to the plane system

$$\begin{cases} \dfrac{d\varphi}{d\xi} = \phi, \\ \dfrac{d\phi}{d\xi} = -\dfrac{2g + 2(a+c)\varphi + \varphi^3 + \varphi\phi^2}{\varphi^2 + \phi^2 + 2c}, \end{cases} \tag{2.9}$$

which has the following first integral:

$$H(\varphi, \phi) = \frac{1}{4}(\varphi^2 + \phi^2 + 2c)^2 + a\varphi^2 + 2g\varphi. \tag{2.10}$$

The system (2.9) possesses very complex dynamical behaviour. In addition, it is not Hamiltonian. However, if we define a new independent variable ζ by setting $(d\xi/d\zeta) = \varphi^2 + \phi^2 + 2c$, then we can obtain the Hamiltonian system as

$$\begin{cases} \dfrac{d\varphi}{d\zeta} = \dfrac{\partial H}{\partial \phi} = P(\varphi, \phi), \\ \dfrac{d\phi}{d\zeta} = -\dfrac{\partial H}{\partial \varphi} = Q(\varphi, \phi), \end{cases} \tag{2.11}$$

where $H(\varphi, \phi)$ is defined by eq. (2.10). System (2.11) has the same topological phase portraits as system (2.9) except for the singular elliptic $\varphi^2 + \phi^2 + 2c = 0$.

3. Phase-space analysis of compactons

The general theory of dynamic systems and mathematical physics shows that, the smooth travelling wave solutions of eq. (1.6) are given by smooth orbits of system (2.9): solitary wave solutions correspond to homoclinic orbits at a single equilibrium point; periodic waves come from periodic orbits; while heteroclinic orbit connecting two equilibrium points yields kink (or antikink) solutions. However, there exist some special cases for singular solitons such as peakon and compacton. It is well-known that a peakon (or compacton) is given by three degenerate heteroclinic orbits (or a degenerate periodic orbit).

We shall now study the qualitative behaviour of system (2.11). In general, the qualitative behaviour of system (2.11) is very complex. Therefore, we suppose that $a < 0, c < 0$

and $g = -a\sqrt{-2c}$. The singular points of system (2.11) obtained from $P(\varphi, \phi) = 0$ and $Q(\varphi, \phi) = 0$ are given by

$$E_1\left(-\frac{1}{2}(\sqrt{-8a-2c}+\sqrt{-2c}), 0\right),\quad E_2\left(\frac{1}{2}(\sqrt{-8a-2c}-\sqrt{-2c}), 0\right),$$
$$E_3(\sqrt{-2c}, 0).$$

We make use of the eigenvalues of the Jacobian matrix

$$\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

for the classification of singular points $p(\varphi_e, \phi_e)$, where

$$a_{11} = \left.\frac{\partial P}{\partial \varphi}\right|_{(\varphi_e,\phi_e)},\quad a_{12} = \left.\frac{\partial P}{\partial \phi}\right|_{(\varphi_e,\phi_e)}$$

and

$$a_{21} = \left.\frac{\partial Q}{\partial \varphi}\right|_{(\varphi_e,\phi_e)},\quad a_{22} = \left.\frac{\partial Q}{\partial \phi}\right|_{(\varphi_e,\phi_e)}.$$

If $\Delta = a_{11}a_{22} - a_{21}a_{12}$ and $T = a_{11} + a_{22}$, then the singular point p is said to be non-degenerate if $\Delta \neq 0$. Then p is an isolated singular point. Moreover, p is a saddle if $\Delta < 0$, a node if $T^2 > 4\Delta > 0$ (stable if $T < 0$, unstable if $T > 0$), a focus if $4\Delta > T^2 > 0$ (stable if $T < 0$, unstable if $T > 0$), and either a weak focus or a centre if $T = 0 < \Delta$. When $\Delta = T = 0$ but the Jacobian matrix at p is not the zero matrix and p is isolated in the set of all singular points, we say that p is nilpotent.

We quote a result on nilpotent singular points that we shall need (for more details, see [16]). Let (0, 0) be an isolated point of the vector field $(y + F(x, y), G(x, y))$, where F and G are analytic functions in a neighbourhood of the origin at least with quadratic terms in the variables x and y. Let $y = f(x)$ is the solution of the equation $y + F(x, y) = 0$ in a neighbourhood of (0, 0). Assume that the development of the function $G(x, f(x))$ is of the form Kx^k + HOT (higher order terms). If k is odd and $K > 0$, then the origin is a saddle. Moreover, the saddle has two separatices tangent to the semiaxis $x < 0$, and other two separatices tangent to the semiaxis $x > 0$.

The eigenvalues of the Jacobian matrix M for singular points E_1 and E_2 are given by two pairs of pure imaginary roots $\lambda_1 = \pm\sqrt{A}i$ and $\lambda_2 = \pm\sqrt{B}i$, respectively, where

$$A = \left(\sqrt{4ac+c^2} - 2a + c\right)\left(3\sqrt{4ac+c^2} - 4a - c\right) \tag{3.1}$$

and

$$B = \left(\sqrt{4ac+c^2} + 2a - c\right)\left(4a + c - 3\sqrt{4ac+c^2}\right). \tag{3.2}$$

Thus singular points E_1 and E_2 are two centres.

For singular point E_3, $\Delta = T = 0$ and the Jacobian matrix is not the zero matrix. Therefore, E_3 is a nilpotent point. Let $\varphi - \sqrt{-2c} \to \varphi/(4c - 2a)$, then system (2.11) changes to

$$\begin{cases} \dfrac{d\varphi}{d\zeta} = G(\varphi, \phi), \\ \dfrac{d\phi}{d\zeta} = \varphi + F(\varphi, \phi), \end{cases} \tag{3.3}$$

where

$$G(\varphi,\phi) = \left(\frac{1}{4c-2a}\varphi^2 + 2\sqrt{-2c}\varphi + \phi^2\right)\phi \tag{3.4}$$

and

$$F(\varphi,\phi) = -\left(\frac{3\sqrt{-2c}}{(4c-2a)^2}\varphi^2 + \frac{1}{(4c-2a)^3}\varphi^3 + \left(\frac{1}{4c-2a}\varphi + \sqrt{-2c}\right)\phi^2\right). \tag{3.5}$$

The equation $\varphi + F(\varphi,\phi) = 0$ implies $\varphi = \sqrt{-2c}\phi^2 + \mathrm{HOT}$. Thus, we obtain

$$G(\varphi(\phi),\phi) = -2a\phi^3 + \mathrm{HOT}. \tag{3.6}$$

Equation (3.6) gives $K = -2a > 0$ and $k = 3$. So, E_3 is a nilpotent saddle point. There exist two close orbits

$$\phi^2 = -\varphi^2 - 2c \pm 2\sqrt{-a}(\varphi - \sqrt{-2c}), \tag{3.7}$$

which are tangent to singular elliptic $\varphi^2 + \phi^2 + 2c = 0$ at point E_3 (see figure 1a). The two close orbits yield two new compactons (see figure 1b)

$$\begin{cases} u(x,t) = \sqrt{-a} - (\sqrt{-2c} - \sqrt{-a})\cos(x-ct), & |x-ct| \le 2\pi, \\ u(x,t) = \sqrt{-2c}, & \text{otherwise} \end{cases} \tag{3.8}$$

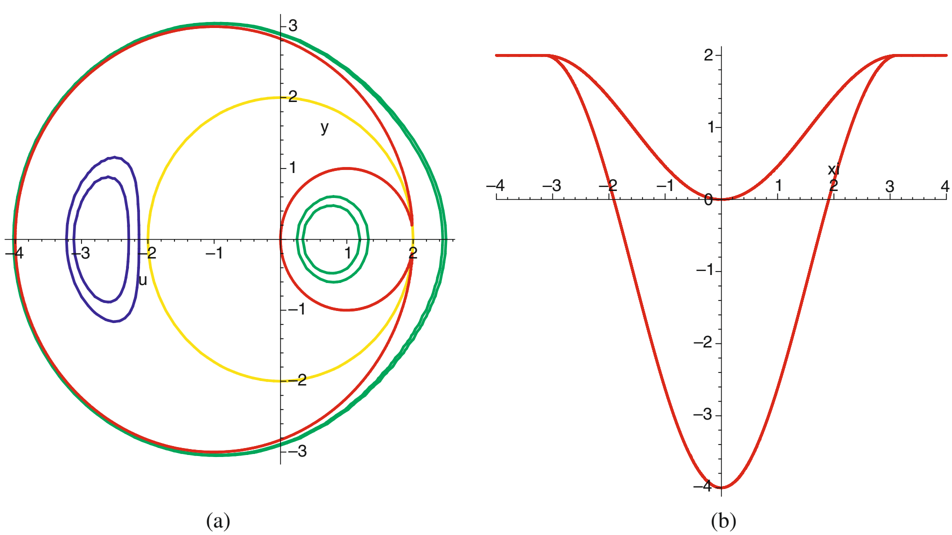

Figure 1. The phase portrait and Olver–Rosenau compacton of eq. (1.6). (**a**) The two close orbits (red) are tangent to the singular elliptic (yellow). (**b**) Profiles of two downward compactons.

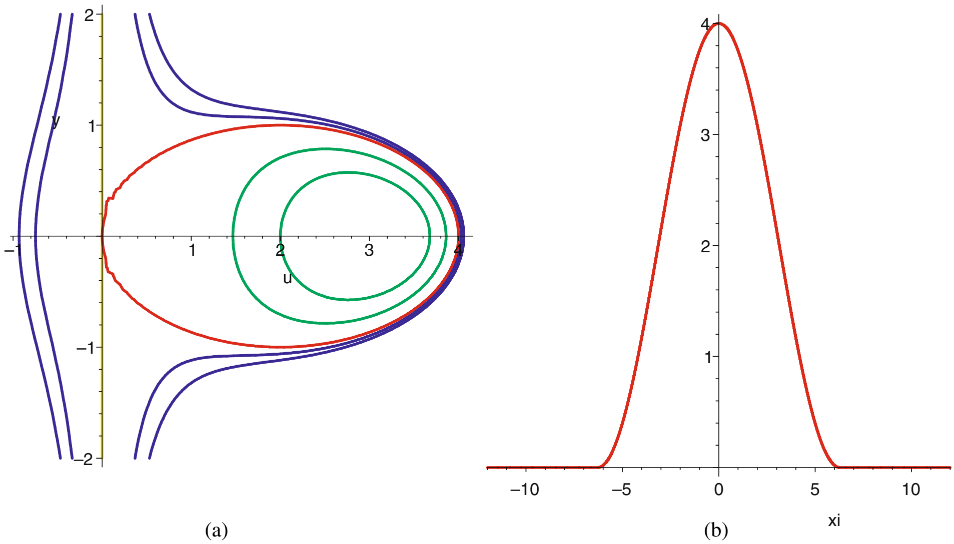

Figure 2. The phase portrait and Rosenau–Hyman compacton of eq. (1.2). (**a**) A close orbit (red) is tangent to the singular straight line (yellow). (**b**) Profile of Rosenau–Hyman compacton.

and

$$\begin{cases} u(x,t) = -\sqrt{-a} - (\sqrt{-2c} + \sqrt{-a})\cos(x - ct), & |x - ct| \le 2\pi, \\ u(x,t) = \sqrt{-2c}, & \text{otherwise.} \end{cases} \tag{3.9}$$

Remark. Using similar method to $K(2,2)$ equation, we can obtain phase portrait of the corresponding system for Rosenau–Hyman compacton (see figure 2a). Figure 2a shows that there exists a close orbit, which is tangent to singular straight line $\varphi = 0$ at the origin. The close orbit gives Rosenau–Hyman compacton shown in figure 2b. In comparison with the Rosenau–Hyman compacton and the CSS compacton, the Olver–Rosenau compactons are induced by a singular elliptic rather than a singular straight line.

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