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Peakons and compactons on the background of periodic wave

CHAO-QING DAI* and CUI-YUN LIU
School of Sciences, Zhejiang Agriculture and Forestry University, Lin'an 311300, People's Republic of China
*Corresponding author. E-mail: dcq424@126.com

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Abstract. In this paper, the extended tanh-function method (ETM) based on the mapping equation is further improved by generalizing the Riccati equation. The new variable separation solutions of the (2+1)-dimensional Broer–Kaup–Kupershmidt (BKK) system are derived. From the periodic wave solution and by selecting appropriate functions, the evolutional behaviours of peakons and compactons on the background of Jacobian elliptic wave are investigated.

Keywords. Extended tanh-function method; Broer–Kaup–Kupershmidt system; novel solitary wave structures; background of Jacobian elliptic wave.

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1. Introduction

Recently, quite a number of localized coherent structures have been studied for (2+1)-dimensional integrable models [1–5]. Peakon and compacton solutions have been found as two weak continuous solitary wave solutions. The so-called peakon solution ($u = c\exp(-|x-ct|)$), which is discontinuous at its crest and refers to as a weak solution of the celebrated (1+1)-dimensional Camassa–Holm equation, was first given by Camassa and Holm [6]. A (2+1)-dimensional peakon may be defined as a special dromion with some derivative singularities at its peak(s) but with continuous value(s). A compacton has nonzero values only in a small region, and it is not derivable at the 'boundary' of the nonzero area. In order to understand the role of nonlinear dispersion, Rosenau and Hyman [7] introduced a class of solitary waves with compact support (called compacton) in fully nonlinear KdV equation $K(m, n)$ and by numerical experiments, found compacton solutions with remarkable properties. In the traditional treatment of nonlinear systems, one usually studies the interaction behaviour among solitons and many methods can provide exact multiple soliton solutions. However, there are few works in the literatures which discuss solitary wave structures in the background of Jacobian elliptic wave.

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There are usually two ways of constructing the N-soliton solution with respect to a given background solution. Both are based on the fact that the underlying Lax operator is reflectionless with respect to the background, but has N additional eigenvalues. One is via the inverse scattering transform by choosing an arbitrary number of eigenvalues (plus corresponding norming constants) and setting the reflection coefficient equal to zero. The other is by inserting the eigenvalues using commutation methods. This works fine in case of a constant background solution and the eigenvalues can be chosen arbitrarily. However, in case of a (quasi-)periodic background solution it turns out that the eigenvalues need to satisfy certain restrictions. To our excitement, based on the obtained periodic wave solution in this paper and by selecting appropriate functions, we can also discuss solitons with respect to a given background solution, i.e. peakons and compactons in the background of the periodic waves.

In recent papers some direct methods, such as the F-expansion method [8], the (G'/G)-expansion method [9], the functional variable method [10], Jacobian elliptic function expansion method [11], the tanh-function expansion method [12], the first integral method [13], the similarity transformation method [14], etc. were developed. Many authors have also obtained variable separation solutions of various (2+1)-dimensional models by the direct method. For example, Yang and some authors have realized variable separation for some (2+1)-dimensional nonlinear systems [2,3] with the help of the tanh-function expansion method. In this paper, with the help of a more general Riccati equation, we discuss peakons and compactons in the background of periodic waves based on the new exact solutions of the (2+1)-dimensional Broer–Kaup–Kupershmidt equation (BKKE)

$$H_{ty} - H_{xxy} + 2(HH_x)_y + 2G_{xx} = 0,$$

$$G_t + G_{xx} + 2(HG)_x = 0. \tag{1}$$

The (2+1)-dimensional BKKE may be derived from the inner parameter-dependent symmetry constraint of the Kadomtsev–Petviashvili (KP) equation [15]. Using some suitable dependent and independent variable transformations, Chen and Li have proved that the (2+1)-dimensional BKKE can be transformed to the (2+1)-dimensional integrable dispersive long wave equation (DLWE) [16] and the (2+1)-dimensional integrable Ablowitz–Kaup–Newell–Segur equation (AKNSE). When we take $y = x$, the (2+1)-dimensional BKKE is reduced to the usual (1+1)-dimensional BKKE, which can be used to describe the propagation of long wave in shallow water [17]. Chaotic behaviours of eq. (1) were reported [18].

2. New solutions of the (2+1)-dimensional Broer–Kaup–Kupershmidt equation

To solve the (2+1)-dimensional BKKE, first, let us make a transformation for eq. (1):

$$G = H_y. \tag{2}$$

Substituting the transformation (2) into eq. (1) yields

$$H_{ty} + 2(H_x H)_y + H_{xxy} = 0. \tag{3}$$

Similar to the mapping method [2,3], we assume that eq. (3) possesses the solutions of the following form:

$$H(x, y, t) = \sum_{i=0}^{n} a_i \phi(w)^i, \tag{4}$$

where $w = w(x, y, t)$, $a_i \equiv a_i(x, y, t), i = 0, 1, \ldots, n$, while $n = 1$ is obtained by balancing the highest-order derivative terms with the nonlinear terms in eq. (3). ϕ satisfies a more general Riccati equation than the one in [2,3], i.e.

$$\phi' = R + P\phi + Q\phi^2, \tag{5}$$

with constants R, P and Q. The Riccati equation (5) has some special solutions shown in table 1.

Inserting eqs (4) and (5) with $n = 1$ into eq. (3), selecting the variable separation ansatz

$$w = \chi(x, t) + \psi(y), \tag{6}$$

and eliminating all the coefficients of polynomials of ϕ, one gets a set of partial differential equations for a_0, a_1, $\chi(x, t)$ and $\psi(y)$, from which we can derive new variable separation solution:

$$H = -\frac{\chi_{xx} + P\chi_x^2 + \chi_t}{2\chi_x} - Q\chi_x\phi(\chi + \psi). \tag{7}$$

From eqs (7), (5) and table 1, we can construct 15 couples of variable separation solutions.

Table 1. Solutions of Riccati equation.

Solution	R	P	Q	$\phi(w)$
1	1/2	0	−1/2	$\coth(w) \pm \text{csch}(w),\ \tanh(w) \pm i\,\text{sech}(w)$
2	1/2	0	1/2	$\tan(w) \pm \sec(w),\ \tan(w)/[1 \pm \sec(w)]$
3	−1/2	0	−1/2	$\cot(w) \pm \csc(w),\ \cot(w)/[1 \pm \csc(w)]$
4	1	0	−1	$\tanh(w), \coth(w)$
5	1	0	1	$\tan(w)$
6	−1	0	−1	$\cot(w)$
7	1	0	−4	$\tanh(w)/[1 + \tanh^2(w)]$
8	1	0	4	$\tan(w)/[1 - \tan^2(w)]$
9	−1	0	−4	$\cot(w)/[1 - \cot^2(w)]$
10	1	−2	2	$\tan(w)/[1 + \tan(w)]$
11	1	2	2	$\tan(w)/[1 - \tan(w)]$
12	−1	2	−2	$\cot(w)/[1 + \cot(w)]$
13	1	−2	−2	$\cot(w)/[1 - \cot(w)]$
14	0	0	$\neq 0$	$-1/[Qw + c_0]$
15	0	$\neq 0$	0	$[\exp(Bw) - A]/B$

For convenience, here we only give the periodic solution from eqs (7), (5) and solution (10) in table 1

$$G = H_y = -2\chi_x\psi_y \left[\frac{1}{\sin(\chi+\psi)+\cos(\chi+\psi)}\right]^2, \tag{8}$$

where χ and ψ are arbitrary functions of $\{x, t\}$ and $\{y\}$, respectively.

3. Novel solitary wave structures in the background of Jacobian elliptic waves

All rich localized coherent structures, such as nonpropagating solitons, dromions, peakons, compactons, foldons, instantons, ghostons, ring solitons, and the interactions between these solitons, can be derived by the quantity G in the form of solitary wave solution from eq. (7) and solution (4) in table 1. These abundant localized coherent structures are reported in [1–3]. Thus, we omit the discussion about them in the present paper. Moreover, from stereotyped thinking pattern, we have not paid much attention to the periodic wave solution (8) before because we guessed the periodic wave solution would not yield important localized excitations. However, the above conjecture may be inappropriate. In the following part, we shall discuss some significant localized excitations derived from the periodic solution (8).

3.1 *Peakons and compactons on the background of Jacobian elliptic wave*

Here we focus on peakons and compactons in the background of Jacobian elliptic waves, which can be constructed by the selection of χ and ψ as the superposition solution of the Jacobian elliptic sine functions and piecewise continuous functions. Here is a special example

$$\chi = -2 - 0.003\,\mathrm{sn}(0.5x, 0.3) - 0.015F_i(x), \tag{9}$$

and

$$\psi = 2 + 0.003\,\mathrm{sn}(0.5y, 0.3) + 0.015H_i(y), \quad i = 1, 2, \tag{10}$$

where $\mathrm{sn}(\cdot,\cdot)$ is the Jacobian elliptic sine function with the modulus 0.3, $F_i(x)$ and $H_i(y)$ are piecewise continuous functions.

From figure 1a, a peakon in the background of the Jacobian elliptic waves can be constructed by the selection of χ and ψ as (9) and (10) with

$$F_1 = \begin{cases} \exp(0.5x) & x \le 0, \\ -\exp(-0.5x) & x > 0, \end{cases}$$

$$H_1 = \begin{cases} \exp(0.5y) & y \le 0, \\ -\exp(-0.5y) & y > 0. \end{cases} \tag{11}$$

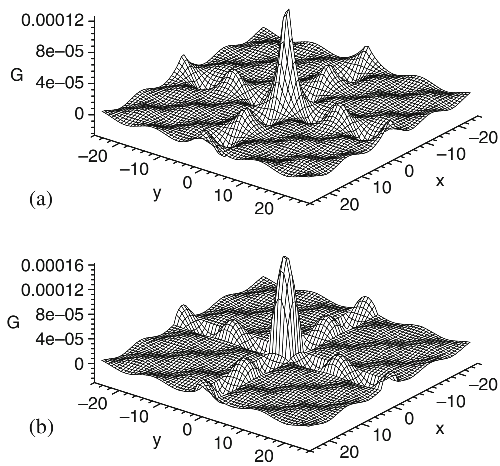

Figure 1. (**a**) Combined peakon and (**b**) compacton in the background of Jacobian elliptic waves.

Seen from figure 1b, a compacton in the background of Jacobian elliptic waves can be constructed by the selection of χ and ψ as (9) and (10) with

$$F_2 = \begin{cases} 0 & x \le -\dfrac{5\pi}{8}, \\ \sin(0.5x) + 1 & -\dfrac{5\pi}{8} < x \le \dfrac{5\pi}{8}, \\ 2 & x > \dfrac{5\pi}{8}, \end{cases}$$

$$H_2 = \begin{cases} 0 & y \le -\dfrac{5\pi}{8}, \\ \sin(0.5y) + 1 & -\dfrac{5\pi}{8} < y \le \dfrac{5\pi}{8}, \\ 2 & y > \dfrac{5\pi}{8}. \end{cases} \tag{12}$$

Moreover, χ and ψ can also be taken as the elliptic functions of different types, and we can even take χ as the form $\sum_{i=1}^{M} \mathrm{sn}(x, m_i)$ and/or ψ as the form $\sum_{j=1}^{N} \mathrm{sn}(y, n_i)$, etc. Of course, we may obtain a diversity of periodic wave solutions in terms of the Jacobian elliptic functions by selecting the arbitrary functions appropriately. It is worth noting that the Jacobian transformation, detailed description can be found in [19,20], implies that any solution found by one Jacobian elliptic function may be transformed into an equivalent one that can be obtained by another. Since other Jacobian elliptic functions have singularities, we consider only the periodic wave solutions in terms of sn functions.

3.2 *Interaction between peakon and compacton on the background of Jacobian elliptic wave*

Due to the arbitrariness of the functions $\chi(x, t)$ and $\psi(y)$ in solution (8), we can discuss the interaction behaviour between peakon and compacton on the background of Jacobian elliptic wave by selecting χ and ψ as

$$\chi = -2 - 0.002\mathrm{sn}(0.5x, 0.3) - 0.01A(x, t) - 0.015B(x, t), \tag{13}$$

and

$$\psi = 2 + 0.002\mathrm{sn}(0.5y, 0.3) + 0.01C(y, t) + 0.015D(y, t), \tag{14}$$

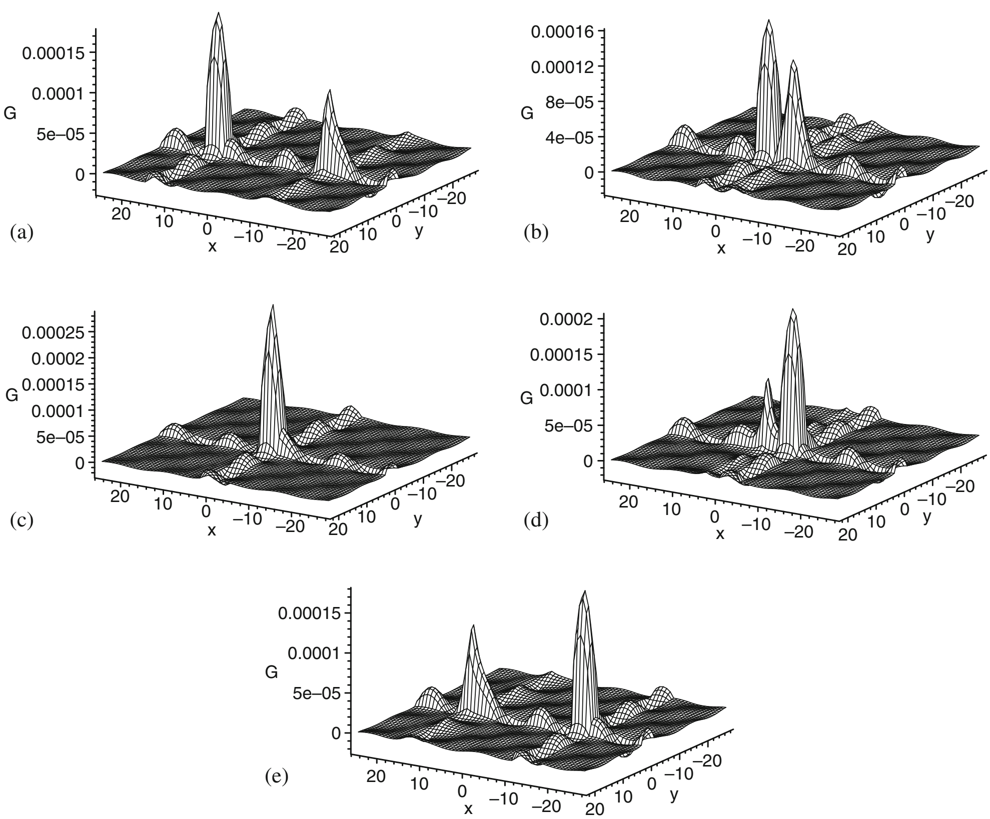

Figure 2. Heading-on interaction between combined peakon and compacton in the background of Jacobian elliptic waves at times: (**a**) $t = -13$ s; (**b**) $t = -3$ s; (**c**) $t = 0$ s; (**d**) $t = 3$ s; (**e**) $t = 13$ s.

with

$$A = \begin{cases} \exp[0.5(x-t)] & x-t \le 0, \\ -\exp[-0.5(x-t)] & x-t > 0, \end{cases}$$

$$B = \begin{cases} 0 & x+t \le -\dfrac{5\pi}{8}, \\ \sin[0.5(x+t)]+1 & -\dfrac{5\pi}{8} < x+t \le \dfrac{5\pi}{8}, \\ 2 & x+t > \dfrac{5\pi}{8}, \end{cases} \tag{15}$$

and

$$C = \begin{cases} \exp[0.5(y-t)] & x-t \le 0, \\ -\exp[-0.5(y-t)] & x-t > 0, \end{cases}$$

$$D = \begin{cases} 0 & y+t \le -\dfrac{5\pi}{8}, \\ \sin[0.5(y+t)]+1 & -\dfrac{5\pi}{8} < y+t \le \dfrac{5\pi}{8}, \\ 2 & y+t > \dfrac{5\pi}{8}. \end{cases} \tag{16}$$

From figure 2, we can see that the compacton moves along the negative x-axis, and the peakon runs along the positive x-axis. Then they interact at $\{x = 0,\ y = 0\}$. After interaction, their shapes and velocities are preserved.

4. Summary and discussion

In summary, using the mapping method along with the variable separation idea, many variable separation solutions of the (2+1)-dimensional Broer–Kaup–Kupershmidt equation are obtained. From the special periodic solution (8) and by selecting appropriate functions, the evolutional behaviours of peakons and compactons on the background of Jacobian elliptic wave are studied.

What we have obtained also further verify that the mapping method is quite useful to generate abundant localized excitations for (2+1)-dimensional nonlinear models. In our future work, on the one hand, we devote to extending this method to other nonlinear systems, such as the differential-difference equations and (1+1)-dimensional nonlinear systems. On the other hand, we shall look for more interesting localized excitations.

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