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Q-S synchronization of the fractional-order unified system

YI CHAI[1], LIPING CHEN[1,*], RANCHAO WU[2] and JUAN DAI[3]

[1]State Key Laboratory of Power Transmission Equipment & System Security and New Technology, School of Automation, Chongqing University, Chongqing 400044, People's Republic of China
[2]School of Mathematics, Anhui University, Hefei 230039, People's Republic of China
[3]School of Automation, Beijing Institute of Technology, Beijing 100081, People's Republic of China
*Corresponding author. E-mail: lip_chen@yahoo.com.cn

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Abstract. Concept of Q-S synchronization for fractional-order systems is introduced and Q-S synchronization of the fractional-order unified system is investigated in this paper. On the basis of the stability theory of the fractional-order system, two suitable control schemes are designed to achieve Q-S synchronization of the fractional-order unified systems under the given observable variables of drive system and the response system. Theoretical analysis and numerical simulations are shown to demonstrate the validity and feasibility of the proposed method.

Keywords. Q-S synchronization; unified system; fractional-order system.

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1. Introduction

In recent years, research on fractional-order systems has gained a lot of attention. In fact, many systems can be described by fractional differential equations, for example, dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, visco-elastic systems, quantitative finance, bioengineering, diffusion wave and nuclear magnetic resonance [1–4]. Its advantage lies in providing an excellent instrument for the description of memory and hereditary properties of various materials and processes. Recently, many authors begin to investigate the chaotic dynamics of fractional dynamical systems. It was proved that many fractional-order differential systems behave chaotically with suitable orders, such as fractional-order Chua's circuit system [5], fractional-order Rössler system [6], fractional-order Chen system [7], fractional-order Lü system [8], fractional-order modified Duffing system [9] and fractional-order unified system [10].

Chaos synchronization phenomena have received growing attention in the study of chaos in fractional-order dynamical systems for their potential applications in some engineering application and information science, in particular in secure communication [11–14]. Since Pecora and Carroll [15] introduced a method to realize synchronization between two identical chaotic systems with different initial values in 1990, many important and fundamental results have been reported on the control and synchronization, and various types of chaos synchronization schemes for fractional-order dynamical systems have been proposed, such as complete synchronization [16], lag synchronization [17], projective synchronization [18], generalization synchronization [19], impulsive synchronization [20], and so on. However, these studies are mainly concerned with state synchronization. As we all know, due to perturbations of varying nature and unavoidable noise, especially in a complicated system or a large system, it is difficult to detect all state variables in real systems. In 1999, Yang proposed the concept of Q-S synchronization for the first time [21], which just requires observable variable synchronization between the response system and the drive system. From then on, Q-S synchronization has received a great deal of attention and a series of works on Q-S synchronization have been published. Some scholars extended the concept of Q-S synchronization and proposed generalized Q-S (lag, anticipated and complete) synchronization and function Q-S synchronization (see refs [22–24]).

On the other hand, recently, chaos, bifurcation behaviours and circuit realization of the fractional-order unified system are investigated numerically, and its synchronization is theoretically and numerically studied. For example, Wu *et al* studied projective synchronization of the fractional-order unified systems [25], Kuntanapreeda discussed synchronization between two fractional-order unified chaotic systems by linear feedback controller [26]. Wang and Zhang designed two schemes to achieve chaos synchronization of fractional-order unified systems [27]. However, to the best of our knowledge, there are few results about Q-S synchronization of fractional-order chaotic systems. Much of the literature on Q-S synchronization focussed on the systems with integer orders [22–24]. Motivated by the above discussions, in this letter, Q-S synchronization of fractional-order system is discussed. Two control laws are derived under the given observable variables. Corresponding theoretical analysis and numerical simulations are presented to verify the validity and feasibility of the proposed method.

The remainder of this paper is organized as follows. In §2, preliminary results are presented and the fractional-order unified system is described. In §3, two control schemes for Q-S synchronization are given. In §4, numerical simulations are given to illustrate the effectiveness of the main results. Finally, conclusions are drawn in §5.

2. Preliminaries and system description

There are some definitions for fractional derivatives [28,29]. The commonly used definitions are Grunwald–Letnikov (GL), Riemann–Liouville (RL) and Caputo (C) definitions.

The Grunwald–Letnikov (GL) derivative with fractional order q is given by

$$ {}_{a}^{\mathrm{GL}}D_t^q f(t) = \lim_{h\to 0} f_h^{(q)}(t) = \lim_{h\to 0} h^{-q} \sum_{i=0}^{[\frac{t-q}{h}]} (-1)^i \binom{q}{i} f(t-ih), \tag{1}$$

where $[\cdot]$ means the integer part.

The Riemann–Liouvill (RL) fractional derivatives is defined as

$$ {}_a^{\mathrm{RL}}D_t^q f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\frac{1}{\Gamma(n-q)}\int_a^t \frac{f(\tau)}{(t-\tau)^{(q-n+1)}}\mathrm{d}\tau, \quad n-1<q<n, \tag{2} $$

where $\Gamma(\cdot)$ is the gamma function, $\Gamma(\tau) = \int_0^\infty t^{\tau-1}\mathrm{e}^{-t}\mathrm{d}t$.

The Caputo (C) fractional derivative is defined as follows:

$$ {}_a^{\mathrm{C}}D_t^q f(t) = \frac{1}{\Gamma(n-q)}\int_a^t (t-\tau)^{(n-q-1)} f^{(n)}(\tau)\mathrm{d}\tau, \qquad n-1<q<n. \tag{3} $$

It should be noted that the advantage of Caputo approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as those for integer-order differentiation, which have well understood physical meanings. Comparing these two formulas, one easily arrives at the fact that Caputo derivative of a constant is equal to zero, which is not the case for the Riemann–Liouville derivative. Therefore, in the rest of this paper, the notation D_*^q is chosen as the Caputo fractional derivative operator ${}_a^{\mathrm{C}}D_t^q$.

Next, we introduce the concept of Q-S synchronization for fractional chaotic system. For two continuous-time dynamical systems with fractional-order q

$$ D_*^q x(t) = F(x,t), \tag{4} $$

$$ D_*^q y(t) = G(y,t) + U(x,y,t), \tag{5} $$

where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ and $y = (y_1, y_2, \ldots, y_n)^T \in R^n$ are the state vectors, F: $R^n \rightarrow R^n$ and G: $R^n \rightarrow R^n$ denote continuous vector functions. Set $Q(x(t)) = (Q_1(x(t)), \ldots, Q_n(x(t)))^T$ and $S(y(t)) = (S_1(y(t)), \ldots, S_n(y(t)))^T$ as observable variables of systems (4) and (5), respectively, where Q, S are continuous smooth vector functions. Let the Q-S synchronization error of the two chaotic systems be

$$ \begin{aligned} e(t) &= (e_1(t), e_2(t), \ldots, e_n(t))^T = Q(x(t)) - S(y(t)) \\ &= (Q_1(x(t)) - S_1(y(t)), \ldots, Q_n(x(t)) - S_n(y(t)))^T. \end{aligned} \tag{6} $$

Q-S synchronization occurs between systems (4) and (5) with respect to $Q(x(t)) = (Q_1(x(t)), \ldots, Q_n(x(t)))^T$ and $S(y(t)) = (S_1(y(t)), \ldots, S_n(y(t)))^T$, if there exists a controller $U(x,y,t)$ such that all trajectories (x,y) in (4) and (5) with any initial conditions $(x(0), y(0))$ approach the manifold $M = \{(x(t), y(t)) | Q_i(x(t)) = S_i(y(t)), i = 1, 2, \ldots, n\}$ as time t goes to infinity, that is to say,

$$ \lim_{t\to\infty} e_i(t) = \lim_{t\to\infty}(Q_i(x(t)) - S_i(y(t)) = 0, \qquad i = 1, 2, \ldots, n. \tag{7} $$

The unified chaotic system unifies Lorenz, Chen and Lü systems which was introduced by Lü *et al* [30] and can be described by

$$ \begin{cases} \dot{x_1} = (25a+10)(x_2 - x_1), \\ \dot{x_2} = (28-35a)x_1 - x_1x_3 + (29a-1)x_2, \\ \dot{x_3} = x_1x_2 - \dfrac{a+8}{3}x_3, \end{cases} \tag{8} $$

where $a \in [0, 1]$. It has been proved that system (8) is chaotic for all $a \in [0, 1]$, particularly, system (8) reduces to Lorenz system for $a = 0$, Lü system for $a = 0.8$ and Chen

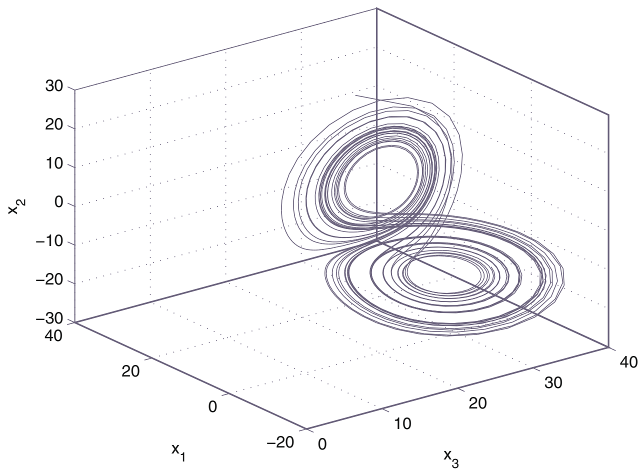

Figure 1. Chaotic attractors of Lorenz-like system in the fractional-order unified system: $a = 0.4$, $q = 0.9$.

system for $a = 1$ respectively. Numerical results have shown that its chaotic attractors are similar to those of the corresponding Lorenz and Chen attractors for $a \in [0, 0.8]$ and $a \in (0.8, 1]$.

In this paper, consider the fractional-order unified system [27] defined as follows:

$$\begin{cases} D_*^q x_1 = (25a + 10)(x_2 - x_1), \\ D_*^q x_2 = (28 - 35a)x_1 - x_1 x_3 + (29a - 1)x_2, \\ D_*^q x_3 = x_1 x_2 - \dfrac{a + 8}{3} x_3, \end{cases} \tag{9}$$

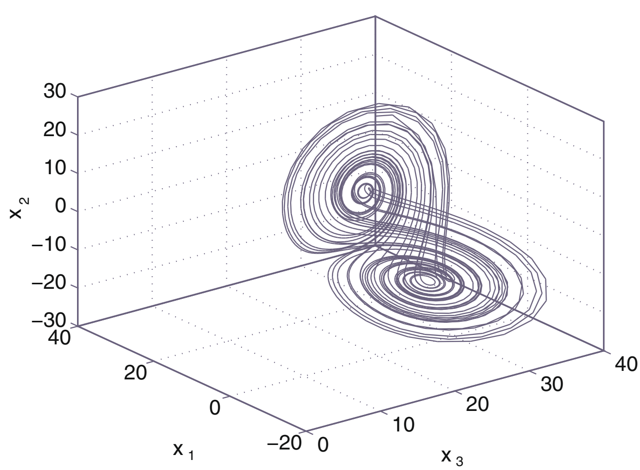

Figure 2. Chaotic attractors of Lü system in the fractional-order unified system: $a = 0.8$, $q = 0.9$.

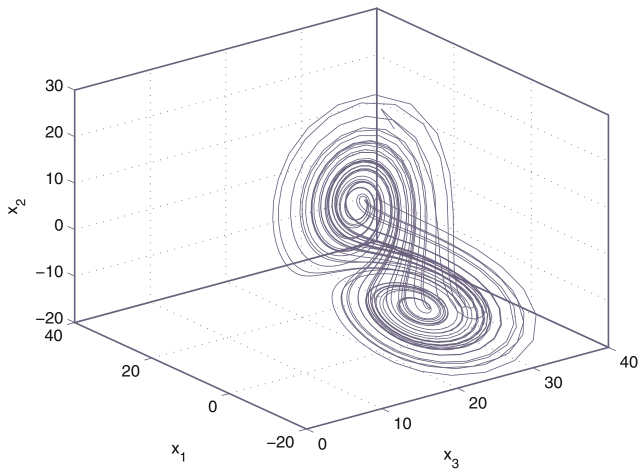

Figure 3. Chaotic attractors of Chen system in the fractional-order unified system: $a = 1, q = 0.9$.

where q is the fractional order, $0 < q < 1$. According to the computation method of the largest Lyapunov exponent proposed by Benettin *et al* [31], when the parameter a are chosen as 0.4, 0.8 and 1, $q = 0.9$, the values of the largest Lyapunov exponents are 0.741, 0.804 and 0.715, respectively [32]. Figures 1, 2 and 3 display these chaotic attractors of the Lorenz-like system ($a = 0.4$), the Lü system ($a = 0.8$) and the Chen system ($a = 1$), respectively.

3. Q-S synchronization

In the section, we shall study Q-S synchronization behaviour of the fractional-order unified system by designing two control schemes. The drive and the response systems are described as follows, respectively:

$$\begin{cases} D_*^q x_1 = (25a + 10)(x_2 - x_1), \\ D_*^q x_2 = (28 - 35a)x_1 - x_1 x_3 + (29a - 1)x_2, \\ D_*^q x_3 = x_1 x_2 - \dfrac{a+8}{3} x_3, \end{cases} \tag{10}$$

and

$$\begin{cases} D_*^q y_1 = (25a + 10)(y_2 - y_1) + u_1, \\ D_*^q y_2 = (28 - 35a)y_1 - y_1 y_3 + (29a - 1)y_2 + u_2, \\ D_*^q y_3 = y_1 y_2 - \dfrac{a+8}{3} y_3 + u_3, \end{cases} \tag{11}$$

where x_i and y_i ($i = 1, 2, 3$) stand for state variables of the master system and the slave system, respectively, u_1, u_2 and u_3 are the nonlinear controllers to be designed later.

Assume the observable variables of systems (10) and (11) to be

$$\begin{cases} Q_1(x_1, x_2, x_3) = x_1 + x_2, \\ Q_2(x_1, x_2, x_3) = x_2 + x_3, \\ Q_3(x_1, x_2, x_3) = x_3 + x_1, \end{cases} \tag{12}$$

and

$$\begin{cases} S_1(y_1, y_2, y_3) = y_1 + y_2, \\ S_2(y_1, y_2, y_3) = y_2 + y_3, \\ S_3(y_1, y_2, y_3) = y_3 + y_1. \end{cases} \tag{13}$$

DEFINITION 1

For given vector functions (12) and (13), Q-S synchronization between the drive system (10) and the response system (11) will be achieved, if there exist a suitable controller $u = (u_1, u_2, u_3)^T$ such that

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|Q(x) - S(y)\| = 0, \tag{14}$$

where$\|\cdot\|$ is the Euclidean norm.

It follows from (10)–(13) that we have the following error dynamical system:

$$\begin{cases} D_*^q e_1 = (54a + 9)e_1 - (114a - 9)(x_1 - y_1) \\ \qquad\qquad - x_1x_3 + y_1y_3 - u_1 - u_2, \\ D_*^q e_2 = \dfrac{88a + 5}{3}e_1 - \dfrac{a + 8}{3}e_2 + \dfrac{79 - 193a}{3}(x_1 - y_1) \\ \qquad\qquad - x_1x_3 + x_1x_2 + y_1y_3 - y_1y_2 - u_2 - u_3, \\ D_*^q e_3 = -\dfrac{a + 8}{3}e_1 + (25a + 10)e_2 - (25a + 10)e_3 \\ \qquad\qquad + \dfrac{a + 8}{3}(x_1 - y_1) + x_1x_2 - y_1y_2 - u_1 - u_3, \end{cases} \tag{15}$$

where $e_1 = x_2 + x_1 - y_1 - y_2$, $e_2 = x_2 + x_3 - y_2 - y_3$, $e_3 = x_1 + x_3 - y_3 - y_1$. Our aim is to find suitable control laws $u_i (i = 1, 2, 3)$ for stabilizing the error dynamics system (15). To this end, the following control laws are proposed:

$$\begin{cases} u_1 = -\dfrac{74a + 22}{3}(x_1 - y_1) + \dfrac{1}{2}ke_1, \\ u_2 = \dfrac{49 - 268a}{3}(x_1 - y_1) - x_1x_3 + y_1y_3 + \dfrac{1}{2}ke_1, \\ u_3 = (25a + 10)(x_1 - y_1) + x_1x_2 - y_1y_2 - \dfrac{1}{2}ke_1, \end{cases} \tag{16}$$

where k is the feedback gain.

Then, we have the following theorem.

Theorem 1. *If observable variables of drive system* (10) *and response system* (11) *are taken as* (12) *and* (13), *control laws are chosen as* (16), *and feedback gain* k *satisfies* $k > 54a + 9$, *then Q-S synchronization between systems* (14) *and* (15) *will be obtained.*

Proof. By substituting eq. (16) into eq. (15), we obtain

$$\begin{cases} D_*^q e_1 = (54a + 9 - k)e_1, \\ D_*^q e_2 = \dfrac{88a + 5}{3} e_1 - \dfrac{a + 8}{3} e_2, \\ D_*^q e_3 = -\dfrac{a + 8}{3} e_1 + (25a + 10)e_2 - (25a + 10)e_3. \end{cases} \tag{17}$$

Error system (17) can be rewritten as the following matrix form:

$$\begin{bmatrix} D_*^q e_1 \\ D_*^q e_2 \\ D_*^q e_3 \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 54a + 9 - k & 0 & 0 \\ \dfrac{88a + 5}{3} & -\dfrac{a + 8}{3} & 0 \\ -\dfrac{a + 8}{3} & 25a + 10 & -(25a + 10) \end{bmatrix}.$$

It is easy to obtain characteristic root of matrix A as

$$\lambda_1 = 54a + 9 - k, \qquad \lambda_2 = -\frac{a + 8}{3}, \qquad \lambda_3 = -25a - 10.$$

In this case, any eigenvalue of matrix A satisfies

$$|\arg(\lambda_{\mathrm{i}})| > \frac{\pi}{2} > \frac{q\pi}{2}, \qquad q < 1, i = 1, 2, 3, \tag{18}$$

if and only if feedback gain k satisfies $k > 54a + 9$ ($a \in [0, 1]$). According to the stability theory of fractional-order systems [33], error system (17) is asymptotically stable, which implies that Q-S synchronization between systems (10) and (11) will be achieved.

Theorem 2. *For given observable variables* (12) *and* (13) *of drive system* (10) *and response system* (11), *Q-S synchronization between systems* (10) *and* (11) *will occur by the following control scheme*:

$$\begin{cases} u_1 = -\dfrac{1}{2} x_1 x_2 + \dfrac{1}{2} y_1 y_2 + (4 - 25a)e_1 \\ \qquad + \dfrac{1}{2}(28 - 35a)e_2 - \dfrac{1}{3}(37a + 11)e_3, \\ u_2 = (29 - 64a)(x_2 - y_2) - x_1 x_3 + y_1 y_3 + \dfrac{1}{2} x_1 x_2 - \dfrac{1}{2} y_1 y_2 \\ \qquad + (25a + 24)e_1 + \dfrac{1}{2}(28 - 35a)e_2 + \dfrac{1}{3}(37a + 11)e_3, \\ u_3 = \dfrac{a + 8}{3}(x_2 - y_2) + \dfrac{1}{2} x_1 x_2 - \dfrac{1}{2} y_1 y_2 \\ \qquad - 14e_1 - \dfrac{1}{2}(28 - 35a)e_2 + \dfrac{1}{3}(37a + 11)e_3. \end{cases} \tag{19}$$

Proof. Define the error signals as

$$\begin{cases} e_1 = x_1 + x_2 - y_1 - y_2, \\ e_2 = x_2 + x_3 - y_2 - y_3, \\ e_3 = x_3 + x_1 - y_3 - y_1. \end{cases} \tag{20}$$

Combine (10) and (11) with (19), then we have the following error dynamics:

$$\begin{cases} D_*^q e_1 = -35ae_1 + (60a - 18)e_2 - (25a + 10)e_3, \\ D_*^q e_2 = (18 - 60a)e_1 - \dfrac{a+8}{3}e_2 - \dfrac{74a+22}{3}e_3, \\ D_*^q e_3 = (25a + 10)e_1 + \dfrac{74a+22}{3}e_3 - (25a + 10)e_3. \end{cases} \tag{21}$$

Transform (21) into matrix form

$$\begin{bmatrix} D_*^q e_1 \\ D_*^q e_2 \\ D_*^q e_3 \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

where

$$A = \begin{bmatrix} -35a & 18 - 60a & -(25a + 10) \\ 18 - 60a & -\dfrac{a+8}{3} & -\dfrac{74a+22}{3} \\ 25a + 10 & \dfrac{74a+22}{3} & -(25a + 10) \end{bmatrix}.$$

Suppose λ is one of the eigenvalues of matrix A and the corresponding non-zero eigenvector is $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^T$, i.e.,

$$A\varepsilon = \lambda\varepsilon. \tag{22}$$

Take conjugate transpose on both sides of eq. (22), and one obtains

$$\overline{(A\varepsilon)^T} = \bar{\lambda}\varepsilon^H. \tag{23}$$

Equation (22) multiplied left by $\frac{1}{2}\varepsilon^H$ plus eq. (23) multiplied right by $\frac{1}{2}\varepsilon$, we derive that

$$\varepsilon^H \left(\frac{1}{2}A + \frac{1}{2}A^H \right) \varepsilon = \frac{1}{2}(\lambda + \bar{\lambda})\varepsilon^H \varepsilon. \tag{24}$$

From eq. (24), we have

$$\frac{1}{2}(\lambda + \bar{\lambda}) = \varepsilon^H \left(\frac{1}{2}A + \frac{1}{2}A^H \right) \frac{\varepsilon}{\varepsilon^H \varepsilon}. \tag{25}$$

By substituting A into eq. (25), we obtain

$$\frac{1}{2}(\lambda + \bar{\lambda}) = \varepsilon^H \frac{1}{\varepsilon^H \varepsilon} \begin{bmatrix} -35a & 0 & 0 \\ 0 & -\dfrac{a+8}{3} & 0 \\ 0 & 0 & -(25a + 10) \end{bmatrix} \varepsilon. \tag{26}$$

$\lambda + \bar{\lambda} < 0$ ($a \in [0, 1]$), i.e., any eigenvalue of matrix A satisfies

$$|\arg(\lambda)| > \frac{\pi}{2} > \frac{q\pi}{2}, \qquad q < 1. \tag{27}$$

According to the stability theorem of linear FDEs [33], error system (21) is asymptotically stable, which implies Q-S synchronization between the drive system (10) and the response system (11) is achieved under the nonlinear controller (19).

4. Numerical simulations

In this section, to verify theoretical results obtained in the previous section, the corresponding numerical simulations will be performed. A predictor–corrector algorithm for fractional-order differential equations is applied [34]. In all simulations, fractional-order q is chosen as 0.9.

Case I: *Fractional-order Lorenz-like system*: When $a = 0.4$, system (9) is the fractional-order Lorenz-like system. Base on Theorem 1, feedback gain k must satisfy $k > 30.6$. In the simulation, we chose feedback gain k as 32. The initial values of the drive and the response systems are chosen as $(x_1(0), x_2(0), x_3(0)) = (1, 2, 3)$ and $(y_1(0), y_2(0), y_3(0)) = (-2, -5, 18)$, respectively. Figures 4 and 5 show the error state time response of system (10) and system (11) with the controllers (16) and (19), respectively.

Case II: *Fractional-order Lü system*: When $a = 0.8$, system (9) reduces to the fractional-order Lü system. In the simulation, we take feedback gain k as 53, which satisfies $k > 52.2$ according to Theorem 1, and choose the initial values of the drive and the response

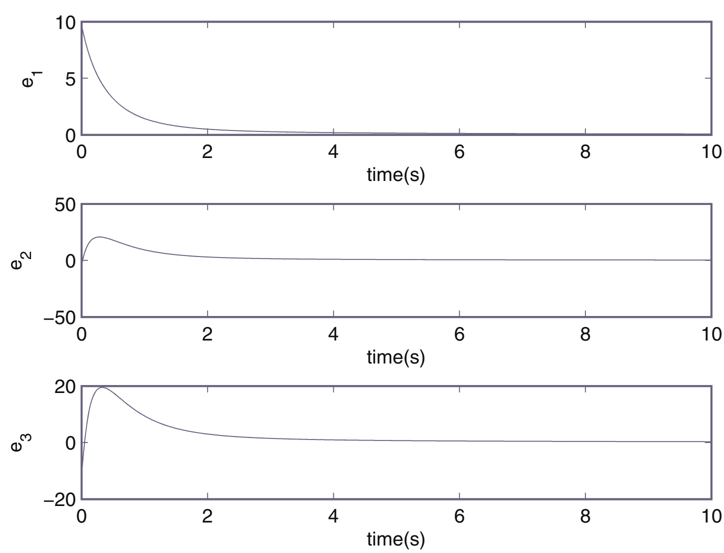

Figure 4. The error time response of systems (10) and (11) with controller (16) ($a = 0.4$, fractional-order Lorenz-like system).

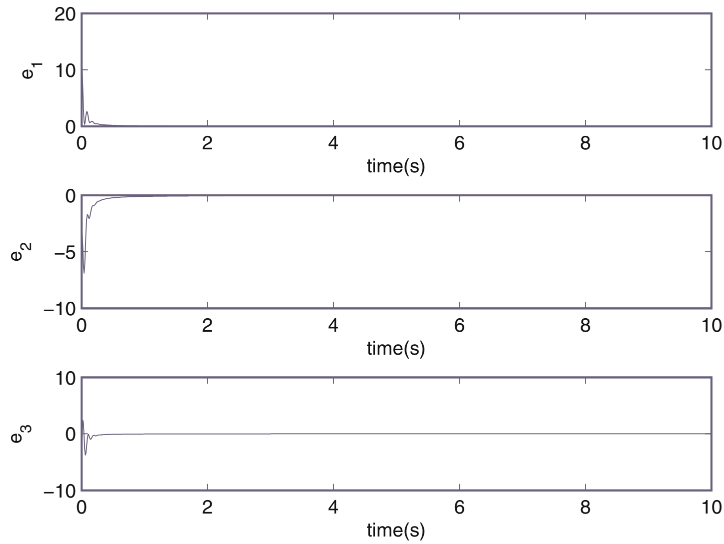

Figure 5. The error time response of systems (10) and (11) with controller (19) ($a = 0.4$, fractional-order Lorenz-like system).

systems as $(x_1(0), x_2(0), x_3(0)) = (2, 3, 4)$ and $(y_1(0), y_2(0), y_3(0)) = (5.5, 7.5, -6.5)$, respectively. The synchronization error states between systems (10) and (11) under the controllers (16) and (19) are displayed in figures 6 and 7, respectively.

Case III: *Fractional-order Chen system*: When $a = 1$, system (9) is the fractional-order Chen system. In the simulation, feedback gain k is chosen as 65, which satisfies

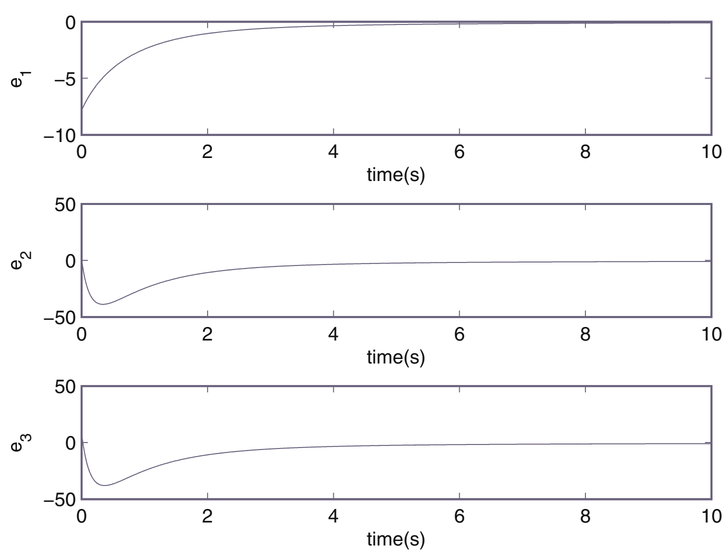

Figure 6. The error time response of systems (10) and (11) with controller (16) ($a = 0.8$, fractional-order Lü system).

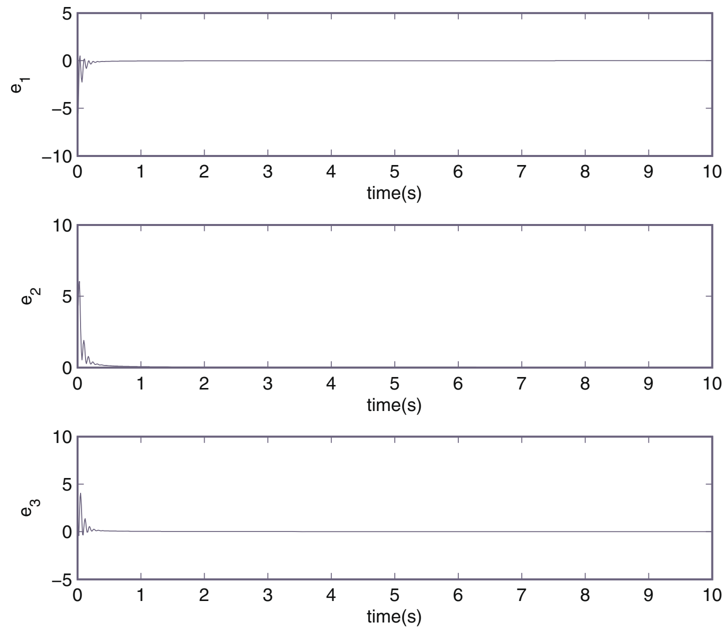

Figure 7. The error time response of systems (10) and (11) with controller (19) ($a = 0.8$, fractional-order Lü system).

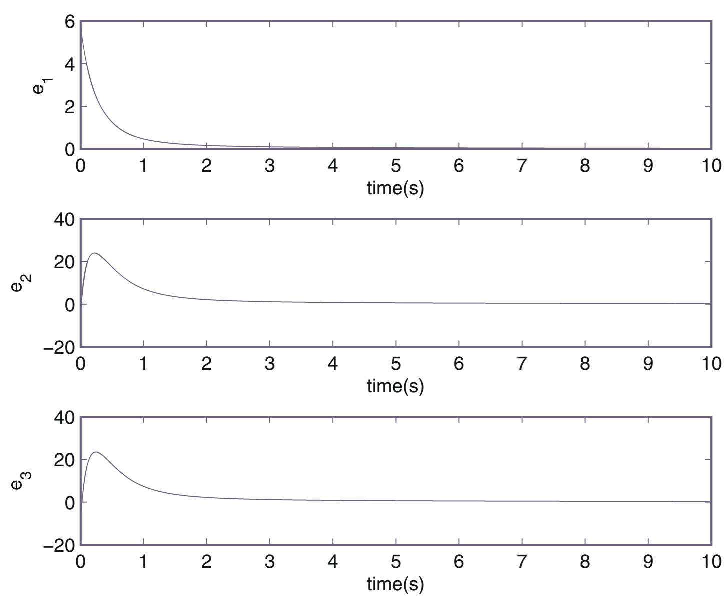

Figure 8. The error time response of systems (10) and (11) with controller (16) ($a = 1$, fractional-order Chen system).

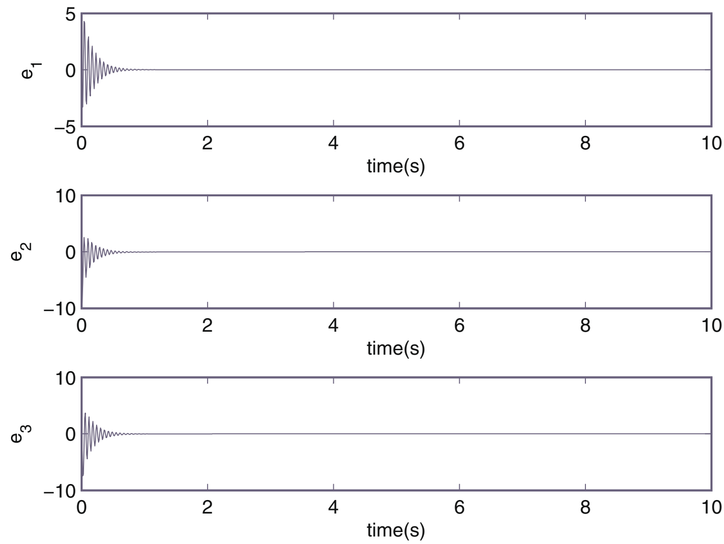

Figure 9. The error time response of systems (10) and (11) with controller (19) ($a = 1$, fractional-order Chen system).

condition $k > 63$ in Theorem 1. The initial values of the drive and the response systems are taken as $(x_1(0), x_2(0), x_3(0)) = (1, 4, 6)$ and $(y_1(0), y_2(0), y_3(0)) = (-8.5, 7.5, 10.5)$, respectively. Figures 8 and 9 display the error state time response between systems (10) and (11) with the controllers (16) and (19).

Remark. From the simulation result we can find that the state response time under controller (19) is shorter than those under controller (16), which also confirms the superiority of the controller (19) over the controller (16).

5. Conclusion

In this paper, we extend the concept of Q-S synchronization for integer-order systems to fractional-order systems and investigate Q-S synchronization of the fractional-order unified system. Based on the stability theory of fractional-order systems, two suitable controllers are designed. Finally, numerical simulations are provided to verify the effectiveness of the results obtained. It has to be noted that as we consider only the case of the given $Q(x)$ and $S(y)$, there may exist some limitations in practical application. Readers can establish more criteria to guarantee Q-S synchronization of fractional-order dynamical system by introducing different observable variable functions. Therefore, the next work to be done is to propose a general scheme for Q-S synchronization of fractional-order chaotic systems, which will be discussed in future papers.

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