PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 80, No. 2
February 2013
pp. 361–367

Soliton solutions of some nonlinear evolution equations with time-dependent coefficients

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MS received 14 March 2012; revised 3 July 2012; accepted 24 July 2012

Abstract. In this paper, we obtain exact soliton solutions of the modified KdV equation, inhomogeneous nonlinear Schrödinger equation and $G(m, n)$ equation with variable coefficients using solitary wave ansatz. The constraint conditions among the time-dependent coefficients turn out as necessary conditions for the solitons to exist. Numerical simulations for dark and bright soliton solutions for the mKdV equation are also given.

Keywords. Dark and bright soliton; KdV equation; nonlinear Schrödinger equation; $G(m, n)$ equation.

PACS Nos 42.81.Dp; 42.65.Tg; 05.45.Yv

1. Introduction

To find exact solutions of the nonlinear evolution equations (NLEEs) is one of the central themes in mathematics and physics. In recent years, many powerful methods have been developed to derive the exact solutions of NLEEs [1–7]. When inhomogeneities of the media and nonuniformity of the boundaries are taken into account in various real physical situations, the variable-coefficient NLEEs provide more powerful and realistic models than their constant-coefficient counterparts. In ref. [8], Tian and Gao put their focus on a variable coefficient higher order nonlinear Schrödinger model, which can be used to describe the femtosecond pulse propagation, which is particularly applicable in the design of ultrafast signal routing and dispersion-managed fibre transmission systems. The modified KdV-type (mKdV) equation, on the other hand, has been discovered recently, to model the dust-ion-acoustic waves in cosmic environments as those of the supernova shells and Saturn's F-ring [9]. The phenomenon of shallow water wave engineering experiment is modelled by Gardener equation (GE), which is basically the KdV equation with dual power-law nonlinearity [10]. In this paper, the mKdV equation, nonlinear Schrödinger (NLS) equation and generalized GE known as $G(m, n)$ equation with time-dependent coefficients will be integrated using a modern technique. It will be

DOI: 10.1007/s12043-012-0467-2; ***e*Publication: 24 January 2013**

seen that for time-dependent coefficients, the criterion for chiral soliton to exist is that dispersion coefficient must be Riemann integrable.

2. Modified KdV equation with time-dependent coefficients

The mKdV equation appears in many fields such as acoustic waves in certain anharmonic lattices, Alfvén waves in a collisionless plasma, transmission lines in Schottky barrier, models of traffic congestion, ion acoustic solitons, elastic media, etc. It possesses many remarkable properties such as Miura transformation, conservation laws, inverse scattering transformation, bilinear transformation, N-solitons, Bäcklund transformation, Painlevé integrability, Darboux transformation, etc. More recently, Pradhan and Panigrahi [11] studied the modified KdV equation with variable coefficients

$$u_t + \alpha(t)u_x - \beta(t)u^2u_x + \gamma(t)u_{xxx} = 0, \tag{1}$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are analytic functions of t.

To find the bright soliton solution for eq. (1), we use the following solitary ansatz [12]:

$$u(x,t) = A\ \mathrm{sech}^p[B(x - vt)], \tag{2}$$

where A, B and v are respectively the amplitude, the inverse width and the velocity of the soliton, and all are time-dependent. The exponent p will be determined during the course of soliton derivation. Substituting eq. (2) into eq. (1), we get an algebraic equation. To determine the value of p, equate the highest exponents of $\mathrm{sech}^{3p}(\theta)\tanh(\theta)$ and $\mathrm{sech}^{p+2}(\theta)\tanh(\theta)$ functions in the resulting equation, we get $p = 1$. On collecting the coefficients of $\mathrm{sech}^p(\theta)$, $\mathrm{sech}^p(\theta)\tanh(\theta)$, $\mathrm{sech}^{3p}(\theta)\tanh(\theta)$ or $\mathrm{sech}^{p+2}(\theta)\tanh(\theta)$, respectively, where each coefficient has to vanish, we obtain the following system of algebraic equations:

$$B_t = 0, \tag{3a}$$

$$xB_t - (Bvt)_t + \alpha(t)B + \gamma(t)B^3 = 0, \tag{3b}$$

$$\beta(t)A^2 + 6\gamma(t)B^2 = 0. \tag{3c}$$

Equation (3a) gives $B(t) = B_0 =$ constant, which is the initial inverse width of solitons. From eq. (3b), the soliton velocity is determined as

$$v(t) = \frac{1}{t}\left\{\int(\alpha(t) + \gamma(t)B_0^2)\mathrm{d}t + C_1\right\}, \tag{4}$$

where C_1 is the integration constant and from eq. (3c), we obtain

$$A(t) = \sqrt{\frac{-6\gamma(t)}{\beta(t)}}B_0, \tag{5}$$

which exists provided $\gamma(t)\beta(t) < 0$. As inverse width $B(t)$ is a constant, eq. (5) implies that the pulse amplitude $A(t)$ is affected by the time dependence of the distributed coefficients $\beta(t)$ and $\gamma(t)$ and for a constant amplitude pulse, the ratio of $(\gamma(t)/\beta(t))$ must be a constant. This in turn gives the bright soliton solution

$$u(x,t) = \sqrt{\frac{-6\gamma(t)}{\beta(t)}}B_0\ \mathrm{sech}[B_0(x - vt)], \tag{6}$$

where the velocity $v(t)$ is given by eq. (4) and it should be noted that the soliton solution eq. (6) exists under the condition $\gamma(t)\beta(t) < 0$. Figure 1 shows the numerical simulation of eq. (6) for the bright 1-soliton. All parameter values are chosen in dimensionless units to perform numerical simulation, but they meet the constraint condition for solitons to exist. In order to find dark solitary wave (topological solitons) solutions for eq. (1), a proper choice for the function $u(x, t)$ would be

$$u(x,t) = A\ \tanh^p[B(x - vt)]. \tag{7}$$

Substituting eq. (7) into eq. (1) and collecting the coefficients of $\tanh^p(\theta)$, $\theta\ \tanh^{p+1}(\theta)$, etc. in the resulting expression, we obtain a set of algebraic equations. The solutions of such algebraic equations provide soliton amplitude A and inverse width B as constants and the soliton velocity as

$$v(t) = \frac{1}{t}\left\{\int(-2\gamma(t)B_0^2 + \alpha(t))\mathrm{d}t + C_2\right\}, \tag{8}$$

where C_2 is an integration constant and the velocity of soliton is not a constant. This velocity will be meaningful provided the functions $\alpha(t)$ and $\gamma(t)$ are Riemann integrable. We also get a constraint relation between A and B as $A(t) = \sqrt{(6\gamma(t)/\beta(t))}\ B(t)$. Since $A(t)$ and $B(t)$ are constants, $(\gamma(t)/\beta(t))$ = constant which shows that it is necessary to have $\beta(t)\gamma(t) > 0$ for soliton solutions to exist. Finally, the dark soliton solution for the time-dependent mKdV eq. (1) from eq. (7) is given as

$$u(x,t) = \sqrt{\frac{6\gamma(t)}{\beta(t)}}B_0\ \tanh[B_0(x - vt)], \tag{9}$$

where the velocity of dark soliton is given by eq. (8). Figure 2 shows the numerical simulation of eq. (9) for different choices of parameters.

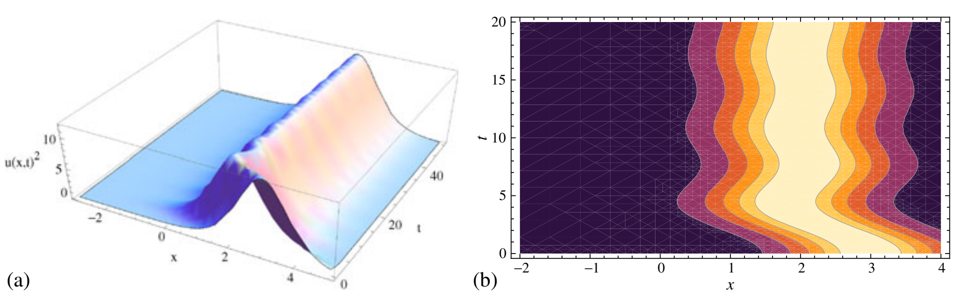

Figure 1. (**a**) The intensity of bright solitary wave $u^2(x, t)$, which satisfies the constraint $\gamma(t)\beta(t) < 0$ for $\alpha(t) = \cos(t)$, $\beta(t) = -1$, $\gamma(t) = 2$, $B_0 = 1$, $C_1 = 0$. (**b**) The corresponding contour plot in x–t plane with the same parameters as in (**a**).

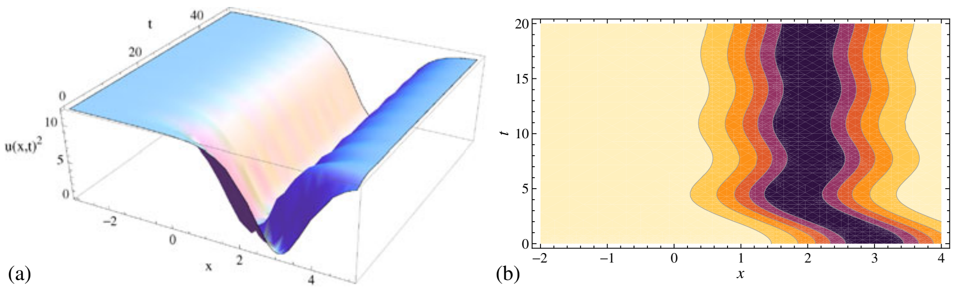

Figure 2. (**a**) The intensity of dark solitary wave $u^2(x,t)$ which satisfies the constraint $\gamma(t)\beta(t) > 0$ for $\alpha(t) = \cos(t)$, $\beta(t) = -1.0$, $\gamma(t) = 2.0$, $B_0 = 1.0$, $C_2 = 0$. (**b**) The corresponding contour plot in x–t plane with the same parameters as in (**a**).

3. Inhomogeneous NLS equation with time-dependent coefficients

The NLS equation describes numerous nonlinear physical phenomena in the field of nonlinear science such as optical solitons in optical fibres, solitons in the mean-field theory of Bose–Einstein condensates, rogue waves in the nonlinear oceanography, etc. In the present work, we extend the NLS equation to the inhomogeneous NLS equation with variable coefficients, including group velocity dispersion $\beta(t)$, linear potential $V(x,t)$, nonlinearity $g(t)$ and the gain/loss term $\gamma(t)$, in the form [13]

$$iu_t + \frac{\beta(t)}{2}u_{xx} + V(x,t)u + g(t)|u|^2u = i\gamma(t)u, \tag{10}$$

and find bright and dark 1-soliton solutions.

In order to determine bright soliton solutions to eq. (10), we make an ansatz of the form

$$u(x,t) = A\cosh^{-p}[B(x - vt)]\mathrm{e}^{i(-kx+\omega t+\theta)}, \tag{11}$$

where k represents the frequency, ω is the wave number, while θ is the phase. It is to be noted that in eq. (10), the coefficients of dispersion and nonlinearity are time-dependent and the soliton parameters A, B, k, ω and θ may be time-dependent and $B(x - vt) = \tau$. Substituting eq. (11) into eq. (10) and requiring that the imaginary and real parts of each term be separately equal to zero, we get two coupled equations. From imaginary part equation, equating the coefficient of $1/\cosh^p\tau$, one gets $A(t) = A_0\mathrm{e}^{\int\gamma(t)\mathrm{d}t}$, where A_0 is an integral constant related to the initial pulse amplitude and setting the coefficient of $\tau\tanh\tau/\cosh^p\tau$ to zero yields $B(t) = \text{constant} = B_0$. Again setting the coefficient of $\tanh\tau/\cosh^p\tau$ to zero yields

$$v(t) = \frac{1}{t}\left\{\int\beta(t)k\,\mathrm{d}t + C_3\right\}, \tag{12}$$

where C_3 is an arbitrary constant. From real part equation, setting the coefficients of the linearly independent functions $1/\cosh^{p+2}\tau$ and $1/\cosh^{3p}\tau$ to zero leads to

$$B(t) = A_0\mathrm{e}^{\int\gamma(t)\mathrm{d}t}\sqrt{\frac{g(t)}{\beta(t)}}. \tag{13}$$

Since inverse width $B(t)$ is a constant, the amplitude $A(t)$ of the bright soliton depends on the ratio of nonlinear and dispersion coefficients and also modelled by the gain/loss coefficient $\gamma(t)$. From eq. (13), one needs to have $\beta(t)g(t) > 0$. Now from eq. (13), for $B(t)$ to be a constant, eq. (13) yields $g(t)\mathrm{e}^{2\int \gamma(t)\mathrm{d}t} = c\beta(t)$, where $c \in R$ is a constant. From real part, equating the coefficients of $1/\cosh^p \tau$ to zero gives

$$\omega(t) = \frac{1}{t}\left\{ \int \frac{1}{2} A_0 \mathrm{e}^{\int \gamma(t)\mathrm{d}t} \left[(B_0^2 - k^2)\beta(t) + 2V(x,t) \right] \mathrm{d}t + C_4 \right\}, \tag{14}$$

while the other soliton parameters, namely k and θ remain constants. Thus, finally the 1-soliton solution of the NLS equation is given by eq. (11), where inverse width $B(t)$ stays constant and the velocity $v(t)$ is given by eq. (12). The other conditions that need to hold for the solitons to exist is that $\beta(t)$ and $\gamma(t)$ are Riemann integrable and $\beta(t)g(t) > 0$.

For dark solitons with time-dependent coefficients, the 1-soliton solution is assumed by

$$u(x,t) = A \ \tanh^p[B(x - vt)]\mathrm{e}^{i(-kx+\omega t+\theta)}. \tag{15}$$

Here, as in the case of bright solitons with time-dependent coefficients, the soliton parameters are all time-dependent. By using the recipe described above, we obtain the dark soliton solution in the form

$$u(x,t) = A_0 \mathrm{e}^{\int \gamma(t)\mathrm{d}t} \ \tanh\left[A_0 \mathrm{e}^{\int \gamma(t)\mathrm{d}t} \sqrt{-\frac{g(t)}{\beta(t)}} (x - vt) \right] \mathrm{e}^{i(-kx+\omega t+\theta)}, \tag{16}$$

where the relation between parameters A and B is given by

$$B(t) = A_0 \mathrm{e}^{\int \gamma(t)\mathrm{d}t} \sqrt{-\frac{g(t)}{\beta(t)}},$$

which shows that the solitons will exist for $g(t)\beta(t) < 0$. The velocity and the wave number are determined, respectively from the following equations.

$$v(t) = \frac{1}{t}\left\{ \int -\beta(t)k \ \mathrm{d}t + C_5 \right\}, \tag{17}$$

$$\omega(t) = -\frac{1}{t}\left\{ \int \frac{1}{2} A_0 \mathrm{e}^{\int \gamma(t)\mathrm{d}t} \left[(2B_0^2 + k^2)\beta(t) + 2V(x,t) \right] \mathrm{d}t + C_6 \right\}, \tag{18}$$

where C_5 and C_6 are the integration constants and k and θ are assumed to remain constants.

4. *G*(*m*, *n*) equation with time-dependent coefficients

The generalized GE with full nonlinearity is given by

$$(u^l)_t + [2\alpha(t)u^m + 3\beta(t)u^{2m}]u_x + \gamma(t)(u^n)_{xxx} = 0, \tag{19}$$

where l, m and n are positive real numbers. This generalization is based on the same kind as given by the $K(m,n)$ equation, which is a generalization of the KdV equation. Hence, based on the similarity of the structure, eq. (19) is referred to as the $G(m,n)$ equation with generalized evolution.

In order to solve eq. (19), we consider the 1-soliton solution of the form

$$u(x,t) = A(\lambda + \cosh[B(x - vt)])^{-p}, \tag{20}$$

where $A = A(t)$, $B = B(t)$ and $v = v(t)$ are time-dependent coefficients, which will be determined as functions of the model coefficients α, β and γ. Here, in addition to the usual soliton parameters, λ is a constant that will also be determined. The exponent p is unknown at this stage, and its value will be revealed during the course of derivation of the 1-soliton solution to the $G(m,n)$ equation given by eq. (19). Substituting eq. (20) in eq. (19), we get

$$\begin{aligned}
&\frac{lA^{l-1}A_t}{[\lambda+\cosh(\theta)]^{pl}} - plA^l\frac{[\frac{\theta}{B}B_t - B(v+tv_t)]\sinh(\theta)}{[\lambda+\cosh(\theta)]^{pl+1}} \\
&- \frac{2\alpha(t)pBA^{m+1}\sinh(\theta)}{[\lambda+\cosh(\theta)]^{p(m+1)+1}} - \frac{3\beta(t)pBA^{2m+1}\sinh(\theta)}{[\lambda+\cosh(\theta)]^{p(2m+1)+1}} \\
&+ \gamma(t)\Big[-\frac{pn(pn+1)(pn+2)B^3A^n(\lambda^2-1)\sinh(\theta)}{[\lambda+\cosh(\theta)]^{np+3}} - \frac{(pnB)^3A^n\sinh(\theta)}{[\lambda+\cosh(\theta)]^{np+1}}\Big] \\
&+ \frac{pn(pn+1)(2pn+1)\lambda B^3A^n\sinh(\theta)}{[\lambda+\cosh(\theta)]^{np+2}} = 0,
\end{aligned} \tag{21}$$

where $\theta = B(x - vt)$. Now from eq. (21), equating the exponents $p(2m+1)+1$ and $pn+3$ gives $p(2m+1)+1 = pn+3$, which yields the exponent $p = 2/(2m-n+1)$. It is emphasized to note that the bright soliton solution exists only when $p > 0$. Therefore, we should have $2m+1 > n$ for the existence of the soliton solution. Then, by setting their corresponding coefficients to zero, we find

$$B(t) = \sqrt{-\frac{3\beta(t)A^{2m-n+1}}{\gamma(t)n(pn+1)(pn+2)(\lambda^2-1)}}. \tag{22}$$

Again from eq. (21), equating the exponents $p(m+1)+1$ and $pn+2$ gives $p(m+1)+1 = pn+2$, which yields $p = 1/(m-n+1)$. Also, equating the two values of p, one gets $n = 1$ and therefore, $p = 1/m$. This shows that, for $G(m,n)$ given by eq. (19), the 1-soliton solution exists only for $n = 1$. Now from eq. (21), setting the coefficients of the $\sinh(\theta)/[\lambda+\cosh(\theta)]^{p(m+1)+1}$ and $\sinh(\theta)/[\lambda+\cosh(\theta)]^{np+2}$ terms to zero gives

$$A = \Big[\frac{\gamma(t)(m+1)(m+2)\lambda B^2}{2\alpha(t)m^2}\Big]^{1/m}. \tag{23}$$

Substituting eq. (23) into eq. (22), one finds that

$$\lambda = -\frac{1}{4}\frac{3\beta(t)A^m \pm \sqrt{9\beta(t)^2A^{2m} + 16\alpha(t)^2}}{\alpha(t)}.$$

Equation (22) shows that the solitons will exist for $\beta(t)\gamma(t) < 0$ as long as $\lambda > 0$, which results from expression of λ. By equating the exponents $pl+1$ and $pn+1$ in eq. (21), we get $l = n$. Hence, for the $G(m,n)$ equation with full nonlinearity, soliton solution will exist as long as $l = n = 1$. The soliton velocity v is found by setting the coefficients of

$\sinh(\theta)/[D+\cosh(\theta)]^{pl+1}$ and $\sinh(\theta)/[D+\cosh(\theta)]^{pn+1}$ terms with $l=n$ in eq. (21) to zero:

$$v(t) = \frac{1}{t}\left(\int \frac{n^3 B^3 p^2 A^{n-1}}{l} \mathrm{d}t + C_7\right), \tag{24}$$

where C_7 is an arbitrary integration constant. As the inverse width B and amplitude of soliton pulse A are constants and after substituting the values of n, l, p in eq. (24), the soliton velocity is given by

$$v(t) = \frac{B^2}{m^2} + \frac{C_7}{t}.$$

Thus, the $G(m,n)$ equation is reduced to the KdV equation with dual power-law nonlinearity under the condition $l=n=1$, whose 1-soliton solution is given by

$$u(x,t) = \frac{A}{(\lambda + \cosh[B(x-vt)])^{1/m}},$$

where the inverse width B of the soliton is given by eq. (22) and velocity v is given by eq. (24).

5. Conclusions

This paper integrates the mKdV, inhomogeneous NLS and $G(m,n)$ equations with time-dependent coefficients. The exact topological and non-topological 1-soliton solutions are obtained. It is only necessary that these time-dependent coefficients should be Riemann integrable. The parameter domain is identified and the constraint relation between these time-dependent coefficients is obtained for the solitons to exist. The method used here is far less involved than the other standard techniques used to study these types of problems.

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