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# Studying the cosmological apparent horizon with quasistatic coordinates

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**Abstract.** This article aims at a natural generalization of the static coordinates to the $(n + 1)$-dimensional Friedmann–Lemaître–Robertson–Walker (FLRW) Universe. After demonstrating a no-go theorem, we put forward the quasistatic coordinates for the FLRW Universe. Then, the quasistatic coordinates are utilized to study the unified first law and the scalar-type perturbations on the cosmological apparent horizon.

**Keywords.** Cosmological horizon; thermodynamics of gravity.

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## 1. Motivation

The metric of de Sitter space-time has many forms in various coordinate systems [1]. They are useful for different purposes. In cosmology, two of them are most frequently used. One is written in planar coordinates [1]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{e}^{\pm 2t/l}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{n-1}^2), \tag{1}$$

while the other is in static coordinates [1]

$$\mathrm{d}s^2 = -\left(1 - \frac{\tilde{r}^2}{l^2}\right)\mathrm{d}\tilde{t}^2 + \left(1 - \frac{\tilde{r}^2}{l^2}\right)^{-1}\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega_{n-1}^2. \tag{2}$$

As a generalization of eq. (1) to spatially homogeneous and isotropic Universe, the $(n+1)$-dimensional Friedmann–Lemaître–Robertson–Walker (FLRW) metric is usually written as [1a]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{n-1}^2). \tag{3}$$

However, to our knowledge, the analogue of eq. (2) for such a space-time is hitherto lacking.

Our main concern in the present article is to extend eq. (2) to the FLRW Universe. Actually, FLRW metric, other than de Sitter and Minkowski space-times cannot be written in spherically symmetric static coordinates. This will be proved in §2. But in §3, we shall show that a natural generalization of eq. (2) does exist in non-static coordinates, which will be called quasistatic coordinates. The quasistatic coordinates are useful for describing the cosmological apparent horizon. So in §4 we shall employ them to study the unified first law on the apparent horizon. In cosmology, scalar-type perturbations are of particular interest, especially on the apparent horizon. The equation of motion for scalar fields will be discussed in §5 with the quasistatic coordinates, which lead to a simple interpretation of the effective potential. Section 6 is a summary of our main results and some open problems.

## 2. Isotropic Killing vector

We are interested in static coordinates with spherical symmetry. Therefore, we seek for a time-like isotropic Killing vector of the form

$$k^{\mu}\partial_{\mu} = P(t,r)\partial_t + Q(t,r)\partial_r. \tag{4}$$

To be a Killing vector, $k^{\mu}$ should satisfy

$$(\mathcal{L}_k g)_{\mu\nu} = 0, \tag{5}$$

where the Lie derivative is

$$(\mathcal{L}_k g)_{\mu\nu} = k^{\lambda} g_{\mu\nu,\lambda} + k^{\lambda}{}_{,\mu} g_{\lambda\nu} + k^{\lambda}{}_{,\nu} g_{\lambda\mu}. \tag{6}$$

Corresponding to the FLRW metric (3), condition (5) gives the following system of equations:

$$\begin{aligned}
&\partial_t P = 0,\\
&\partial_r P - a^2\partial_t Q = 0,\\
&HP + \partial_r Q = 0,\\
&rHP + Q = 0,
\end{aligned} \tag{7}$$

where the Hubble parameter

$$H = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \tag{8}$$

is a function of $t$ or $a$. Solving this system of equations, we find that the vector $k^{\mu}$ is a Killing vector if and only if both $P$ and $H$ are constants. This implies that such a Killing vector does not exist in another FLRW Universe than de Sitter and Minkowski space-times.

We also notice that

$$k_{\mu}k^{\mu} = a^2Q^2 - P^2 = (a^2r^2H^2 - 1)P^2. \tag{9}$$

So the Killing vector $k^{\mu}$ is time-like in the region $r < (aH)^{-1}$.

In the absence of a time-like Killing vector, one cannot write the metric in a static form. On the other hand, if there is a time-like Killing vector, the metric can be written in the static form, like the de Sitter case. The lesson from this section is as follows: preserving the spherical symmetry, there is no static analog of metric (2) unless the FLRW Universe reduces to the de Sitter or Minkowski space-time.

## 3. Quasistatic coordinates

Based on the lesson of §2, we turn to a non-static generalization of metric (2) for the FLRW Universe. Specifically, we keep structures of the spatial part of eq. (2) and look for a metric of the form

$$\mathrm{d}s^2 = -\left(1 - H^2\tilde{r}^2\right) f^2 \mathrm{d}\tilde{t}^2 + \left(1 - H^2\tilde{r}^2\right)^{-1} \mathrm{d}\tilde{r}^2 + \tilde{r}^2 \mathrm{d}\Omega_{n-1}^2. \tag{10}$$

This metric will be called quasistatic metric. Here $H$ is the Hubble parameter defined by eq. (8). Both $H$ and $f$ are functions of the newly introduced temporal and radial coordinates $(\tilde{t}, \tilde{r})$.

Since the quasistatic metric (10) is equivalent to the standard FLRW metric (3), they are related by the coordinate transformation

$$\tilde{r} = ar, \qquad \frac{\partial t}{\partial \tilde{r}} = \frac{H\tilde{r}}{H^2\tilde{r}^2 - 1}, \qquad \frac{\partial t}{\partial \tilde{t}} = f(a, r) = f\left(a, \frac{\tilde{r}}{a}\right). \tag{11}$$

These relations should be used consistently. For instance, $\partial^2 t/\partial\tilde{t}\partial\tilde{r} = \partial^2 t/\partial\tilde{r}\partial\tilde{t}$ imposes a consistency condition

$$\frac{\partial f}{\partial \tilde{r}} = \frac{\partial}{\partial \tilde{t}}\left(\frac{H\tilde{r}}{H^2\tilde{r}^2 - 1}\right). \tag{12}$$

In particular, if $f = 1$, then the consistency condition demands that $H$ be a constant. For that special case, the metric becomes a de Sitter or Minkowski space-time in static coordinates.

From eq. (11), it is easy to get more relations as follows:

$$\begin{aligned}
&\frac{\partial r}{\partial \tilde{t}} = -Hrf, \frac{\partial r}{\partial \tilde{r}} = \frac{1}{a(1 - H^2\tilde{r}^2)},\\
&\frac{\partial \tilde{t}}{\partial t} = \frac{1}{f(1 - H^2\tilde{r}^2)}, \frac{\partial \tilde{t}}{\partial r} = \frac{Ha^2 r}{f(1 - H^2\tilde{r}^2)}.
\end{aligned} \tag{13}$$

The equations in the second line can be combined to give

$$Ha^2\frac{\partial \tilde{t}}{\partial t} - \frac{1}{r}\frac{\partial \tilde{t}}{\partial r} = H^2a^3\frac{\partial \tilde{t}}{\partial a} - \frac{1}{r}\frac{\partial \tilde{t}}{\partial r} = 0. \tag{14}$$

A straightforward solution to this equation is

$$C\mathrm{e}^{2\tilde{t}/l} = \frac{1}{l^2}\left(r^2 + \int^a \frac{2}{H^2a^3}\mathrm{d}a\right) = \frac{1}{l^2}\left(r^2 + \int^t \frac{2}{Ha^2}\mathrm{d}t\right), \tag{15}$$

in which $C$ is a dimensionless constant and $l$ is a constant length. With this solution, one directly finds that

$$C\mathrm{e}^{2\tilde{t}/l}\frac{\partial \tilde{t}}{\partial t} = \frac{1}{Hla^2}, \qquad C\mathrm{e}^{2\tilde{t}/l}\frac{\partial \tilde{t}}{\partial r} = \frac{r}{l}. \tag{16}$$

Comparing them with eq. (13), we get

$$f = \frac{Hla^2 C\mathrm{e}^{2\tilde{t}/l}}{1 - H^2\tilde{r}^2}. \tag{17}$$

As a result, the quasistatic metric (10) can be incarnated as

$$\begin{aligned} \mathrm{d}s^2 &= -H^2 l^2 a^4 C^2 \mathrm{e}^{4\tilde{t}/l} (1 - H^2 \tilde{r}^2)^{-1} \mathrm{d}\tilde{t}^2 \\ &\quad + (1 - H^2 \tilde{r}^2)^{-1} \mathrm{d}\tilde{r}^2 + \tilde{r}^2 \mathrm{d}\Omega_{n-1}^2, \end{aligned} \tag{18}$$

where $H$ is a function of $a$ determined by the Friedmann equations

$$H^2 = \frac{16\pi G_N}{n(n-1)}\rho, \qquad \frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{8\pi G_N}{n-1}(\rho + p), \tag{19}$$

and $a$ is dictated by eq. (15) as an implicit function of $(\tilde{t}, \tilde{r})$. Since $C$ and $l$ are adjustable constants, we set $C^2 = 1$ for simplicity from now on.

If the equation-of-state parameter $w = p/\rho$ is a constant, then according to the continuity equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + nH(\rho + p) = 0, \tag{20}$$

we have $\rho \propto a^{-n(1+w)}$. In the case that $w \neq (2-n)/n$, we obtain

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{4}{H^2 l^2 (n-2+nw)^2} \left(1 + \frac{n-2+nw}{2} H^2 \tilde{r}^2\right)^2 \left(1 - H^2 \tilde{r}^2\right)^{-1} \mathrm{d}\tilde{t}^2 \\ &\quad + \left(1 - H^2 \tilde{r}^2\right)^{-1} \mathrm{d}\tilde{r}^2 + \tilde{r}^2 \mathrm{d}\Omega_{n-1}^2. \end{aligned} \tag{21}$$

Especially, for $w = -1$, the quasistatic metric reduces to static metric (2) of the de Sitter space-time after fixing $l = 1/H$.

## 4. Unified first law

It is interesting to reconsider the unified first law of cosmological apparent horizon in light of the quasistatic coordinates above. This law has been repeatedly studied by many authors in recent years. As a partial list, see refs [2–5] and references therein. Here, we shall follow the formulae in ref. [3] and make full use of the two-dimensional metric

$$h_{ab} = \mathrm{diag}\left(-\frac{H^2 l^2 a^4 \mathrm{e}^{4\tilde{t}/l}}{1 - H^2 \tilde{r}^2}, \frac{1}{1 - H^2 \tilde{r}^2}\right) \tag{22}$$

normal to the $(n-1)$-sphere. Corresponding to this metric, the energy–momentum tensor projected to the normal directions is

$$\begin{pmatrix} T_{\tilde{t}}^{\ \tilde{t}} & T_{\tilde{t}}^{\ \tilde{r}} \\ T_{\tilde{r}}^{\ \tilde{t}} & T_{\tilde{r}}^{\ \tilde{r}} \end{pmatrix} = \frac{1}{1 - H^2 \tilde{r}^2} \begin{bmatrix} -\rho - pH^2 \tilde{r}^2 & -(\rho + p) H^2 \tilde{r} l a^2 \mathrm{e}^{2\tilde{t}/l} \\ \dfrac{(\rho + p)\tilde{r}}{l a^2 C \mathrm{e}^{2\tilde{t}/l}} & \rho H^2 \tilde{r}^2 + p \end{bmatrix}. \tag{23}$$

For spherically symmetric space-time, it proves most useful to follow Misner and Sharp's definition of energy [6]

$$E = \frac{n(n-1)}{16\pi G_N} \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} \tilde{r}^{n-2} \left(1 - h^{ab} \partial_a \tilde{r} \partial_b \tilde{r}\right) \tag{24}$$

inside a sphere of radius $\tilde{r}$. Following the line of ref. [3], one may define the work density by

$$W = -\frac{1}{2}T^{ab}h_{ab} = \frac{1}{2}(\rho - p), \tag{25}$$

and the energy supply vector by

$$\Psi_a = T_a{}^b\partial_b\tilde{r} + W\partial_a\tilde{r} = \frac{\rho + p}{1 - H^2\tilde{r}^2}\begin{bmatrix} -H^2\tilde{r}la^2\mathrm{e}^{2\tilde{t}/l} \\ \frac{1}{2}(1 + H^2\tilde{r}^2) \end{bmatrix}. \tag{26}$$

With the above results in hand, it is straightforward to prove

$$\mathrm{d}E = A\Psi + W\mathrm{d}V, \tag{27}$$

where $A = n\pi^{n/2}\tilde{r}^{n-1}/\Gamma(\frac{n}{2}+1)$ and $V = \pi^{n/2}\tilde{r}^n/\Gamma(\frac{n}{2}+1)$. When doing this, we should make use of the Friedmann equations (19) and relations

$$\frac{\partial H}{\partial \tilde{t}} = \frac{\mathrm{d}H}{\mathrm{d}t}\frac{\partial t}{\partial \tilde{t}}, \quad \frac{\partial H}{\partial \tilde{r}} = \frac{\mathrm{d}H}{\mathrm{d}t}\frac{\partial t}{\partial \tilde{r}}. \tag{28}$$

The dynamical apparent horizon is dictated by $h^{ab}\partial_a\tilde{r}\partial_b\tilde{r} = 0$, that is $\tilde{r}_A = 1/H$. On the apparent horizon, equality (27) is usually interpreted as the unified first law of thermodynamics [2–5].

## 5. Scalar-type perturbations

To study scalar-type perturbations, one should write down the Klein–Gordon equation in curved space-time

$$\frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}g^{\alpha\beta}\partial_\beta\chi\right) - \mu^2\chi = 0, \tag{29}$$

in which $\mu$ is the mass of the scalar field $\chi$. It would be interesting to take a look at the Klein–Gordon equation when the metric is given by eq. (18).

For concreteness, let us take the four-dimensional case as an example. Making use of the spherical symmetry, the scalar field in four dimensions can be decomposed as

$$\chi(\tilde{t}, \tilde{r}, \theta, \phi) = \sum_{\ell m}\frac{\chi_\ell(\tilde{r}, \tilde{t})}{\tilde{r}}Y_{\ell m}(\theta, \phi). \tag{30}$$

After some calculation, we find inside the horizon $H\tilde{r} < 1$, the equation of motion for $\chi$ takes the form of wave equation

$$\frac{1 - H^2\tilde{r}^2}{Hla^2\mathrm{e}^{2\tilde{t}/l}}\frac{\partial}{\partial \tilde{t}}\left(\frac{1}{Hla^2\mathrm{e}^{2\tilde{t}/l}}\frac{\partial \chi_\ell}{\partial \tilde{t}}\right) - \frac{1 - H^2\tilde{r}^2}{Hla^2}\frac{\partial}{\partial \tilde{r}}\left(Hla^2\frac{\partial \chi_\ell}{\partial \tilde{r}}\right) + U_\ell\chi_\ell = 0 \tag{31}$$

with an effective potential

$$U_\ell = \frac{\ell(\ell + 1)}{\tilde{r}^2} + \mu^2 + \frac{1 - H^2\tilde{r}^2}{Hla^2\tilde{r}}\frac{\partial}{\partial \tilde{r}}\left(Hla^2\right). \tag{32}$$

This form of equation is still too complicated, hopefully it will be solved by numerical approaches in the future. At the moment, we note that the last term in the effective potential can be written as $2\kappa/\tilde{r}$ in terms of the surface gravity [3]

$$\kappa = \frac{1}{2\sqrt{-h}}\partial_a\left(\sqrt{-h}h^{ab}\partial_b\tilde{r}\right) = \frac{1-H^2\tilde{r}^2}{2Hla^2}\frac{\partial}{\partial\tilde{r}}\left(Hla^2\right). \quad (33)$$

This reveals the physical significance of eq. (32): it is contributed by a term from the spherical harmonic expansion, the mass of scalar field, and the surface gravity. The surface gravity on the apparent horizon is interpreted as temperature in ref. [4].

## 6. Summary

After ruling out a static generalization of eq. (2) to the FLRW Universe, we proposed a system of quasistatic coordinates. In quasistatic coordinates, the FLRW metric is rewritten as eq. (10) or eq. (18). The new metric preserves the spherical symmetry as well as the spatial structure of eq. (2). It provides a new platform to study the physics of cosmological apparent horizon, including the unified first law and the scalar-type perturbations.

Several open problems are left for future investigations. First, in this article, we have concentrated on spatially flat Universe. It would be useful to extend our results to the open or closed Universe. Second, we have written down the Klein–Gordon equation in terms of the quasistatic coordinates and found a simple physical interpretation for the resulted effective potential. But the equation is too difficult to be solved analytically. Further efforts are needed to solve it numerically. Third, the static metric for de Sitter or anti-de Sitter space-time has many other applications such as in the gauge/gravity duality [1]. It is interesting to utilize our quasistatic coordinates for similar applications.

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