PRAMANA — journal of physics © Indian Academy of Sciences Vol. 80, No. 2 February 2013 pp. 223–235

Complete switched modified function projective synchronization of a five-term chaotic system with uncertain parameters and disturbances

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MS received 23 May 2012; revised 6 August 2012; accepted 22 August 2012

Abstract. A five-term three-dimensional (3D) autonomous chaotic system with an exponential nonlinear term is reported in this paper. Basic dynamical behaviours of the chaotic system are further investigated. Then a new synchronization phenomenon, complete switched modified function projective synchronization (CSMFPS), for this novel five-term chaotic system with uncertain parameters and disturbances is investigated. This paper extends previous work, where CSMFPS of chaotic systems means that all the state variables of the drive system synchronize with different state variables of the response system. As the synchronization scheme has many combined forms, it is a promising type of synchronization and can provide greater security in secure communication. Based on Lyapunov stability theory, a robust adaptive controller is contrived to acquire CSMFPS, parameter identification and suppress disturbances simultaneously. Finally, the Lorenz system and the new five-term chaotic system are taken as examples and the corresponding numerical simulations are presented to verify the effectiveness and feasibility of the proposed control scheme.

Keywords. Five-term three-dimensional chaotic system; complete switched modified function projective synchronization; adaptive control; uncertain parameters; disturbances.

PACS Nos 05.45.Ac; 05.45.Pq; 05.45.Xt

1. Introduction

Chaos analysis and applications in dynamical systems have been studied extensively in many fields such as secure communication, bioengineering and fluid mechanics [1–3]. Since Lorenz found the first chaotic attractor when he studied the atmospheric convection in 1963 [4], considerable research has been conducted to search for new chaotic attractors. For example, Rössler systems [5], Chen system [6], Lü system [7], Liu systems

[8] and the generalized Lorenz system family [9] were reported and analysed. But these chaotic systems are characterized by too many terms at the right-hand side. For example, Lorenz, Rössler, Chen and Celikovsky attractors are characterized by seven terms, whilst Lü, Liu attractors are characterized by six terms. For some of the practical applications, the cost of calculation will be more important than the number of terms in the equation [10]. A chaotic system has only five terms at the right-hand side and is worth studying. In 1997, Sprott first presented a dissipative chaotic flow with only five terms [11], and then several three-dimensional (3D) autonomous chaotic systems with five terms were proposed [12,13]. It can be seen that all these systems are algebraically simpler than the classic Lorenz and Rossler examples. But, the complicated dynamic properties of all these chaotic systems are obtained by quadratic cross-product nonlinearity terms. Recently, Wei and Wang proposed an autonomous chaotic system equipped with a nonlinear term in the form of exponential function [14], but this chaotic system was also characterized by six terms.

Since Pecora and Carroll first introduced a method to synchronize two identical chaotic systems in 1990 [15], chaos synchronization has been extensively studied due to its potential application in technological application and many methods and techniques for handling chaos control and synchronization of various chaotic systems have been developed, such as linear feedback control [16], OGY approach [17], active control [18], sliding mode control [19], adaptive synchronization [20], observer-based synchronization [21,22], backstepping approach [23], and so on. Recently, projective synchronization (PS) received much attention which says that the states of the drive and response are proportional to a constant scaling factor [18,24], and it includes the complete synchronization (CS) and the antisynchronization (AS). More recently, the extended and modified types of PS, such as modified projective synchronization (MPS) [25], function projective synchronization (FPS) [26], modified function projective synchronization (MFPS) [27,28] and generalized function projective synchronization (GFPS) [29] have been proposed and developed. Compared to PS or MPS, FPS (or MFPS and GFPS) can be used to obtain more secure communication in application to secure communications, because the unpredictability of the scaling function in FPS (or MFPS and GFPS) can additionally enhance the security of communication.

Further analysis showed that most of the PS schemes are involved mainly with chaotic systems in which all the state variables of the drive system synchronize with the corresponding state variables of the response system. Recently, Sudheer *et al* studied switched modified function projective synchronization (SMFPS) of two identical *Qi* hyperchaotic systems using adaptive control method [30]. Switched synchronization of chaotic systems in which a state variable of the drive system synchronizes with a different state variable of the response system is a promising type of synchronization as it provides greater security in secure communication [30,31]. Moreover, the parameters of chaotic systems cannot be exactly known *a priori* and are perturbed by external real-world factors and cannot be known exactly. Furthermore, noise disturbance is inevitable in practical situations. Therefore, synchronization of two chaotic systems with uncertain parameters and disturbances is more essential.

Inspired by the above discussions, in this paper, a five-term 3D chaotic system with an exponential nonlinear term is first reported. Basic dynamical properties of the chaotic system are also investigated. Then, we propose a new synchronization, complete switched modified function projective synchronization (CSMFPS), for two different chaotic systems, where the drive and response systems can be completely switched synchronized to a function matrix. The unpredictability of the function matrix in CSMFPS can additionally strengthen the security of communications. Based on Lyapunov stability theory, the adaptive controller with the parameter update law is also designed that makes the proposed scheme robust to external disturbances and can identify uncertain parameters simultaneously, so that the proposed scheme is more essential and realistic in actual applications. Finally, the Lorenz chaotic system and the new five-term chaotic system are taken as examples and simulation results are presented to demonstrate the effectiveness of the proposed method.

2. Five-term 3D chaotic system and its basic properties

Consider a new five-term 3D chaotic system with a nonlinear quadratic exponential term which is given by

$$\begin{cases} \dot{x} = a(y - x)\,, \\ \dot{y} = -bxz, \\ \dot{z} = \mathrm{e}^{xy} - c, \end{cases} \tag{1}$$

where a, b, c are the system parameters and x, y, z are the state variables. Compared to the other existing three-dimensional chaotic systems, system (1) has only five terms on the right-hand side and it mainly relies on two quadratic nonlinearities, one is a quadratic

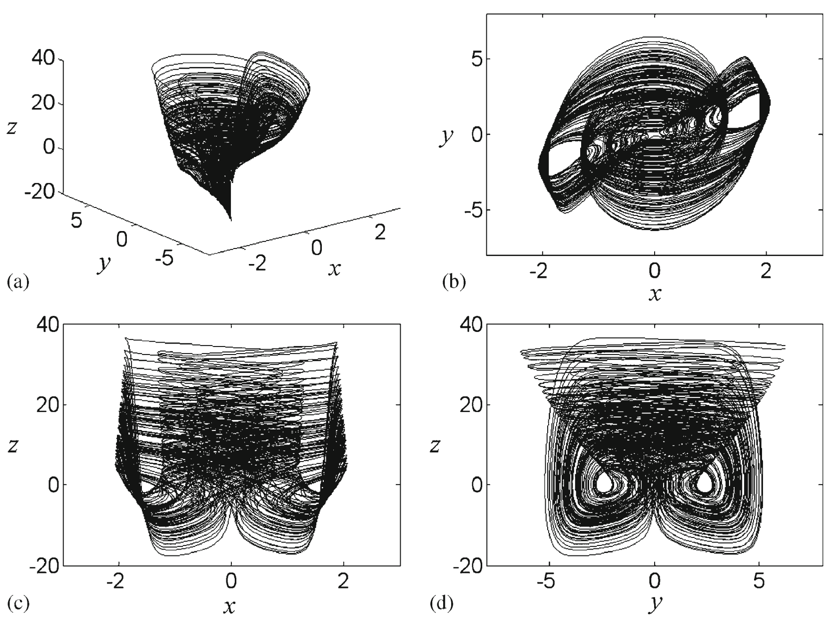

Figure 1. Phase portraits of system (1). (**a**) x–y–z view, (**b**) x–y plane, (**c**) x–z plane, and (**d**) y–z plane.

exponential nonlinear term and the other is a quadratic cross-product term. System (1) can generate a two-scroll chaotic attractor for the parameters $a = 8, b = 4$ and $c = 50$. The chaotic attractors are displayed in figure 1. It appears that the new attractor exhibits an interesting complex of the chaotic dynamics behaviour, which is similar to Lorenz chaotic attractor, but is different from that of the Lorenz system or any existing system.

It is noted that the system has a symmetry because the transformation $(x, y, z) \to (-x, -y, z)$, which permits system (1) to be invariant for all values of the parameters a, b and c. Apparently, the z-axis itself is an orbit. Moreover, the orbit on the z-axis tends to the origin as $t \to \infty$, and the transformation indicates that the system is symmetrical on the z-axis. The divergence of the vector field is

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -a. \tag{2}$$

Note that $-a = -8$ is a negative value, and so the system is a dissipative system and the exponential rate is:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \mathrm{e}^{-a} = \mathrm{e}^{-8}. \tag{3}$$

From eq. (3), it can be seen that a volume element V_0 is contracted by the flow into a volume element $V_0\mathrm{e}^{-8t}$ in time t. This means that each volume containing the system trajectory shrinks to zero as $t \to \infty$ at an exponential rate -8. Therefore, all system orbits are ultimately confined to a specific subset having zero volume and the asymptotic motion settles onto an attractor [8].

The Lyapunov exponents generally refer to the average exponential rates of divergence or convergence of the nearby trajectories in the phase-space. If there is at least one positive Lyapunov exponent, the system can be defined to be chaotic. According to the detailed numerical as well as theoretical analyses, the Lyapunov exponents of the system (1) are found to be $l_1 = 2.733$, $l_2 = 0$ and $l_3 = -10.733$. Therefore, the Lyapunov dimension of this system is

$$D_{\mathrm{L}} = j + \frac{\sum_{i=1}^{j} l_i}{|l_{j+1}|} = 2 + \frac{l_1 + l_2}{|l_3|} = 2 + \frac{2.733}{|-10.733|} = 2.255. \tag{4}$$

Equation (4) means that system (1) is really a dissipative system, and the Lyapunov dimensions of the system are fractional. Having a strange attractor and positive Lyapunov exponent, it is obvious that the system is really a three dimension chaos system. The equilibria of system (1) can be found by solving the following algebraic equations simultaneously:

$$\begin{cases} a(y - x) = 0, \\ -bxz = 0, \\ \mathrm{e}^{xy} - c = 0. \end{cases} \tag{5}$$

If $c > 1$, the system has two equilibrium points, which are respectively described as $E^{+}\left(\sqrt{\ln c}, \sqrt{\ln c}, 0\right)$ and $E^{-}\left(-\sqrt{\ln c}, -\sqrt{\ln c}, 0\right)$. When $a = 8, b = 4$ and $c = 50$, we operate the above nonlinear algebraic equations and obtain E^{+} (1.978, 1.978, 0) and E^{-} (−1.978, −1.978, 0).

Table 1. Eigenvalues of the Jacobin matrix for the two equilibriums.

E^+	E^-
$\lambda_1 = -12.7622$	$\lambda_1 = -12.7622$
$\lambda_2 = 2.3811 + 17.1658i$	$\lambda_2 = 2.3811 + 17.1658i$
$\lambda_3 = 2.3811 - 17.1658i$	$\lambda_3 = 2.3811 - 17.1658i$

In order to investigate the stability of all the equilibria, we consider the Jacobian matrix corresponding to each equilibrium and calculate its eigenvalue. The results are shown in table 1. Based on the eigenvalues, we know that the equilibria of system (1) are saddle focus nodes which means that all the equilibrium points are unstable.

The Poincaré maps can reflect bifurcation and folding properties of chaos. The Poincaré maps of system (1) are shown in figure 2. It can be seen that the Poincaré maps consist of virtually symmetrical branches and a number of nearly symmetrical twigs. We can further find that the section of the attractor looks similar to some vortexes from the Poincaré maps. The spectrum of system (1) exhibits a continuous broadband feature as shown in figure 3. In order to investigate the impact of parameters on the dynamics of the chaotic system, here we take parameter b as an example and extend the range of b to an interval (0, 246].

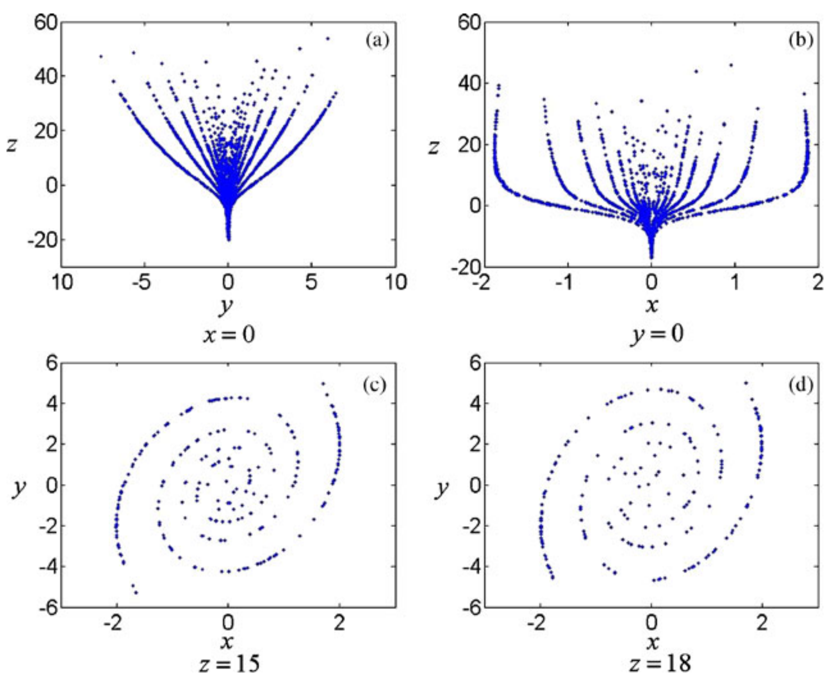

Figure 2. Poincaré maps in planes. (**a**) $x = 0$, (**b**) $y = 0$, (**c**) $z = 15$, (**d**) $z = 18$.

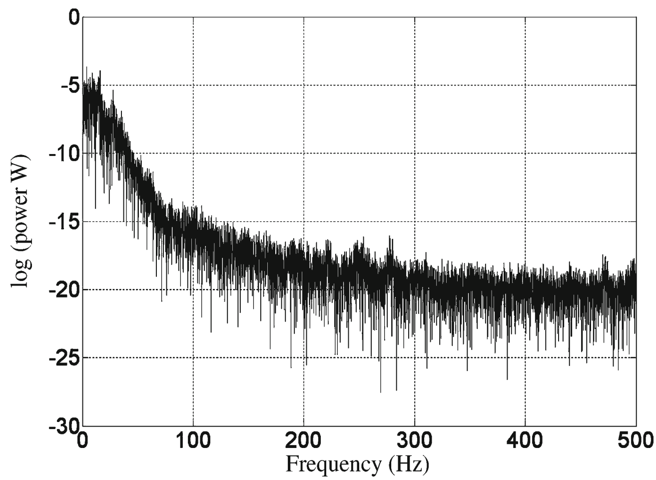

Figure 3. Spectrum map of $\log|x|$.

Lyapunov exponent spectrum vs. parameter b is given in figure 4. It is found that the system presents chaos characteristics in a large range. Figure 5 shows the time domain waveform. It can be seen that the time domain waveform has non-cyclical characteristics and every variable of all orbits can freely move across the boundary to the opposite side.

3. CSMFPS of two different chaotic systems

In this section, CSMFPS between Lorenz system and the new five-term chaotic system is presented and the main results and numerical simulations are demonstrated as well.

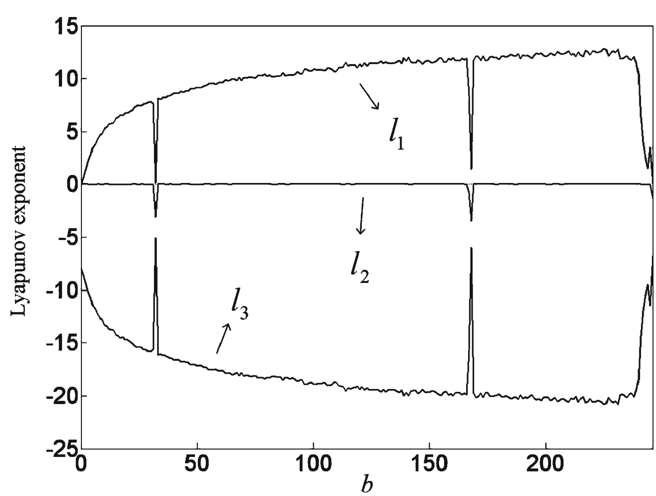

Figure 4. Lyapunov exponent spectrum of system (1).

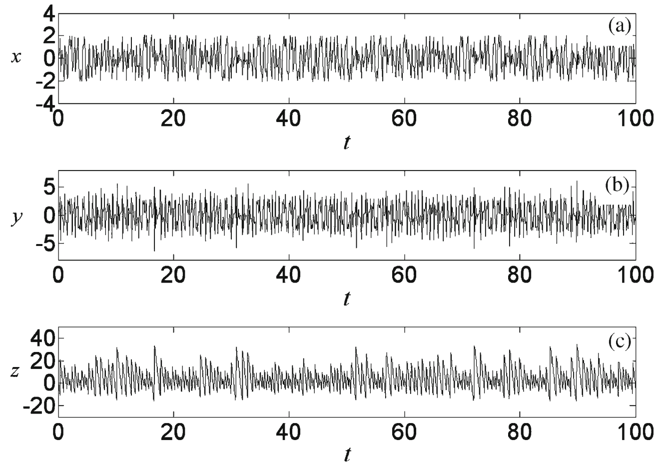

Figure 5. Time domain of system (1). (**a**) t–x wave, (**b**) t–y wave, (**c**) t–z wave.

3.1 *Problem formulation*

Consider the following drive and response systems:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) + \mathbf{F}(t, \mathbf{x})\,\mathbf{p} + \mathbf{D}'(t), \tag{6}$$

$$\dot{\mathbf{y}} = \mathbf{g}(t, \mathbf{y}) + \mathbf{G}(t, \mathbf{y})\mathbf{q} + \mathbf{D}''(t) + \mathbf{U}(t, \mathbf{x}, \mathbf{y}), \tag{7}$$

where $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are the state variables of the drive and response systems, respectively; $\mathbf{f}(t, \mathbf{x})$, $\mathbf{g}(t, \mathbf{y})$: $\mathbf{R}^n \rightarrow \mathbf{R}^n$, $\mathbf{F}(t, \mathbf{x}) \in \mathbf{R}^{n\times k}$, $\mathbf{G}(t, \mathbf{y}) \in \mathbf{R}^{n\times l}$ are the continuous nonlinear vector functions and $\mathbf{p} \in \mathbf{R}^k$, $\mathbf{q} \in \mathbf{R}^l$ are the parameters of the drive system (6) and the response system (1), respectively; and $\mathbf{D}' = [d_{11}, d_{12}, \ldots, d_{1n}]^T$, $\mathbf{D}'' = [d_{21}, d_{22}, \ldots, d_{2n}]^T \in \mathbf{R}^n$ represents the external disturbances with $|d_{1i}| \leq \alpha_{1i}$, $|d_{2i}| \leq \beta_{2i}$ $(i = 1, 2, \ldots, n)$ and assume $\boldsymbol{\alpha} = [\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}]^T$, $\boldsymbol{\beta} = [\beta_{21}, \beta_{22}, \ldots, \beta_{2n}]^T$, while $\mathbf{U} = [u_1, u_2, \ldots, u_n]^T \in \mathbf{R}^n$ is the control input vector.

Then, the dynamic equation of synchronization error can be expressed as follows:

$$\mathbf{e} = \mathbf{x}_i - \boldsymbol{\lambda}(t)\mathbf{y}_j, \tag{8}$$

where $\boldsymbol{\lambda}(t) = \mathrm{diag}(\lambda_1(t), \lambda_2(t), \ldots, \lambda_i(t), \ldots, \lambda_n(t))$ is an n-order diagonal matrix and $\lambda_i(t)$ is a continuously differentiable function with bounded, $\lambda_i(t) \neq 0$ for all t. x_i, y_j $(i, j = 1, 2, \ldots, n)$ are the state variables of the drive system and the response system, respectively.

DEFINITION 1

For the drive system (6) *and the response system* (7), *if there exists a scaling function matrix* $\boldsymbol{\lambda}(t)$ *such that* $\lim_{t\rightarrow+\infty} \|\mathbf{e}\| = \lim_{t\rightarrow+\infty} \left\|\mathbf{x}_i - \boldsymbol{\lambda}(t)\mathbf{y}_j\right\| = 0$, *where* $\|\circ\|$ *represents a vector norm induced by the matrix norm and when* $i \neq j$, *systems* (6) *and* (7) *are CSMFPS.*

Remark 1. On the basis of ref. [31], we can get the following combined numbers of the error vector of CSMFPS:

$$
\begin{aligned}
N_{\mathrm{CS}} &= n! - n! \cdot \left(1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{n-1}\frac{1}{n!}\right) \\
&= n! \cdot \sum_{p=1}^{n} (-1)^{p-1}\frac{1}{p!},\ (i \neq j, i, j = 1, 2, \ldots, n, n \geq 3). \qquad (9)
\end{aligned}
$$

Remark 2. If $i = j$ $(i, j = 1, 2, \ldots, n, n \geq 3)$, the problem of CSMFPS becomes MFPS. And if $\boldsymbol{\lambda}(t) = 1$, it turns out to be the MPS.

3.2 *Main results*

From eq. (8), we can obtain the error dynamical system as

$$
\begin{aligned}
\dot{\mathbf{e}} &= \dot{\mathbf{x}}_i - \dot{\boldsymbol{\lambda}}(t)\mathbf{y}_j - \boldsymbol{\lambda}(t)\dot{\mathbf{y}}_j \\
&= \mathbf{f}(t, \mathbf{x}_i) + \mathbf{F}(t, \mathbf{x}_i)\,\mathbf{p} + \mathbf{D}'(t) - \dot{\boldsymbol{\lambda}}(t)\mathbf{y}_j \\
&\quad -\boldsymbol{\lambda}(t)[\mathbf{g}(t, \mathbf{y}_j) + \mathbf{G}(t, \mathbf{y}_j)\mathbf{q} + \mathbf{D}''(t) + \mathbf{U}(t)]. \qquad (10)
\end{aligned}
$$

Theorem 1. *For the given scaling function matrix* $\boldsymbol{\lambda}(t)$, *if the controller input vector* $\mathbf{U}$ *is determined as*

$$
\begin{aligned}
\mathbf{U} = &\ \frac{1}{\boldsymbol{\lambda}(t)}[\mathbf{f}(t, \mathbf{x}_i) + \mathbf{F}(t, \mathbf{x}_i)\hat{\mathbf{p}} + \mathrm{sgn}(\mathrm{diag}(\lambda_1(t)e_1, \ldots, \lambda_n(t)e_n))\boldsymbol{\alpha} \\
&- \dot{\lambda}(t)\mathbf{y}_j + ke] - \mathbf{g}(t, \mathbf{y}_j) - \mathbf{G}(t, \mathbf{y}_j)\mathbf{q} \\
&- \mathrm{sgn}(\mathrm{diag}(\lambda_1(t)e_1, \ldots, \lambda_n(t)e_n))\boldsymbol{\beta}, \qquad (11)
\end{aligned}
$$

where $\hat{\mathbf{p}}$ *is the estimation of the parameters of the drive system* $\mathbf{p}$, $\mathrm{sgn}(\circ)$ *denotes the sign function and* k *is a positive constant, which would influence the rate of convergence. The update laws of the estimation of the unknown parameters is determined by*

$$
\dot{\hat{\mathbf{p}}} = \mathbf{F}^T(t, \mathbf{x}_i)\,\mathbf{e}. \qquad (12)
$$

Then the error dynamical system (10) *will be stabilized at the zero equilibrium asymptotically, i.e. CSMFPS between drive system* (6) *and response system* (7) *with external disturbances will occur.*

Proof. Let the error of the estimations of the parameters be $\tilde{\mathbf{p}} = \mathbf{p} - \hat{\mathbf{p}}$. Choose the following Lyapunov function:

$$
\mathbf{V} = \frac{1}{2}\left(\mathbf{e}^T\mathbf{e} + \tilde{\mathbf{p}}^T\tilde{\mathbf{p}}\right). \qquad (13)
$$

By substituting eqs (11) and (12) into eq. (13), and calculating the derivative of $\mathbf{V}$ along the solutions of error system (8), one has

$$\dot{\mathbf{V}} = -k\mathbf{e}^T\mathbf{e} + \mathbf{e}^T\mathbf{D}' - \mathbf{e}^T \mathrm{sgn}(\mathrm{diag}(e_1\lambda_1(t), \ldots, e_n\lambda_n(t)))\boldsymbol{\alpha} \\ - \left[\boldsymbol{\lambda}(t)\mathbf{e}^T\mathbf{D}'' - \boldsymbol{\lambda}(t)\mathbf{e}^T \mathrm{sgn}(\mathrm{diag}(e_1\lambda_1(t), \ldots, e_n\lambda_n(t)))\boldsymbol{\beta}\right]. \tag{14}$$

Let

$$\begin{cases} \psi_1 = \mathrm{sgn}(\mathrm{diag}(e_1\lambda_1(t), e_2\lambda_2(t), \ldots, e_n\lambda_n(t)))\boldsymbol{\alpha}, \\ \psi_2 = \mathrm{sgn}(\mathrm{diag}(e_1\lambda_1(t), e_2\lambda_2(t), \ldots, e_n\lambda_n(t)))\boldsymbol{\beta}, \\ \eta_1 = \mathbf{e}^T\mathbf{D}' - \mathbf{e}^T\psi_1, \\ \eta_2 = \boldsymbol{\lambda}(t)\mathbf{e}^T\mathbf{D}'' - \lambda(t)\mathbf{e}^T\psi_2, \end{cases} \tag{15}$$

where $\eta_1, \eta_2 \in \mathbf{R}$ and $\eta_2 \geq 0$. According to the definition and assumption of $\mathbf{D}'$, $\mathbf{D}''$ and $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, it is guaranteed that $\eta_1 \leq \eta_2$. Therefore

$$\dot{\mathbf{V}} = -k\mathbf{e}^T\mathbf{e} + \eta_1 - \eta_2 \leq -k\mathbf{e}^T\mathbf{e} \leq 0. \tag{16}$$

So $\dot{\mathbf{V}}$ is negative semidefinite, and since $\mathbf{V}$ is positive definite, it follows that $\mathbf{e} \in L_\infty$. Thus, $\dot{\mathbf{e}} \in L_\infty$, and

$$\int_0^t \|\mathbf{e}\|^2 \mathrm{d}t = \int_0^t \mathbf{e}^T\mathbf{e}\,\mathrm{d}t \leq -\frac{1}{k}\int_0^t \dot{\mathbf{V}}\,\mathrm{d}t = \frac{1}{k}\left[\mathbf{V}(0) - \mathbf{V}(t)\right] \leq \frac{1}{k}\mathbf{V}(0). \tag{17}$$

Thus, $\dot{\mathbf{e}} \in L_\infty$, and according to the Barbalat's lemma, we have $e_1(t), e_2(t), \ldots, e_n(t) \to 0$ as $t \to \infty$. Therefore, the error dynamical system (10) will be stabilized at the zero equilibrium asymptotically. This completes the proof. □

Remark 3. ψ_1 and ψ_2 in eq. (15) are the compensators that are introduced to eliminate the influence of external uncertainties.

3.3 *Illustrative example*

We take the new five-term chaotic system as the drive system which is given by

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) + d_{11}, \\ \dot{y}_1 = -bx_1z_1 + d_{12}, \\ \dot{z}_1 = \mathrm{e}^{x_1y_1} - c + d_{13}. \end{cases} \tag{18}$$

Suppose the parameters of the drive system (18) are fully uncertain. Here, we take the classical Lorenz chaotic system as a response system which is given by

$$\begin{cases} \dot{x}_2 = 10(y_2 - x_2) + d_{21} + u_1, \\ \dot{y}_2 = 28x_2 - x_2z_2 - y_2 + d_{22} + u_2, \\ \dot{z}_2 = x_2y_2 - 8z_2/3 + d_{23} + u_3, \end{cases} \tag{19}$$

where d_{ij} $(i = 1, 2,\ j = 1, 2, 3)$ are the exotic disturbances of systems (18) and (19), respectively, which satisfy the bounded condition $\|d_{1j}\| \leq \alpha_{1j}, \|d_{2j}\| \leq \beta_{2j}$ for all t, where α_{1j}, β_{2j} are the known positive constants. u_1, u_2, u_3 are nonlinear controllers to be designed so that the two chaotic systems (18) and (19) can be synchronized. According to Remark 1, if $n = 3$, the CSMFPS has two combined forms of error vectors, without loss of generality, and we choose one of the error vector as follows:

$$\begin{cases} e_1 = x_1 - \lambda_1(t) y_2, \\ e_2 = y_1 - \lambda_2(t) z_2, \\ e_3 = z_1 - \lambda_3(t) x_2. \end{cases} \tag{20}$$

The time derivative of eq. (20) is

$$\begin{cases} \dot{e}_1 = \dot{x}_1 - \lambda_1(t) \dot{y}_2 - \dot{\lambda}_1(t) y_2, \\ \dot{e}_2 = \dot{y}_1 - \lambda_2(t) \dot{z}_2 - \dot{\lambda}_2(t) z_2, \\ \dot{e}_3 = \dot{z}_1 - \lambda_3(t) \dot{x}_2 - \dot{\lambda}_3(t) x_2. \end{cases} \tag{21}$$

Substituting eqs (18) and (19) in (21), the error dynamics becomes

$$\begin{cases} \dot{e}_1 = a\,(y_1 - x_1) + d_{11} - \lambda_1(t)\,(28x_2 - x_2 z_2 - y_2 + d_{22} + u_2) \\ \qquad - \dot{\lambda}_1(t) y_2, \\ \dot{e}_2 = -b x_1 z_1 + d_{12} - \lambda_2(t)(x_2 y_2 - 8z_2/3 + d_{23} + u_3) - \dot{\lambda}_2(t) z_2, \\ \dot{e}_3 = \mathrm{e}^{x_1 y_1} - c + d_{13} - \lambda_3(t)[10(y_2 - x_2) + d_{21} + u_1] - \dot{\lambda}_3(t) x_2. \end{cases} \tag{22}$$

In terms of Theorem 1, the controllers $\mathbf{U} = (u_1, u_2, u_3)^T$ are determined by

$$\begin{cases} u_1 = \dfrac{1}{\lambda_3(t)}[\mathrm{e}^{x_1 y_1} - \hat{c} - \dot{\lambda}_3(t) x_2 + \alpha_{13}\,\mathrm{sgn}(\lambda_3(t) e_3) + k e_3] \\ \qquad - 10(y_2 - x_2) - \beta_{21}\,\mathrm{sgn}(\lambda_3(t) e_3), \\ u_2 = \dfrac{1}{\lambda_1(t)}[\hat{a}(y_1 - x_1) - \dot{\lambda}_1(t) y_2 + \alpha_{11}\,\mathrm{sgn}(\lambda_1(t) e_1) + k e_1] \\ \qquad - (28x_2 - x_2 z_2 - y_2) - \beta_{22}\,\mathrm{sgn}(\lambda_1(t) e_1), \\ u_3 = \dfrac{1}{\lambda_2(t)}[-\hat{b} x_1 z_1 - \dot{\lambda}_2(t) z_2 + \alpha_{12}\,\mathrm{sgn}(\lambda_2(t) e_2) + k e_2] \\ \qquad - (x_2 y_2 - 8z_2/3) - \beta_{23}\,\mathrm{sgn}(\lambda_2(t) e_2), \end{cases} \tag{23}$$

and the parameter update rule is determined by

$$\dot{\hat{a}} = (y_1 - x_1)\, e_1, \quad \dot{\hat{b}} = -x_1 z_1 e_2, \quad \dot{\hat{c}} = -e_3, \tag{24}$$

where $\hat{a}, \hat{b}, \hat{c}$ are the estimates of a, b, c respectively. $\mathrm{sgn}(\circ)$ denotes the sign function and k the positive control coefficient, which will influence the rate of convergence, by the way that the larger k, the faster the rate, and so we can adjust k according to the desirable convergent rate [28]. The corresponding block diagram representation of the CSMFPS scheme is shown in figure 6.

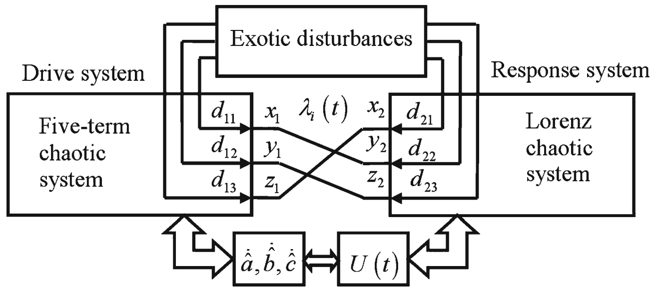

Figure 6. Block diagram of one of the proposed CSMFPS schemes.

According to Theorem 1, we have the result that for the given scaling function matrix $\boldsymbol{\lambda}(t)$, the CSMFPS between systems (18) and (19) will be obtained by the adaptive control laws in system (23) and the parameter update rules in eq. (24) with any initial conditions.

3.4 *Numerical simulations*

To verify the effectiveness of the proposed synchronization methods, the fourth-order Runge–Kutta method is used to solve the systems with a time step size of 0.001. The parameters are chosen to be $a = 8, b = 4, c = 50$ in all simulations so that the systems exhibit chaotic behaviours if no controls are applied. The initial conditions of the drive and response systems are $x_1(0) = 0.2, y_1(0) = 0.4, z_1(0) = 4$ and $x_2(0) = 3, y_2(0) = 1, z_2(0) = -3$. In addition, the initial conditions of the adaptive update laws of system

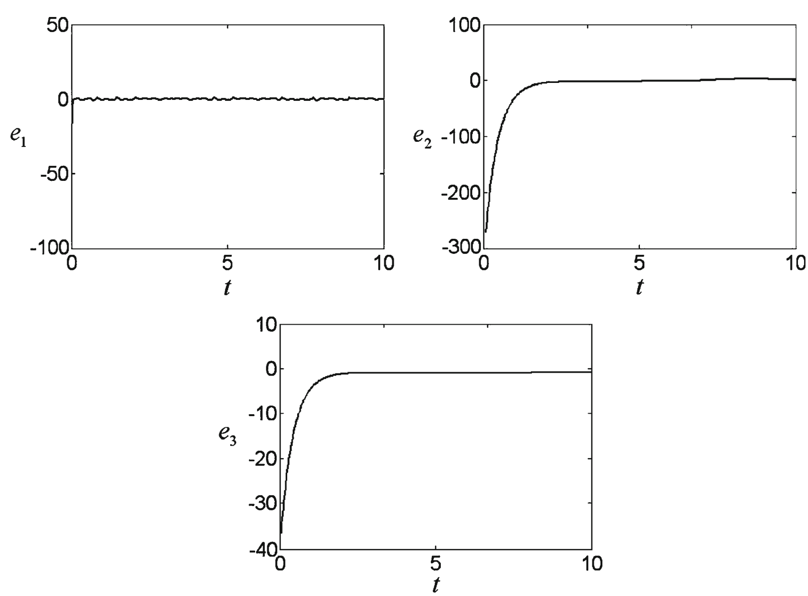

Figure 7. Dynamics of the variables e_1, e_2 and e_3 for error system (20) with time t.

parameters are $\hat{a}(0) = \hat{b}(0) = \hat{c}(0) = 0.1$, and the external disturbances are supposed as

$$\begin{cases} \mathbf{D}' = [d_{11}, d_{12}, d_{13}]^T = [0.3\cos(10t), -0.4\cos(20t), 0.5\sin(10t)]^T, \\ \mathbf{D}'' = [d_{21}, d_{22}, d_{23}]^T = [-1.4\sin(10t), 1.2\cos(20t), -1.6\sin(20t)]^T. \end{cases} \tag{25}$$

Thus, the boundaries of the uncertainties are

$$\begin{cases} \boldsymbol{\alpha} = [\alpha_{11}, \alpha_{12}, \alpha_{13}]^T = [0.3, 0.4, 0.5]^T, \\ \boldsymbol{\beta} = [\beta_{21}, \beta_{22}, \beta_{23}]^T = [1.4, 1.2, 1.6]^T. \end{cases} \tag{26}$$

We choose $k = 8$ and the scaling matrix is taken as

$$\begin{cases} \lambda_1(t) = \frac{1}{140}x_1 + 20, \\ \lambda_2(t) = \frac{1}{160}x_2 + 50, \\ \lambda_3(t) = \frac{1}{180}x_3 + 10. \end{cases} \tag{27}$$

Figure 7 displays the CSMFPS errors between systems (18) and (19). Figure 8 shows that the estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ of the uncertain parameters converge to $a = 8, b = 4, c = 50$ as $t \to \infty$.

Remark 4. The proposed scheme of CSMFPS is also applicable to two hyperchaotic systems.

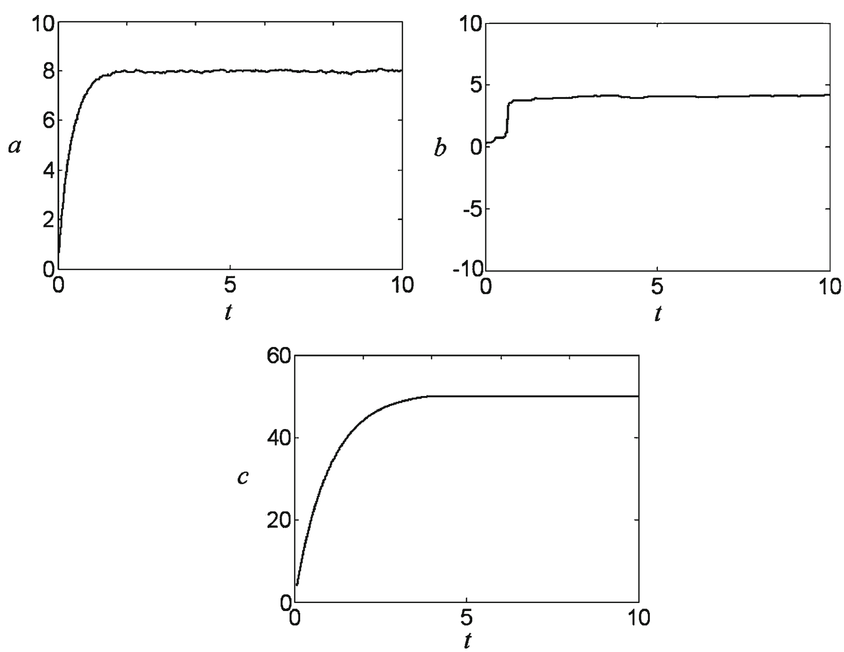

Figure 8. Estimation of uncertain parameters.

4. Conclusions

In this paper, a simple five-term 3D chaotic system with an exponential nonlinear term is proposed. Basic dynamical behaviours of the chaotic system are also investigated. Then, CSMFPS of the five-term chaotic system with uncertain parameters and disturbances is studied. The designed adaptive controller and the adaptive update laws of the system parameters ensure a globally asymptotically stable synchronization between two different chaotic systems. Numerical simulations are also given to verify and test the correctness and effectiveness of the control scheme we proposed. Our future work is to establish a secure communication scheme based on this adaptive CSMFPS.

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