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Fermionic particles with position-dependent mass in the presence of inversely quadratic Yukawa potential and tensor interaction

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Abstract. Approximate solutions of the Dirac equation with position-dependent mass are presented for the inversely quadratic Yukawa potential and Coulomb-like tensor interaction by using the asymptotic iteration method. The energy eigenvalues and the corresponding normalized eigenfunctions are obtained in the case of position-dependent mass and arbitrary spin-orbit quantum number k state and approximation on the spin-orbit coupling term.

Keywords. Dirac equation; position-dependent mass; inversely quadratic Yukawa potential; asymptotic iteration method; tensor interaction.

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1. Introduction

Quantum mechanical systems with position-dependent mass are proved to be useful in modelling the physical and electronic properties of semiconductors [1], quantum wells and quantum dots [2], quantum liquids [3] and semiconductor heterostructures [4] and they have received great attention in recent years [1–17]. Analytical solutions of the non-relativistic Schrödinger equation with position-dependent mass for solvable potentials have been addressed by a number of methods [5–11]. Furthermore, it is very important to investigate relativistic effects in quantum mechanical systems such as heavy atoms and heavy ion doping [13]. Thus, investigation of the relativistic Dirac equation with position-dependent mass is very important for these systems. Recently, many studies have been devoted to the research of the exact or quasiexact solutions of the Dirac equation with position-dependent mass and properties of these solutions for different potentials and mass distributions [12–20].

In this paper, we have studied the approximate solutions of the Dirac equation with spatially-dependent mass in the presence of the inversely quadratic Yukawa (IQY) potential and Coulomb-like tensor potential. This potential is important in different branches of physics, for instance, in plasma physics, atomic physics and nuclear physics. Energy eigenvalues and the corresponding eigenfunctions have been obtained using the exponential approximation of the centrifugal term within the framework of the asymptotic iteration method (AIM), which is based on solving second-order homogeneous linear differential equations [21–23].

This paper is arranged as follows: In §2, the basic equations of the AIM are presented. The relativistic energy eigenvalues of the Dirac equation with position-dependent mass for the IQY and Coulomb-like tensor potential are given in §3. In §4, the eigenfunctions are calculated and normalization of these eigenfunctions is acquired. Finally, conclusions and discussions are given in §5.

2. Basic equations of the AIM

The AIM is briefly outlined here; the details can be found in refs [21–23]. The AIM was proposed to solve second-order differential equations of the form

$$y'' = \lambda_0(x)y' + s_0(x)y, \tag{1}$$

where $\lambda_0(x) \neq 0$ and $s_0(x)$ are in $C_\infty(a, b)$ and these variables are sufficiently differentiable. The differential eq. (1) has a general solution as follows:

$$y(x) = \exp\left(-\int^x \alpha\, \mathrm{d}x'\right)\left[C_2 + C_1\int^x \exp\left(\int^{x'}[\lambda_0(x'') + 2\alpha(x'')]\mathrm{d}x''\right)\mathrm{d}x'\right] \tag{2}$$

if $n > 0$, for sufficiently large n,

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} = \alpha_k, \tag{3}$$

where

$$\begin{aligned} \lambda_n(x) &= \lambda'_{n-1}(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x) \\ s_n(x) &= s'_{n-1}(x) + s_0(x)\lambda_{n-1}(x), \quad n = 1, 2, 3, \ldots . \end{aligned} \tag{4}$$

The termination condition of the method together with eq. (4) can also be written as follows:

$$\delta(x) = \lambda_{n+1}(x)s_n(x) - \lambda_n(x)s_{n+1}(x) = 0. \tag{5}$$

For a given potential, the idea is to convert the relativistic wave equation to the form of eq. (1). Then, s_0 and λ_0 are determined and s_n and λ_n parameters are calculated. The energy eigenvalues are obtained by the termination condition given by eq. (5). However, the exact eigenfunctions can be derived from the following wave function generator:

$$y_n(x) = C_2 \exp\left(-\int^x \alpha_k\, \mathrm{d}x'\right), \tag{6}$$

where $n = 0, 1, 2, \ldots$ and k is the iteration step number and it is greater than n.

3. Energy eigenvalues

In the absence of the scalar IQY potential and units $\hbar = c = 1$, the Dirac equation including tensor interaction for a single particle in the vector IQY potential $V(r)$ using position-dependent mass formalism ($\mu(r)$) is given by

$$\left[\vec{\boldsymbol{\alpha}} \cdot \vec{\mathbf{p}} + \beta\mu(r) - i\beta\vec{\boldsymbol{\alpha}} \cdot \hat{r} U(r)\right] \psi(\vec{r}) = (E - V(r))\psi(\vec{r}), \tag{7}$$

$$\mathbf{p} = -i\nabla, \quad \vec{\boldsymbol{\alpha}} = \begin{pmatrix} 0 & \vec{\boldsymbol{\sigma}} \\ \vec{\boldsymbol{\sigma}} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{8}$$

where σ and I are the vector Pauli spin and the identity matrix, respectively. $\mathbf{p}$ is the momentum operator, α and β are 4×4 Dirac matrices, $V(r)$ is the IQY potential and E is the total relativistic energy of the system. The Dirac eigenfunction can be written according to $F_{nk}(r)$ and $G_{nk}(r)$ as follows:

$$\psi(r) = \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \dfrac{F_{nk}(r)}{r} Y^{\ell}_{jm}(\theta,\phi) \\ \dfrac{iG_{nk}(r)}{r} Y^{\tilde{\ell}}_{jm}(\theta,\phi) \end{pmatrix}, \tag{9}$$

where $Y^{\ell}_{jm}(\theta,\phi)$ and $Y^{\tilde{\ell}}_{jm}(\theta,\phi)$ are the spin and pseudospin spherical harmonic functions and $F_{nk}(r)/r$ and $iG_{nk}(r)/r$ are radial functions for the upper and lower components, respectively. n is the radial quantum number and m is the projection of the angular momentum on the z-axis. The orbital angular momentum quantum numbers ℓ and $\tilde{\ell}$ refer to the spin and pseudospin quantum numbers, respectively. For a given spin-orbit coupling term $k = \pm 1, \pm 2, \ldots$, the total angular momentum, orbital angular momentum and pseudoorbital angular momentum are given by $j = |k| - 1/2$, $\ell = |k + 1/2| - 1/2$ and $\tilde{\ell} = |k - 1/2| - 1/2$, respectively. By substituting eqs (8) and (9) in eq. (7) and by using the following relations [22]

$$(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot (\vec{A} x \vec{B}) \tag{10a}$$

$$(\vec{\sigma} \cdot \vec{P}) = \vec{\sigma} \cdot \hat{r} \left(\hat{r} \cdot \vec{P} + i\frac{\vec{\sigma} \cdot \vec{L}}{r} \right) \tag{10b}$$

and properties

$$(\vec{\sigma} \cdot \vec{L}) Y^{\tilde{l}}_{jm}(\theta,\phi) = (k-1)\, Y^{\tilde{l}}_{jm}(\theta,\phi) \tag{11a}$$

$$(\vec{\sigma} \cdot \vec{L}) Y^{\tilde{l}}_{jm}(\theta,\phi) = -\,(k-1)\, Y^{l}_{jm}(\theta,\phi) \tag{11b}$$

$$(\vec{\sigma} \cdot \hat{r}) Y^{\tilde{l}}_{jm}(\theta,\phi) = -Y^{l}_{jm}(\theta,\phi) \tag{11c}$$

$$(\vec{\sigma} \cdot \hat{r}) Y^{l}_{jm}(\theta,\phi) = -Y^{\tilde{l}}_{jm}(\theta,\phi)\,. \tag{11d}$$

The Dirac equation reduces to the following coupled differential equations:

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{k}{r}-U(r)\right)F_{nk}(r)=(\mu(r)+E_{nk}-V(r))G_{nk}(r) \tag{12}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{k}{r}+U(r)\right)G_{nk}(r)=(\mu(r)-E_{nk}+V(r))F_{nk}(r)\,. \tag{13}$$

Using the wave function $G_{nk}(r)$ in eq. (13) and substituting it in eq. (12), we obtain the following second-order differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{k(k+1)}{r^2}+\frac{2k}{r}U(r)-\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U(r)^2-(\mu(r)+E_{nk}\right.$$
$$-V(r))\,(\mu(r)-E_{nk}+V(r))$$
$$\left.-\frac{\left(\frac{\mathrm{d}\mu(r)}{\mathrm{d}r}-\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{k}{r}-U(r)\right)}{\mu(r)+E_{nk}-V(r)}\right)F_{nk}(r)=0. \tag{14}$$

In a similar way, we have

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{k(k-1)}{r^2}+\frac{2k}{r}U(r)+\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U(r)^2\right.$$
$$-\,(\mu(r)+E_{nk}-V(r))\,(\mu(r)-E_{nk}+V\,(r))$$
$$\left.-\frac{\left(\frac{\mathrm{d}\mu(r)}{\mathrm{d}r}+\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{k}{r}+U(r)\right)}{\mu(r)-E_{nk}+V(r)}\right)G_{nk}(r)=0. \tag{15}$$

The IQY potential is important in different branches of physics, for instance, in plasma physics, atomic physics and nuclear physics. This potential is defined as

$$V(r)=-\frac{V_0}{r^2}\mathrm{e}^{-2\alpha r}, \tag{16}$$

where α is the screening parameter and V_0 is the depth of the potential profile. This attractive potential acts between two particles and this is well-known in nuclear physics as the dominant central part of nucleon–nucleon interaction [24–27]. In addition to the IQY potential for Dirac equation, the tensor interaction potential is defined as the Coulomb-like potential as follows:

$$U(r)=-\frac{H}{r},\qquad H=\frac{Z_aZ_be^2}{4\pi\varepsilon_0},\quad r\geq R_\mathrm{c}, \tag{17}$$

where $R_\mathrm{c}=7.78$ fm is the Coulomb radius and Z_a and Z_b denote the charges of the projectile a and the target nuclei b, respectively [28]. In the presence of vector potential and while scalar potential is zero, to solve eq. (14), firstly, we eliminate the last term in eq. (14) by using equality $(\mathrm{d}\mu(r)/\mathrm{d}r)-(\mathrm{d}V(r)/\mathrm{d}r)=0$, which gives position-dependent mass function

$$\mu(r)=m_0-V_0\frac{\mathrm{e}^{-2\alpha r}}{r^2}, \tag{18}$$

where m_0 is a constant (real and positive) and corresponds to the rest mass of the fermionic particle. In eq. (18), $m(r)$ is the mass function rather than real mass. The second term in mass function (eq. (18)) corresponds to the location dependence of the effective mass. By substituting eq. (18) in eq. (14), an effective potential constituted by mass function, IQY potential and tensor potential occurs in the system. Mass function creates a shift in the potential profile of the system. Therefore, localizations of the wave functions change. The mentioned effect of the position-dependent mass is seen more clearly in semiconductor physics applications [2].

If eqs (16)–(18) are inserted in eq. (14), the following equation is obtained:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{k(k+1)+2kH+H+H^2}{r^2} + \frac{2\,(m_0+E_{nk})V_0\mathrm{e}^{-2\alpha r}}{r^2} - (m_0^2 - E_{nk}^2)\right]F_{nk}\,(r) = 0. \tag{19}$$

In this study, to solve the Dirac equation with position-dependent mass for the IQY potential and tensor interaction, we should use an approximation as discussed in the following. To obtain analytical approximate solutions for the IQY potential with spin-orbit coupling term, we have to use an approximation in the following form [25,26,29,30]:

$$\frac{1}{r^2} \sim 4\alpha^2 \frac{\mathrm{e}^{-2\alpha r}}{\left(1-\mathrm{e}^{-2\alpha r}\right)^2}, \tag{20}$$

which is valid for $\alpha r \ll 1$.

Using the approximation given in eq. (20) for spin-orbit coupling term and also by transforming a new variable $z = \mathrm{e}^{-2\alpha r}$, eq. (19) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z} - \frac{\Lambda}{z\,(1-z)^2} + \frac{\varepsilon_0}{(1-z)^2} - \frac{\varepsilon_1}{z^2}\right]F_{nk}(z) = 0, \tag{21}$$

where

$$\Lambda = k(k+1)+2kH+H+H^2, \quad \varepsilon_0 = 2\,(m_0+E_{nk})\,V_0, \quad \varepsilon_1 = \frac{m_0^2 - E_{nk}^2}{4\alpha^2}.$$

In order to solve eq. (21) using AIM, we should transform this equation to the form of eq. (1). The wave function should respect the boundary conditions, i.e. $F_{nk}\,(0) = 0$ at $z = 0$ for $r \to \infty$ and $F_{nk}(1) = 0$ at $z = 1$ for $r \to 0$. Therefore, the reasonable physical wave function is proposed as follows:

$$F_{nk}(z) = z^{\sqrt{\varepsilon_1}}\,(1-z)^{\frac{1}{2}\left(1+\sqrt{1-4\varepsilon_0+4\Lambda}\right)}\,f(z)\,. \tag{22}$$

If eq. (22) is inserted in eq. (21), we have second-order homogeneous linear differential equation that is solvable by AIM in the following form:

$$f''(z) = -\frac{\left(-1-2\sqrt{\varepsilon_1}+z\left(2+2\sqrt{\varepsilon_1}+\sqrt{1-4\varepsilon_0+4\Lambda}\right)\right)}{z\,(-1+z)}f'(z) - \frac{\left(1+2\Lambda+\sqrt{1-4\varepsilon_0+4\Lambda}+2\sqrt{\varepsilon_1}\left(1+\sqrt{1-4\varepsilon_0+4\Lambda}\right)\right)}{2z\,(-1+z)}f(z)\,. \tag{23}$$

By comparing eq. (23) with eq. (1), $\lambda_0(z)$ and $s_0(z)$ values can be obtained and with the aid of eq. (4), we may calculate $\lambda_n(z)$ and $s_n(z)$. In this way:

$$\begin{aligned}
\lambda_0(z) &= -\frac{-1-2\sqrt{\varepsilon_1}+z(2+2\sqrt{\varepsilon_1}+\sqrt{1-4\varepsilon_0+4\Lambda})}{z\,(-1+z)},\\
s_0(z) &= -\frac{1+2\sqrt{\varepsilon_1}+2\Lambda+\sqrt{1-4\varepsilon_0+4\Lambda}+2\sqrt{\varepsilon_1}\sqrt{1-4\varepsilon_0+4\Lambda}}{2z\,(-1+z)},\\
\lambda_1(z) &= \frac{4+8\,(-1+z)^2\,\varepsilon_1+z(-11+2\Lambda-3\sqrt{1-4\varepsilon_0+4\Lambda})}{2z^2\,(-1+z)^2}\\
&\quad+\frac{z^2(11-2\Lambda+9\sqrt{1-4\varepsilon_0+4\Lambda}+2\,(1-4\varepsilon_0+4\Lambda))}{2z^2\,(-1+z)^2},\\
&\quad+\frac{6\,(-1+z)\,\sqrt{\varepsilon_1}(-2+z(3+\sqrt{1-4\varepsilon_0+4\Lambda}))}{2z^2\,(-1+z)^2}\\
s_1(z) &= \frac{(1+2\sqrt{\varepsilon_1}+2\Lambda+\sqrt{1-4\varepsilon_0+4\Lambda}+2\sqrt{\varepsilon_1}\sqrt{1-4\varepsilon_0+4\Lambda})}{\sqrt{2}z\,(-1+z)}\\
&\quad\times\frac{-2+2\,(-1+z)\,\sqrt{\varepsilon_1}+z(4+\sqrt{1-4\varepsilon_0+4\Lambda})}{\sqrt{2}z\,(-1+z)}.
\end{aligned} \tag{24}$$

Combining the results obtained by the AIM with quantization condition given by eq. (5) yields

$$s_0\lambda_1-s_1\lambda_0=0\Rightarrow\varepsilon_{10}=\frac{1}{4}\left(\frac{1+2\Lambda+\sqrt{1-4\varepsilon_0+4\Lambda}}{1+\sqrt{1-4\varepsilon_0+4\Lambda}}\right)^2,\quad\text{for } n=0, \tag{25a}$$

$$s_1\lambda_2-s_2\lambda_1=0\Rightarrow\varepsilon_{11}=\frac{1}{4}\left(\frac{5+2\Lambda+3\sqrt{1-4\varepsilon_0+4\Lambda}}{3+\sqrt{1-4\varepsilon_0+4\Lambda}}\right)^2,\quad\text{for } n=1, \tag{25b}$$

$$s_2\lambda_3-s_3\lambda_2=0\Rightarrow\varepsilon_{12}=\frac{1}{4}\left(\frac{13+2\Lambda+5\sqrt{1-4\varepsilon_0+4\Lambda}}{5+\sqrt{1-4\varepsilon_0+4\Lambda}}\right)^2,\quad\text{for } n=2. \tag{25c}$$

If the set of eq. (25) is generalized, indirectly the energy eigenvalues statement turns out to be

$$\varepsilon_{1n}=\frac{1}{4}\left(\frac{1+2n\,(n+1)+2\Lambda+(1+2n)\,\sqrt{1-4\varepsilon_0+4\Lambda}}{1+2n+\sqrt{1-4\varepsilon_0+4\Lambda}}\right)^2,\quad n=0,1,2,\ldots. \tag{26}$$

By comparing eq. (26) with ε_1, energy eigenvalues of the position-dependent mass Dirac equation with the IQY potential and Coulomb-like tensor potential can be obtained by means of the following expression:

$$E_{nk}^2 = m_0^2 - \alpha^2 \left(\frac{1 + 2n\,(n+1) + 2\left(k(k+1) + 2kH + H + H^2\right)}{1 + 2n + \sqrt{1 - 8(m_0 + E_{nk})\,V_0 + 4\left(k(k+1) + 2kH + H + H^2\right)}} + \frac{(1+2n)\sqrt{1 - 18(m_0 + E_{nk})V_0 + 4\left(k(k+1) + 2kH + H + H^2\right)}}{1 + 2n + \sqrt{1 - 8(m_0 + E_{nk})\,V_0 + 4\left(k(k+1) + 2kH + H + H^2\right)}} \right)^2 . \tag{27}$$

4. Eigenfunctions

Exact eigenfunctions can be derived from the following generator:

$$f_n(z) = C_2 \exp\left(- \int^z \alpha_k \, \mathrm{d}z' \right). \tag{28}$$

Using eqs (3) and (28), the eigenfunctions are obtained as follows:

$$\begin{aligned}
f_0(z) &= C_2, \\
f_1(z) &= -C_2(-2\sqrt{\varepsilon_1} - 1)\left(1 - \frac{2\sqrt{\varepsilon_1} + \sqrt{1 - 4\varepsilon_0 + 4\Lambda} + 2}{-2\sqrt{\varepsilon_1} - 1} z\right), \\
f_2(z) &= C_2(1 + 2\sqrt{\varepsilon_1})(2\sqrt{\varepsilon_1}) \\
&\quad \times \left(1 - \frac{2(2\sqrt{\varepsilon_1} + \sqrt{1 - 4\varepsilon_0 + 4\Lambda} + 3)}{-1 - 2\sqrt{\varepsilon_1}} z + \frac{(2\sqrt{\varepsilon_1} + \sqrt{1 - 4\varepsilon_0 + 4\Lambda} + 3)(2\sqrt{\varepsilon_1} + \sqrt{1 - 4\varepsilon_0 + 4\Lambda} + 4)}{(1 + 2\sqrt{\varepsilon_1})(2\sqrt{\varepsilon_1})} z^2 \right), \\
f_3(z) &= -C_2(1 + 2\sqrt{\varepsilon_1})(2\sqrt{\varepsilon_1})(1 - 2\sqrt{\varepsilon_1}) \\
&\quad \times \left(1 + \frac{3\left(2\sqrt{\varepsilon_1} + \sqrt{1 - 4\varepsilon_0 + 4\Lambda} + 4\right)}{1 + 2\sqrt{\varepsilon_1}} z + \frac{3\left(2\sqrt{\varepsilon_1} + \sqrt{1 - 4\varepsilon_0 + 4\Lambda} + 4\right)\left(2\sqrt{\varepsilon_1} + \sqrt{1 - 4\varepsilon_0 + 4\Lambda} + 5\right)}{\left(1 + 2\sqrt{\varepsilon_1}\right)\left(2\sqrt{\varepsilon_1}\right)} z^2 \right. \\
&\quad \left. - \frac{\left(2\sqrt{\varepsilon_1} + \sqrt{1 - 4\varepsilon_0 + 4\Lambda} + 4\right)\left(2\sqrt{\varepsilon_1} + \sqrt{1 - 4\varepsilon_0 + 4\Lambda} + 5\right)\left(2\sqrt{\varepsilon_1} + \sqrt{1 - 4\varepsilon_0 + 4\Lambda} + 6\right)}{\left(1 + 2\sqrt{\varepsilon_1}\right)\left(2\sqrt{\varepsilon_1}\right)\left(1 - 2\sqrt{\varepsilon_1}\right)} z^3 \right).
\end{aligned} \tag{29}$$

Finally, the following general formula for the exact solutions $f_n(z)$ is acquired as:

$$\begin{aligned}
f_n(z) \;=\; & (-1)^n \, C_2 \frac{\Gamma(2\sqrt{\varepsilon_1} + n)}{\Gamma(-2\sqrt{\varepsilon_1} - 1)} {}_2F_1(-n, 1 + 2\sqrt{\varepsilon_1} \\
& + \sqrt{1 - 4\varepsilon_0 + 4\Lambda} + n; -1 - 2\sqrt{\varepsilon_1}; z).
\end{aligned} \tag{30}$$

Hence, we write the upper component of the total radial wave function as follows:

$$\begin{aligned}
F_{nk}(z) \;=\; & N z^{\sqrt{\varepsilon_1}} \, (1 - z)^{(1/2)\left(1 + \sqrt{1 - 4\varepsilon_0 + 4\Lambda}\right)} \, {}_2F_1(-n, 1 + 2\sqrt{\varepsilon_1} \\
& + \sqrt{1 - 4\varepsilon_0 + 4\Lambda} + n; -1 - 2\sqrt{\varepsilon_1}; z),
\end{aligned} \tag{31}$$

where N is the normalization constant.

Similarly, in order to obtain another component $G_{nk}(z)$ of the Dirac spinor, we consider that $\mu(r) = m_0 + V_0(\mathrm{e}^{-2\alpha r}/r^2)$ and then defined by $z = \mathrm{e}^{-2\alpha r}$ transformation if the processes to determine eq. (31) are repeated, we obtain that

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z} - \frac{\varepsilon_1}{z^2} - \frac{\Delta}{z(1-z)^2} - \frac{\varepsilon_2}{(1-z)^2}\right] G_{nk}(z) = 0, \tag{32}$$

where $\varepsilon_2 = 2(m_0 - E_{nk})V_0$, $\Delta = k(k-1) + 2kH - H + H^2$. Once again, if the processes to determine $F_{n,\kappa}(z)$ are repeated, we obtain

$$\begin{aligned} G_{nk}(z) = {} & N z^{\sqrt{\varepsilon_1}} (1-z)^{(1/2)\left(1+\sqrt{1+4\Delta+4\varepsilon_2}\right)} {}_2F_1(-n, 1 + 2\sqrt{\varepsilon_1} \\ & + \sqrt{1+4\Delta+4\varepsilon_2} + n; -1 - 2\sqrt{\varepsilon_1}; z). \end{aligned} \tag{33}$$

Owing to eqs (31) and (33), radial Dirac spinor has been obtained.

The total radial wave functions are obtained in terms of confluent hypergeometric functions. In this part of our work, we calculate the normalization constant N in eqs (31) and (33). Calculation of the normalization constant, which has not been explicitly worked out in most of the studies, is necessary. How to compute the normalization constant is given by some of the studies [31]. To compute the normalization constant N, firstly, we start with the normalization condition, $\int_0^\infty (f^2 + g^2) r^2 \,\mathrm{d}r = 1$. According to the transformation of $z = \mathrm{e}^{-2\alpha r}$, $z \to 1$ for $r \to 0$ and $z \to 0$ for $r \to \infty$. Therefore,

$$\int_0^1 \frac{F_{nk}(z)^2}{2z\alpha}\mathrm{d}z + \int_0^1 \frac{G_{nk}(z)^2}{2z\alpha}\mathrm{d}z = 1.$$

So, we have the following equation by means of the above normalization condition, that is,

$$(N)^2 \begin{pmatrix} \int_0^1 z^{2\sqrt{\varepsilon_1}-1} (1-z)^{1+\sqrt{1-4\varepsilon_0+4\Lambda}} \left({}_2F_1\left(-n, 1+2\sqrt{\varepsilon_1}+\sqrt{1-4\varepsilon_0+4\Lambda}+n; -1-2\sqrt{\varepsilon_1}; z\right)\right)^2 \mathrm{d}z \\ + \int_0^1 z^{2\sqrt{\varepsilon_1}-1} (1-z)^{1+\sqrt{1+4\varepsilon_2+4\Delta}} \left({}_2F_1\left(-n, 1+2\sqrt{\varepsilon_1}+\sqrt{1+4\varepsilon_2+4\Delta}+n; -1-2\sqrt{\varepsilon_1}; z\right)\right)^2 \mathrm{d}z \end{pmatrix} = 2\alpha. \tag{34}$$

The series representation of the confluent hypergeometric function ${}_2F_1$ is

$${}_pF_q\left(a_1, \ldots, a_p; c_1, \ldots, c_q; s\right) = \sum_{i=0}^{\infty} \frac{(a_1)_i \cdots (a_p)_i}{(c_1)_i \cdots (c_q)_i} \frac{s^i}{i!}. \tag{35}$$

Being Pochammer symbols $(a_1)_i$, $(c_1)_i$ and a polynomial of degree n in s, we obtain the normalization constant in the following form:

$$(N)^2 \begin{pmatrix} \sum_{i=0}^{n} \frac{(-n)_i \left(1+2\sqrt{\varepsilon_1}+\sqrt{1-4\varepsilon_0+4\Lambda}+n\right)_i \left(2\sqrt{\varepsilon_1}\right)_i}{\left(-1-2\sqrt{\varepsilon_1}\right)_i i! \left(1+2\sqrt{\varepsilon_1}+\sqrt{1-4\varepsilon_0+4\Lambda}\right)_i} B\left(2\sqrt{\varepsilon_1}, 1+\sqrt{1-4\varepsilon_0+4\Lambda}\right) \\ \times {}_3F\left(2-n, 1+2\sqrt{\varepsilon_1}+\sqrt{1-4\varepsilon_0+4\Lambda}+n, 2\sqrt{\varepsilon_1}+i; -1-2\sqrt{\varepsilon_1}, 1+2\sqrt{\varepsilon_1}+\sqrt{1-4\varepsilon_0+4\Lambda}+i; 1\right) \\ + \sum_{i=0}^{n} \frac{(-n)_i \left(1+2\sqrt{\varepsilon_1}+\sqrt{1+4\varepsilon_2+4\Delta}+n\right)_i \left(2\sqrt{\varepsilon_1}\right)_i}{\left(-1-2\sqrt{\varepsilon_1}\right)_i i! \left(1+2\sqrt{\varepsilon_1}+\sqrt{1+4\varepsilon_2+4\Delta}\right)_i} B\left(2\sqrt{\varepsilon_1}, 1+\sqrt{1+4\varepsilon_2+4\Delta}\right) \\ \times {}_3F_2\left(-n, 1+2\sqrt{\varepsilon_1}+\sqrt{1+4\varepsilon_2+4\Delta}+n, 2\sqrt{\varepsilon_1}+i; -1-2\sqrt{\varepsilon_1}, 1+2\sqrt{\varepsilon_1}+\sqrt{1+4\varepsilon_2+4\Delta}+i; 1\right) \end{pmatrix} = 2\alpha. \tag{36}$$

5. Conclusions

In this study, the bound state solutions of the relativistic Dirac equation with the vector IQY potential and Coulomb-like tensor were calculated using advantages of the AIM in the case of the position-dependent mass in detail. We have obtained eq. (27) for the energy eigenvalues and eqs (31), (33), (36) for the corresponding eigenfunctions in the case of position-dependent mass and arbitrary spin-orbit quantum number k state and approximation on the spin-orbit coupling term by AIM. The corresponding eigenfunctions have been obtained in terms of confluent hypergeometric functions and have also been normalized.

Appendix

To normalize the wave functions, some of the special procedures for the beta function is given in the following form [32–38]:

(i)

$$B_q(x+1, y) = \frac{x}{x+y} B_q(x, y) - \frac{q^x(1-q)^y}{x+y},$$

(ii)

$$\begin{aligned}
I_q(x, y) &= I_q(x-1, y) - \frac{q^{x-y}(1-q)^y}{x+y} \\
&= I_q(x-2, y) - \frac{q^{x-2}(1-q)^y}{(x-2)B(x-2, y)} - \frac{q^{x-1}(1-q)^y}{(x-1)B(x-1, y)} \\
&= I_q(x-3, y) - \frac{q^{x-3}(1-q)^y}{(x-3)B(x-3, y)} - \frac{q^{x-2}(1-q)^y}{(x-2)B(x-2, y)} \\
&\quad - \frac{q^{x-1}(1-q)^y}{(x-1)B(x-1, y)} \\
&= \cdots \\
&= I_q(x-m, y) - q^x (1-q)^y \sum_{k=1}^{m} \frac{q^{-k}}{(x-k)\, B(x-k, y)}, \\
m &= 1, 2, \ldots,
\end{aligned}$$

(iii)

$$\begin{aligned}
B_q(x, y) &= \frac{q^x(1-q)^{y-1}}{x} \sum_{k=0}^{\infty} \frac{(1-y)_k}{(1+x)_k} \left(\frac{q}{q-1}\right)^k \\
&= \frac{q^x(1-q)^{y-1}}{x} {}_2F_1\left(1, 1-y; 1+x; \frac{q}{q-1}\right)
\end{aligned}$$

for $q \in (-\infty, 0) \cup (0, \frac{1}{2})$ and

$$B_q(x, y) = B(x, y) - \frac{q^{x-1}(1-q)^y}{y} \sum_{k=0}^{\infty} \frac{(1-x)_k}{(1+y)_k} \left(\frac{q-1}{q}\right)^k$$
$$= B(x, y) - \frac{q^{x-1}(1-q)^y}{y} {}_2F_1\left(1, 1-x; 1+y; \frac{q-1}{q}\right),$$

(iv)

$$(a)_{i+j} = (a)_i \, (a+i)_j \, .$$

References

[1] G Bastard, *Wave mechanics applied to semiconductor heterostructure* (Les Editions de Physique, Les Ulis, 1988)
[2] P Harrison, *Quantum wells, wires and dots* (Wiley, New York, 2000)
L Serra and E Lipparini, *Europhys. Lett.* **40**, 667 (1997)
[3] F Arias de Saavedra, J Boronat, A Polls and A Fabrocini, *Phys. Rev.* **B50**, 4248 (1994)
[4] T Gora and F Williams, *Phys. Rev.* **177**, 1179 (1969)
[5] A R Plastino, A Rigo, M Casas, F Garcias and A Plastino, *Phys. Rev.* **A60**, 4318 (1999)
[6] A de Souza Dutra and C A S Almeida, *Phys. Lett.* **A275**, 25 (2000)
[7] A D Alhaidari, *Phys. Rev.* **A66**, 042116 (2002)
A D Alhaidari, *Int. J. Theor. Phys.* **42**, 2999 (2003)
[8] B Roy and P Roy, *J. Phys.* **A35**, 3961 (2002)
[9] R Koç, M Koca and E Körcük, *J. Phys.* **A35**, L527 (2002)
R Koç, M Koca and G Sahinoglu, *Eur. Phys. J.* **B48**, 583 (2005)
[10] B Gönül, O Özer, B Gönül and F Üzgün, *Mod. Phys. Lett.* **A17**, 2453 (2002)
B Gönül, B Gönül, D Tutcu and O Özer, *ibid.* **17**, 2057 (2002)
[11] B Bagchi, P Gorain, C Quesne and R Roychoudhury, *Mod. Phys. Lett.* **A19**, 2765 (2004)
C Quesne and V M Tkachuk, *J. Phys.* **A37**, 4267 (2004)
C Quesne, *Ann. Phys. N.Y.* **321**, 1221 (2006)
[12] O Panella, S Biondini and A Arda, *J. Phys. A: Math. Theor.* **43**, 325302 (2010)
[13] A D Alhaidari, *Phys. Rev.* **A75**, 042707 (2007)
A D Alhaidari, H Bahlouli, A Al Hasan and M S Abdelmonem, *Phys. Rev.* **A75**, 062711 (2007)
A Alhaidari, *Phys. Lett.* **A322**, 72 (2004)
[14] A de Souza Dutra and C-S Jia, *Phys. Lett.* **352**, 484 (2006)
I O Vakarchuk, *J. Phy. A: Math. Gen.* **38**, 4727 (2005)
[15] S M Ikhadair and R Sever, *Appl. Math. Comput.* **216**, 911 (2010)
[16] C-S Jia, T Chen and L-G Cui, *Phys. Lett.* **A373**, 1621 (2009)
[17] C-S Jia and A de Souza Dutra, *Ann. Phys.* **323**, 566 (2008)
[18] L Dekar, L Chetouani and T F J Hammann, *Math. Phys.* **39**, 2551 (1998)
[19] X-L Penz, J-Y Liu and C-S Jia, *Phys. Lett.* **A352**, 478 (2006)
[20] D Agboola, math-ph/arXiv:1011.2368v1, preprint
[21] H Ciftci, R L Hall and N Saad, *J. Phys.* **A36**, 11807 (2003)
[22] H Ciftci, R L Hall and N Saad, *J. Phys.* **A38**, 1147 (2005)
[23] H Ciftci, R L Hall and N Saad, *Phys. Rev.* **A72**, 022101 (2005)
[24] O Aydogdu and R Sever, *Phys. Scr.* **84**, 025005 (2011)
[25] M R Setare and S Haidari, *Phys. Scr.* **81**, 065201 (2010)
[26] B Gulveren, A Demirtas and R Ogul, *Phys. Scr.* **64**, 277 (2001)
[27] M Hamzavi, S M Ikhdair and B I Ita, *Phys. Scr.* **85**, 045009 (2012)

[28] S M Ikhdair and R Sever, *Appl. Math. Com.* **216**, 911 (2010)
[29] L H Zhang, X P Li and C S Jia, *Phys. Lett.* **A372**, 2201 (2008)
[30] A Soylu, O Bayrak and I Boztosun, *J. Phys. A: Math. Theor.* **41**, 065308 (2008)
[31] N Saad, *Phys. Scr.* **76**, 623 (2007)
[32] N M Temme, *Special functions: An introduction to the classical functions of mathematical physics* (Wiley, New York, 1996) §11.3
[33] R L Hall, *Phys. Lett.* **A372**, 12 (2007)
[34] A D Alhaidari, H Bahlouli and A Al-Hasan, *Phys. Lett.* **A349**, 87 (2006)
[35] T Barakat, M Odeh and O Mustafa, *J. Phys.* **A31**, 3469 (1998)
[36] S Ikhdair, *Eur. Phys. J.* **A40**, 143 (2009)
[37] A D Alhaidari, *Phys. Lett.* **A322**, 72 (2004)
[38] G B Arfken and H J Weber, *Mathematical methods for physicists* (Academic Press, San Diego 1995)