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Bifurcation analysis and the travelling wave solutions of the Klein–Gordon–Zakharov equations

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Abstract. In this paper, we investigate the bifurcations and dynamic behaviour of travelling wave solutions of the Klein–Gordon–Zakharov equations given in Shang *et al*, *Comput. Math. Appl.* **56**, 1441 (2008). Under different parameter conditions, we obtain some exact explicit parametric representations of travelling wave solutions by using the bifurcation method (Feng *et al*, *Appl. Math. Comput.* **189**, 271 (2007); Li *et al*, *Appl. Math. Comput.* **175**, 61 (2006)).

Keywords. Klein–Gordon–Zakharov equations; travelling wave solutions; bifurcation analysis.

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1. Introduction

In this paper, we consider the Klein–Gordon–Zakharov equation [1]

$$u_{tt} - u_{xx} + u + \alpha n u = 0, \tag{1.1}$$

$$n_{tt} - n_{xx} = \beta(|u|^2)_{xx}, \tag{1.2}$$

where $u(x,t)$ denotes the fast time-scale component of the electric field raised by electrons and $n(x,t)$ denotes the deviation of ion density from its equilibrium. Here $u(x,t)$ is a complex function, $n(x,t)$ is a real function and α, β are two nonzero real parameters. This system describes the interaction of the Langmuir wave and the ion acoustic wave in a high-frequency plasma. More details are presented in ref. [1] and the references therein.

Recently, applying trigonometric function series method, Zhang [2] studied the new exact travelling wave solutions of the Klein–Gordon equation

$$u_{tt} - u_{xx} + \alpha u - \beta u^3 = 0. \tag{1.3}$$

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Equation (1.3) describes the propagation of dislocations within crystals, the Bloch wall motion of magnetic crystals, the propagation of a 'splay wave' along a lied membrane, the unitary theory for elementary particles, the propagation of magnetic flux on a Josephson line, etc. (more details are given in ref. [2]). More recently, some exact solutions for the Zakharov equations are obtained using different methods [3–12]. In [1], using the extended hyperbolic functions method presented in [13], Shang obtained the multiple exact explicit solutions of the KGZEs (1.1) and (1.2). These solutions include the solitary wave solutions of bell-type for u and n, the solitary wave solutions of kink-type for u and bell-type for n, the solitary wave solutions of a compound of the bell-type and the kink-type for u and n, the singular travelling wave solutions, the periodic travelling wave solutions of triangle functions type, and solitary wave solutions of rational function type. For higher dimensional KGZEs and $\alpha = 1$, $\beta = 1$, by using the methods of dynamical systems, Li [14] considered the existence of exact explicit bounded travelling wave solutions of the following equations:

$$\phi_{tt} - \Delta\phi + \phi + \phi\psi u = 0, \quad \psi_{tt} - c^2\Delta\psi = \Delta|\phi|^2, \tag{1.4}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is the Laplacian operator, $x \in R^n$, c is the propagation speed of a wave (more details are given in ref. [14]). Based on the modified trigonometric function series method (MTFSM) [15], Zhang *et al* [16] studied the travelling wave solutions of KGZEs (1.1) and (1.2). More precisely, we combined the trigonometric function series method with the exp-function method (more details are given in ref. [16]).

Quite recently, based on the trigonometric-function series method [2] and the exp-function method [17], Zhang *et al* [15] proposed a new method called the modified trigonometric function series method (MTFSM) to construct travelling wave solutions to the perturbed nonlinear Schrödinger's equation (NLSE) with Kerr law nonlinearity

$$iu_t + u_{xx} + \alpha|u|^2u + i[\gamma_1 u_{xxx} + \gamma_2|u|^2u_x + \gamma_3(|u|^2)_x u] = 0. \tag{1.5}$$

Remark 1.1. If we show the Klein–Gordon–Zakharov system in nondimensional variables, it reads as

$$\begin{cases} c^{-2}u_{tt} - \Delta u + c^2 u + nu = 0, \\ \quad \gamma^{-2}n_{tt} - \Delta n = \Delta|u|^2, \end{cases} \tag{1.6}$$

where u: $R^{1+3} \to R^3$ is the electric field, n: $R^{1+3} \to R$ is the density fluctuation of ions, c^2 is the plasma frequency and γ is the ion sound speed, then system (1.6) describes the interaction between Langmuir waves [18] and ion sound waves in a plasma (see Dendy [19] and Bellan [20]). Indeed, eq. (1.1) is the special case of (1.6). Taking $v = \mathrm{e}^{ic^2t}u$, system (1.6) reduces to

$$\begin{cases} c^{-2}v_{tt} + 2iv_t - \Delta v + nv = 0, \\ \quad \gamma^{-2}n_{tt} - \Delta n = \Delta|v|^2. \end{cases} \tag{1.7}$$

Its formal limit as $c, \gamma \to \infty$ is given by the nonlinear Schrödinger's equation:

$$2iv_t - \Delta v = |v|^2 v, \quad n = -|v|^2. \tag{1.8}$$

In fact, eq. (1.8) is the Schrödinger's equation with Kerr law nonlinearity. Recently, Biswas and co-workers [21–24] studied the solitons of eq. (1.8).

Remark 1.2. Obviously, eq. (1.5) is the perturbation of eq. (1.8). For eq. (1.5) (in 1D case), it is worth mentioning that Zhang *et al* [15,25–27] considered the NLSE (1.5) with Kerr law nonlinearity and obtained some new exact travelling wave solutions of eq. (1.5). In ref. [15], by using the modified trigonometric function series method, Zhang *et al* studied some new exact travelling wave solutions of eq. (1.5). In ref. [25], by using the modified mapping method and the extended mapping method, Zhang *et al* derived some new exact solutions of eq. (1.5), which are the linear combinations of two different Jacobi elliptic functions and investigated the solutions in the limit cases. In ref. [26], by using the dynamical system approach, Zhang *et al* obtained the travelling wave solutions in terms of bright and dark optical solitons and the cnoidal waves. The authors found that eq. (1.5) has only three types of bounded travelling wave solutions, namely, bell-shaped solitary wave solutions, kink-shaped solitary wave solutions and Jacobi elliptic function periodic solutions. Moreover, we pointed out the region where these periodic wave solutions lie in. We showed the relation between the bounded travelling wave solution and the energy level h. We observed that these periodic wave solutions tend to the corresponding solitary wave solutions as h increases or decreases. Finally, for some special selections of the energy level h, it is shown that the exact periodic solutions evolute into solitary wave solution. In ref. [27], by using the modified (G'/G)-expansion method, Miao and Zhang obtained the travelling wave solutions, which are expressed by the hyperbolic functions, trigonometric functions and rational functions. In ref. [28], by using the theory of bifurcations, Zhang *et al* investigated the bifurcations and dynamic behaviour of travelling wave solutions of eq. (1.5). Under the given parametric conditions, all possible representations of explicit exact solitary wave solutions and periodic wave solutions are obtained.

Remark 1.3. For system (1.6), if we take the limit $c \to \infty$, then we get the usual Zakharov system:

$$\begin{cases} 2iv_t - \Delta v + nv = 0, \\ \gamma^{-2} n_{tt} - \Delta n = \Delta |v|^2. \end{cases} \tag{1.9}$$

If we take the limit $c \to \infty$, then we get the Klein–Gordon system:

$$c^{-2} u_{tt} - \Delta u + c^2 u + |u|^2 u = 0. \tag{1.10}$$

In fact, eq. (1.3) is the special case of eq. (1.10). It is classically known that when γ goes to infinity in the Zakharov system (1.8) leads to the cubic nonlinear Schrödinger's equation (1.9) and that the limit when c goes to infinity in the cubic nonlinear Klein–Gordon system (1.10) also leads to the cubic nonlinear Schrödinger's equation.

In this paper, we investigate the bifurcation analysis and dynamic behaviour of travelling wave solutions of the Klein–Gordon–Zakharov equations (1.1) by using the bifurcation method and dynamical systems approach [1,29,30]. Under different parameter conditions, some exact explicit parametric representations of travelling wave solutions are obtained.

2. Bifurcation analysis and exact travelling wave solutions of (1.1) and (1.2)

In this section, we investigate the bifurcation analysis and dynamic behaviour of travelling wave solutions of the Klein–Gordon–Zakharov equations (1.1) and (1.2) using the bifurcation method and qualitative theory of dynamical systems [1,29,30].

To facilitate further on our analysis, we assume that eq. (1.1) has travelling wave solutions in the form [1]

$$u(x,t) = \phi(x,t)\exp(i(kx + \omega t + \xi_0)), \tag{2.1}$$

where $\phi(x,t)$ is a real-valued function, k, ω are two real constants to be determined and ξ_0 is an arbitrary constant. Substituting (2.1) into (1.1)–(1.2) yields

$$\phi_{tt} - \phi_{xx} + (k^2 - \omega^2 + 1)\phi + \alpha n\phi = 0, \tag{2.2}$$

$$\omega\phi_t - k\phi_x = 0, \tag{2.3}$$

$$n_{tt} - n_{xx} = \beta(\phi^2)_{xx}. \tag{2.4}$$

By virtue of (2.3), we assume

$$\phi(x,t) = \phi(\xi) = \phi(\omega x + kt + \xi_1), \tag{2.5}$$

where ξ_1 is an arbitrary constant. Substituting eq. (2.5) into (2.2), we have

$$n(x,t) = \frac{(\omega^2 - k^2)\phi''(\xi)}{\alpha\phi(\xi)} + \frac{(\omega^2 - 1 - k^2)}{\alpha}. \tag{2.6}$$

Hence, we can also assume

$$n(x,t) = \psi(\xi) = \psi(\omega x + kt + \xi_1). \tag{2.7}$$

Substituting eq. (2.7) into (2.4) and integrating the resultant equation twice with respect to ξ, we obtain

$$\psi(\xi) = \frac{\beta\omega^2\phi^2(\xi)}{k^2 - \omega^2} + C, \tag{2.8}$$

where C is an integration constant. For simplicity, we choose $C = 0$. It follows from eqs (2.2) and (2.8) that

$$\phi''(\xi) + \frac{k^2 - \omega^2 + 1}{k^2 - \omega^2}\phi(\xi) + \frac{\alpha\beta\omega^2}{(k^2 - \omega^2)^2}\phi^3(\xi) = 0. \tag{2.9}$$

For simplicity, we assume

$$A = \frac{k^2 - \omega^2 + 1}{k^2 - \omega^2}, \quad B = \frac{\alpha\beta\omega^2}{(k^2 - \omega^2)^2},$$

thus (2.9) leads to ordinary differential equation (ODE)

$$\phi''(\xi) + A\phi(\xi) + B\phi^3(\xi) = 0. \tag{2.10}$$

Before proceeding with our analysis, we need the following phase plane analysis for eq. (2.10). Phase plane analysis is a technique for the analysis of the qualitative behaviour of second-order systems. It provides physical insights. For second-order systems, solution trajectories can be represented by curves in the plane, which allow the visualization of the qualitative behaviour of the system. In particular, it is interesting to consider the behaviour of systems around equilibrium points. Moreover, the solution trajectories represented by curves in the plane provide easy visualization of the system's qualitative behaviour. The resulting curve is known as the trajectory of the solution pair $(x(t), y(t))$ and the xy-plane is called the phase plane. The direction field of the phase plane equation may also give important information about the behaviour of the trajectories.

Let $x = \phi(\xi)$, $y = \phi'(\xi)$. Then eq. (2.10) reduces to the following planar dynamic system:

$$\begin{cases} \dfrac{\mathrm{d}\phi}{\mathrm{d}\xi} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\xi} = -A\phi - B\phi^3 \end{cases} \tag{2.11}$$

which admits the following Hamiltonian function:

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{A}{2}\phi^2 + \frac{B}{4}\phi^4 = h, \tag{2.12}$$

where $h \in R$ is an integral constant. Thus from (2.12), we have

$$y = \pm\sqrt{\phi^2(-A - (B/2)\phi^2) + 2h}. \tag{2.13}$$

Substituting (2.13) into $y = \phi'(\xi)$, it yields

$$\frac{\mathrm{d}\phi}{\sqrt{\phi^2(-\frac{B}{A} - \frac{C}{2A}\phi^2) + 2h}} = \pm\mathrm{d}\xi. \tag{2.14}$$

Now we investigate the bifurcation phase portraits of system (2.11) in the parameter space (A, B). Assume

$$f(\phi) = -A\phi - B\phi^3. \tag{2.15}$$

It is well known that in ϕ–y plane all the singular points of system (2.11) are on ϕ-axis. Therefore, to study the distribution of singular points of system (2.11), we only need to investigate the fixed points of (2.15). From eq. (2.15), we study two cases as follows:

(1) If $AB > 0$, then $f(\phi)$ has only one fixed point.
(2) If $AB < 0$, then $f(\phi)$ has two fixed points.

Thus, we can easily draw the graphics of function $f(\phi)$ in figure 1 (when $AB > 0$) and figure 2 (when $AB < 0$).

Suppose that $(\phi, 0)$ is a singular point of system (2.11), then at the singular point $(\phi, 0)$ the characteristic values of the linearized system of (2.11) is

$$\lambda_{1,2} = \pm\sqrt{f'(\phi)}. \tag{2.16}$$

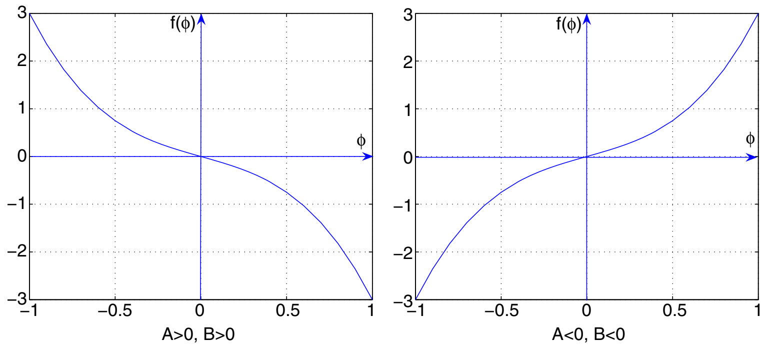

Figure 1. The graphics of function (2.15) with $AB > 0$.

According to the qualitative theory of dynamical systems [29], we conclude that

(1) If $f'(\phi) < 0$, $(\phi, 0)$ is a centre point.
(2) If $f'(\phi) > 0$, $(\phi, 0)$ is a saddle point.
(3) If $f'(\phi) = 0$, $(\phi, 0)$ is a degenerate saddle point.

Thus, we obtain the phase portraits of system (2.11) in figure 3 (when $AB > 0$) and figure 4 (when $AB < 0$).

Case 1. If $A < 0$, $B < 0$ and $h = 0$, the orbits connected at the saddle point $(0, 0)$ are given as

$$y = \pm\phi\sqrt{-A - \frac{B}{2}\phi^2}. \tag{2.17}$$

Substituting eq. (2.17) into $y = \phi'(\xi)$, and integrating them along the orbits, we get

$$\int_{\phi}^{+\infty} \frac{1}{s\sqrt{-A - (B/2)s^2}} \mathrm{d}s = |\xi|, \quad \phi \geq 0, \tag{2.18}$$

and

$$\int_{\phi}^{-\infty} \frac{1}{s\sqrt{-A - (B/2)s^2}} \mathrm{d}s = |\xi|, \quad \phi \leq 0. \tag{2.19}$$

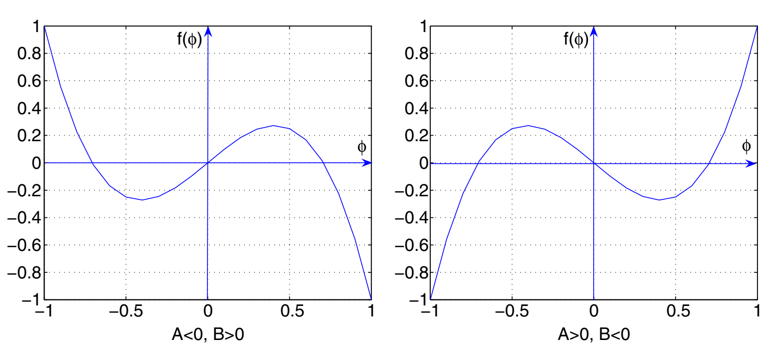

Figure 2. The graphics of function (2.15) with $AB < 0$.

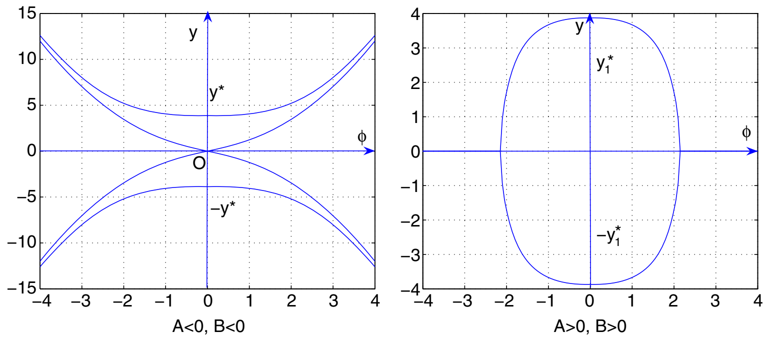

Figure 3. The phase portraits of system (2.11) with $AB > 0$. For simplicity, we denote $-y^* = -\phi_*$, $y^* = \phi_*$, $-y_1^* = -\bar{\phi}_*$ and $y_1^* = \bar{\phi}_*$.

It follows from eqs (2.18) and (2.19) that

$$\phi = \pm\sqrt{\frac{2A}{B}}\text{csch}[\sqrt{-A}\xi]. \tag{2.20}$$

Observing that $\phi = \phi(\xi)$, $\xi = kx + \omega t + \xi_0$ and together with eqs (2.1), (2.7) and (2.8), we get the exact solutions

$$|u_1(x,t)| = \left|\sqrt{\frac{2A}{B}}\text{csch}[\sqrt{-A}(kx + \omega t + \xi_0)]\right|,$$

$$|n_1(x,t)| = \left|\frac{\beta\omega^2}{k^2 - \omega^2}\frac{2A}{B}\text{csch}^2[\sqrt{-A}(kx + \omega t + \xi_0)]\right|. \tag{2.21}$$

Here, $|u|$ is the norm of u. Setting $k = \sqrt{2}$, $\omega = 1$, $\xi_0 = 0$, $A = -1$, $B = -2$ and $\beta = 1$, we obtain the solution of (2.21), i.e.

$$|u_1(x,t)| = |\text{csch}(2x + \sqrt{2}t)|,$$

$$|n_1(x,t)| = |\text{csch}^2(2x + \sqrt{2}t)|. \tag{2.22}$$

Figure 5 shows the graphics of solution (2.22).

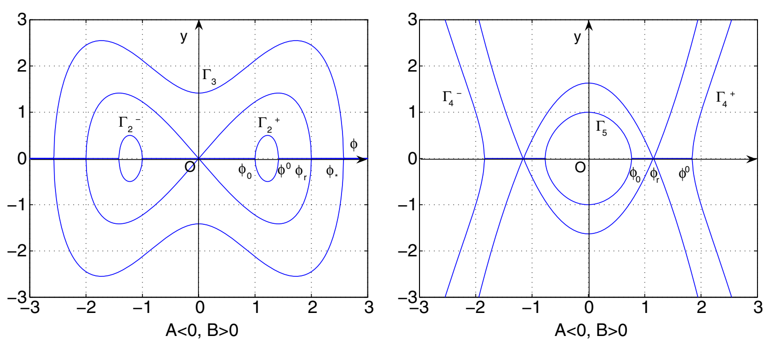

Figure 4. The phase portraits of system (2.11) with $AB < 0$. For simplicity, we denote $\phi^0 = \bar{\phi}_0$ and $\phi_* = \bar{\phi}_*$.

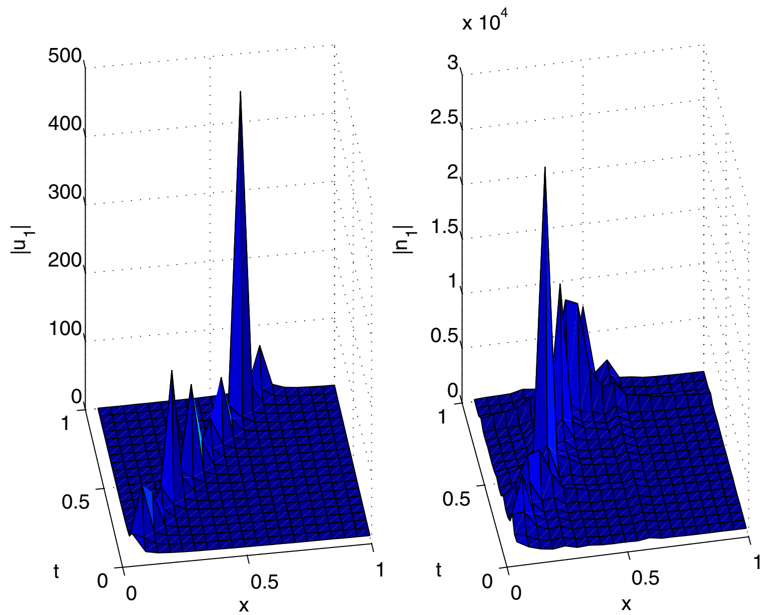

Figure 5. The graphics of solution (2.22).

Case 2. If $A < 0$, $B < 0$ and $h < 0$, from the phase portrait in figure 3, the orbits $\Gamma_1^{\pm}$ are given as

$$y = \pm\sqrt{-\frac{B}{2}(\phi^2 - (\phi_*)^2)(\phi^2 + (\phi^*)^2)}, \tag{2.23}$$

where

$$\phi_* = \sqrt{\frac{\sqrt{A^2 + 4Bh} + A}{-B}}, \quad \phi^* = \sqrt{\frac{\sqrt{A^2 + 4Bh} - A}{-B}}.$$

Substituting (2.23) into $y = \phi'(\xi)$, and integrating them along the orbits $\Gamma_1^{\pm}$, it follows that

$$\int_{\phi}^{+\infty} \frac{1}{(s^2 - (\phi_*)^2)(s^2 + (\phi^*)^2)} \mathrm{d}s = \sqrt{-\frac{B}{2}}|\xi|, \quad \phi \geq 0, \tag{2.24}$$

and

$$\int_{\phi}^{-\infty} \frac{1}{(s^2 - (\phi_*)^2)(s^2 + (\phi^*)^2)} \mathrm{d}s = \sqrt{-\frac{B}{2}}|\xi|, \quad \phi \leq 0. \tag{2.25}$$

From eqs (2.24) and (2.25), we have

$$\phi = \pm\sqrt{\frac{(\phi_*)^2 + (\phi^*)^2}{\mathrm{sn}^2[\sqrt{-(B/2)}\sqrt{(\phi_*)^2 + (\phi^*)^2}\xi]}}. \tag{2.26}$$

Observing that $\phi = \phi(\xi)$ and $\xi = kx + \omega t + \xi_0$ and together with eqs (2.1), (2.7) and (2.8), we get the exact solutions

$$|u_2(x,t)| = \left|\sqrt{\frac{(\phi_*)^2 + (\phi^*)^2}{\mathrm{sn}^2[\sqrt{-(B/2)}\sqrt{(\phi_*)^2 + (\phi^*)^2}(kx + \omega t + \xi_0)]}}\right|,$$

$$|n_2(x,t)| = \left|\frac{\beta\omega^2}{k^2 - \omega^2}\frac{(\phi_*)^2 + (\phi^*)^2}{\mathrm{sn}^2[\sqrt{-(B/2)}\sqrt{(\phi_*)^2 + (\phi^*)^2}(kx + \omega t + \xi_0)]}\right|. \tag{2.27}$$

Setting $k = \sqrt{2}$, $\omega = 1$, $\xi_0 = 0$, $A = -1$, $B = -2$ and $\beta = 1$, $h = -3/8$, we obtain the solution of (2.27), i.e.

$$|u_2(x,t)| = \left|\sqrt{\frac{2}{\mathrm{sn}^2[2(\sqrt{2}x + t)]}}\right|,$$

$$|n_2(x,t)| = \left|\frac{2}{\mathrm{sn}^2[2(\sqrt{2}x + t)]}\right|. \tag{2.28}$$

Figure 6 shows the graphics of solution (2.28).

Case 3. If $A > 0$, $B > 0$ and $h > 0$, the orbit is given as

$$y = \pm\sqrt{-\frac{B}{2}(\phi^2 - (\bar{\phi}_*)^2)(\phi^2 + (\bar{\phi}^*)^2)}, \tag{2.29}$$

where

$$\bar{\phi}_* = \sqrt{-\frac{A - \sqrt{A^2 + 4Bh}}{-B}}, \quad \bar{\phi}^* = \sqrt{\frac{-A - \sqrt{A^2 + 4Bh}}{-B}}.$$

Substituting (2.29) into $y = \phi'(\xi)$, and integrating them along the orbit, we get

$$\int_{\phi}^{\bar{\phi}_*} \frac{1}{\sqrt{((\bar{\phi}_*)^2 - s^2)(s^2 + (\bar{\phi}^*)^2)}}\mathrm{d}s = \sqrt{-\frac{B}{2}}|\xi|. \tag{2.30}$$

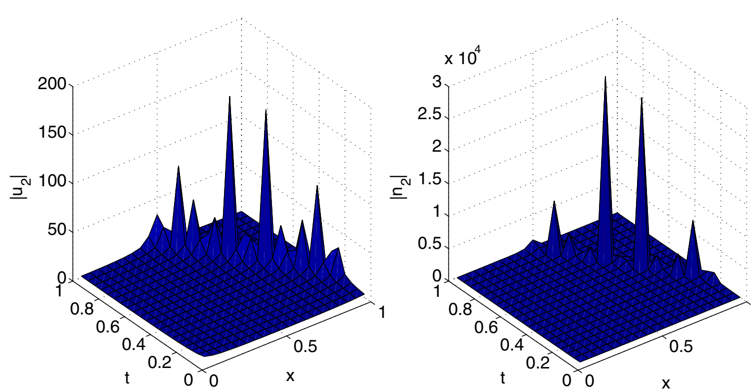

Figure 6. The graphics of solution (2.28).

It follows from (2.30) that

$$\phi = \bar{\phi}_* \mathrm{cn}\left[\sqrt{\frac{B}{2}}\sqrt{(\bar{\phi}_*)^2 + (\bar{\phi}^*)^2}\xi\right]. \tag{2.31}$$

Observing that $\phi = \phi(\xi)$ and $\xi = kx + \omega t + \xi_0$ and together with eqs (2.1), (2.7) and (2.8), we get the exact solutions

$$|u_3(x,t)| = \left|\bar{\phi}_* \mathrm{cn}\left[\sqrt{\frac{B}{2}}\sqrt{(\bar{\phi}_*)^2 + (\bar{\phi}^*)^2}(kx + \omega t + \xi_0)\right]\right|,$$

$$|n_3(x,t)| = \left|\frac{\beta\omega^2}{k^2 - \omega^2}\bar{\phi}_*^2 \mathrm{cn}^2\left[\sqrt{\frac{B}{2}}\sqrt{(\bar{\phi}_*)^2 + (\bar{\phi}^*)^2}(kx + \omega t + \xi_0)\right]\right| \tag{2.32}$$

Setting $k = \sqrt{2}$, $\omega = 1$, $\xi_0 = 0$, $A = 2$, $B = 2$ and $\beta = 1$, $h = 5/8$, we obtain the solution of (2.32), i.e.

$$|u_3(x,t)| = \sqrt{\frac{1}{2}}|\mathrm{cn}[\sqrt{6}(\sqrt{2}x + t)]|,$$

$$|n_3(x,t)| = \frac{1}{2}|\mathrm{cn}^2[\sqrt{6}(\sqrt{2}x + t)]|. \tag{2.33}$$

Figure 7 shows the graphics of solution (2.33).

Case 4. If $A < 0$, $B > 0$ and $h = 0$, from the phase portrait in figure 4, the two symmetric homoclinic orbits connected at the saddle point are given as

$$y = \pm\phi\sqrt{-A - \frac{B}{2}\phi^2}. \tag{2.34}$$

Substituting eq. (2.34) into $y = \phi'(\xi)$, and integrating them along the homoclinic orbits, it follows that

$$\int_{\phi}^{\phi_r} \frac{1}{s\sqrt{-A - (B/2)s^2}}\mathrm{d}s = |\xi|, \quad \phi \geq 0, \tag{2.35}$$

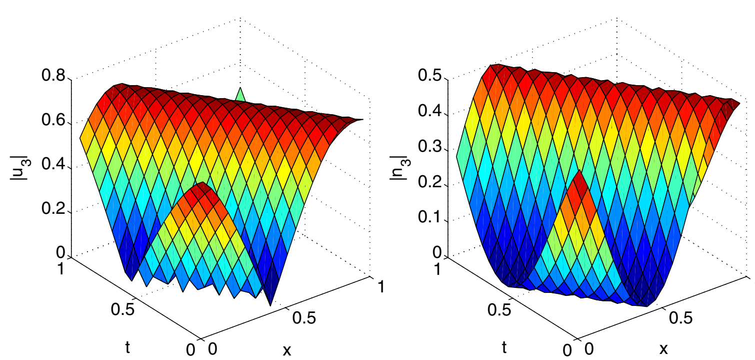

Figure 7. The graphics of solution (2.33).

and

$$\int_{\phi}^{-\phi_r} \frac{1}{s\sqrt{-A-(B/2)s^2}} \mathrm{d}s = |\xi|, \quad \phi \le 0. \tag{2.36}$$

From eqs (2.35) and (2.36), we have

$$\phi = \pm\sqrt{-\frac{2A}{B}} \operatorname{sech}[\sqrt{-B}\xi]. \tag{2.37}$$

Observing that $\phi = \phi(\xi)$ and $\xi = kx + \omega t + \xi_0$ and together with eqs (2.1), (2.7) and (2.8), we get the exact solutions

$$|u_4(x,t)| = \left|\sqrt{-\frac{2A}{B}} \operatorname{sech}\left[\sqrt{-A}(kx+\omega t+\xi_0)\right]\right|,$$

$$|n_4(x,t)| = \left|\frac{\beta\omega^2}{k^2-\omega^2}\left(-\frac{2A}{B}\right)\operatorname{sech}^2\left[\sqrt{-A}(kx+\omega t+\xi_0)\right]\right|. \tag{2.38}$$

Setting $k = \sqrt{2}$, $\omega = 1$, $\xi_0 = 0$, $A = -\frac{1}{2}$, $B = 1$ and $\beta = 1$, $h = 0$, we obtain the solution of (2.38), i.e.

$$|u_4(x,t)| = \left|\operatorname{sech}\left(x + \frac{\sqrt{2}}{2}t\right)\right|,$$

$$|n_4(x,t)| = \left|\operatorname{sech}^2\left(x + \frac{\sqrt{2}}{2}t\right)\right|. \tag{2.39}$$

Figure 8 is the graphics of solution (2.39).

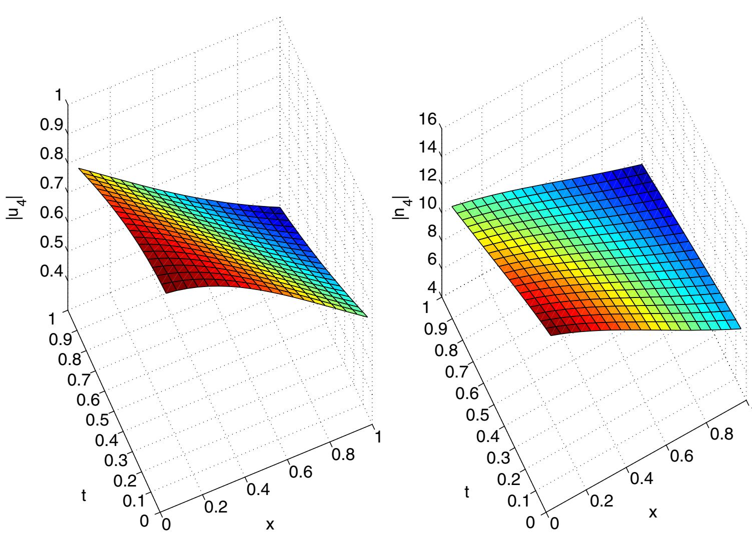

Figure 8. The graphics of solution (2.39).

Case 5. If $A < 0$, $B > 0$ and $h < 0$, from the phase portrait in figure 4, the two symmetric periodic orbits $\Gamma_2^{\pm}$ are given as

$$y = \pm\sqrt{\frac{B}{2}(\phi^2 - \phi_0^2)((\bar{\phi}_0)^2 - \phi^2)}, \tag{2.40}$$

where $\bar{\phi}_0 = \sqrt{-(2A/B) - \phi_0^2}$, and ϕ_0 will be given later.

Substituting eq. (2.40) into $y = \phi'(\xi)$, and integrating them along the orbits $\Gamma_2^{\pm}$, it follows that

$$\int_{\phi_0}^{\phi} \frac{1}{\sqrt{(s^2 - \phi_0^2)((\bar{\phi}_0)^2 - s^2)}} \mathrm{d}s = \sqrt{\frac{B}{2}}|\xi|, \tag{2.41}$$

and

$$\int_{\phi}^{-\phi_0} \frac{1}{\sqrt{(s^2 - \phi_0^2)((\bar{\phi}_0)^2 - s^2)}} \mathrm{d}s = \sqrt{\frac{B}{2}}|\xi|. \tag{2.42}$$

From eqs (2.41) and (2.42), we have

$$\phi = \pm\frac{\bar{\phi}_0\phi_0}{\sqrt{(\bar{\phi}_0)^2 - ((\bar{\phi}_0)^2 - (\phi_0)^2)\mathrm{sn}[\sqrt{(-2A/B)}\bar{\phi}_0|\xi|]}}. \tag{2.43}$$

Observing that $\phi = \phi(\xi)$ and $\xi = kx + \omega t + \xi_0$ and together with eqs (2.1), (2.7) and (2.8), we get exact solutions

$$|u_5(x,t)| = \left|\frac{\bar{\phi}_0\phi_0}{\sqrt{(\bar{\phi}_0)^2 - ((\bar{\phi}_0)^2 - (\phi_0)^2)\mathrm{sn}[\sqrt{(-2A/B)}\bar{\phi}_0|kx + \omega t + \xi_0|]}}\right|,$$

$$|n_5(x,t)| = \left|\frac{\beta\omega^2}{k^2 - \omega^2}\frac{(\bar{\phi}_0\phi_0)^2}{(\bar{\phi}_0)^2 - ((\bar{\phi}_0)^2 - (\phi_0)^2)\mathrm{sn}[\sqrt{(-2A/B)}\bar{\phi}_0|kx + \omega t + \xi_0|]}\right| \tag{2.44}$$

Setting $k = \sqrt{2}$, $\omega = 1$, $\xi_0 = 0$, $A = -\frac{1}{2}$, $B = 1$ and $\beta = 1$, $h = -1$, we obtain the solution of (2.44), i.e.

$$|u_5(x,t)| = \left|\frac{\sqrt{3}/4}{\sqrt{(3/4) - \frac{1}{2}\mathrm{sn}[\frac{1}{2}|\sqrt{2}x + t|]}}\right|,$$

$$|n_5(x,t)| = \left|\frac{\sqrt{3}/16}{\frac{3}{4} - \frac{1}{2}\mathrm{sn}[\frac{1}{2}|\sqrt{2}x + t|]}\right|. \tag{2.45}$$

Figure 9 shows the graphics of eq. (2.45).

Case 6. If $A < 0$, $B > 0$ and $h > 0$, from the phase portrait in figure 4, the periodic orbit Γ_3 is given as

$$y = \pm\sqrt{\frac{-B}{2}(\phi^2 - (\bar{\phi}_*)^2)(\phi^2 + (\bar{\phi}^*)^2)}. \tag{2.46}$$

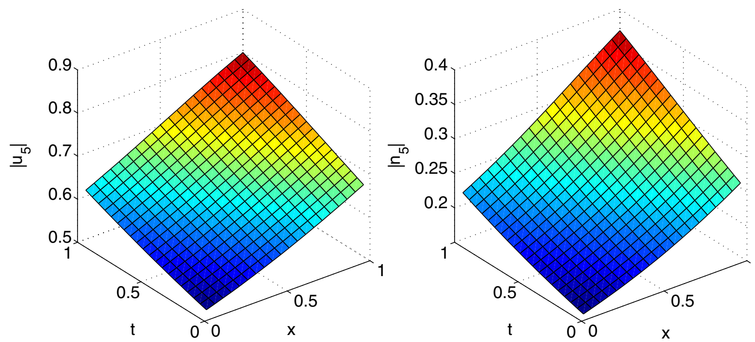

Figure 9. The graphics of solution (2.45).

Substituting eq. (2.46) into $y = \phi'(\xi)$, and integrating them along the periodic orbit Γ_3, it follows that

$$\int_{\phi}^{\bar{\phi}_*} \frac{1}{\sqrt{((\bar{\phi}_*)^2 - s^2)(s^2 + (\bar{\phi}^*)^2)}} \mathrm{d}s = \sqrt{\frac{B}{2}} |\xi|. \tag{2.47}$$

From eq. (2.47), we have

$$\phi = \bar{\phi}_* \mathrm{cn}\left[\sqrt{\frac{B}{2}} \sqrt{(\bar{\phi}_*)^2 + (\bar{\phi}^*)^2} \xi\right]. \tag{2.48}$$

Observing that $\phi = \phi(\xi)$ and $\xi = kx + \omega t + \xi_0$ and together with eqs (2.1), (2.7) and (2.8), we get the exact solutions

$$|u_6(x,t)| = \left|\bar{\phi}_* \mathrm{cn}\left[\sqrt{\frac{B}{2}} \sqrt{(\bar{\phi}_*)^2 + (\bar{\phi}^*)^2} (kx + \omega t + \xi_0)\right]\right|,$$

$$|n_6(x,t)| = \left|\frac{\beta\omega^2}{k^2 - \omega^2} \bar{\phi}_*^2 \mathrm{cn}^2\left[\sqrt{\frac{B}{2}} \sqrt{(\bar{\phi}_*)^2 + (\bar{\phi}^*)^2} (kx + \omega t + \xi_0)\right]\right|. \tag{2.49}$$

Setting $k = \sqrt{2}$, $\omega = 1$, $\xi_0 = 0$, $A = -1$, $B = 2$ and $\beta = 1$, $h = 3/8$, we obtain the solution of (2.49), i.e.

$$|u_6(x,t)| = \sqrt{\frac{3}{2}} \left|\mathrm{cn}\left[\sqrt{\frac{5}{2}} (\sqrt{2}x + t)\right]\right|,$$

$$|n_6(x,t)| = \frac{3}{2} \left|\mathrm{cn}^2\left[\sqrt{\frac{5}{2}} (\sqrt{2}x + t)\right]\right|. \tag{2.50}$$

Figure 10 shows the graphics of solution (2.50).

Case 7. If $A > 0$, $B < 0$ and $h = H(\phi_r, 0)$, from the phase portrait in figure 4, the heteroclinic orbits connected at saddle points $(-\phi_r, 0)$ and $(\phi_r, 0)$ are given as

$$y = \pm\sqrt{-\frac{B}{2}} (\phi_r - \phi)(\phi_r + \phi). \tag{2.51}$$

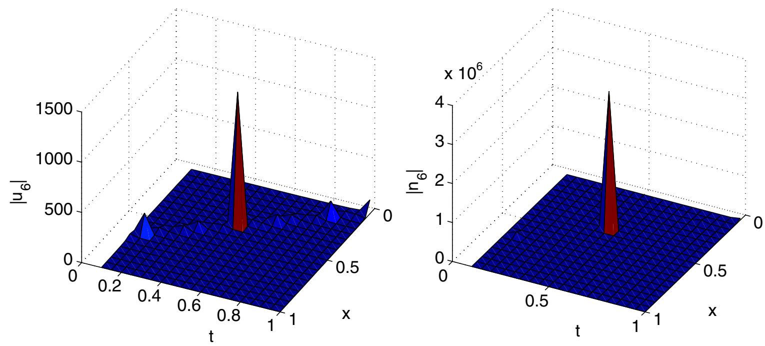

Figure 10. The graphics of solution (2.50).

Substituting eq. (2.51) into $y = \phi'(\xi)$, and integrating them along heteroclinic orbits, it follows that

$$\int_0^{\phi} \frac{1}{(\phi_r - s)(\phi_r + s)} \mathrm{d}s = \sqrt{-\frac{B}{2}}|\xi| \tag{2.52}$$

and

$$\int_{\phi}^{+\infty} \frac{1}{(\phi_r - s)(\phi_r + s)} \mathrm{d}s = \sqrt{-\frac{B}{2}}|\xi|. \tag{2.53}$$

From eqs (2.52) and (2.53), we have

$$\phi = \pm\phi_r \tanh\left[\phi_r \sqrt{-\frac{B}{2}}\xi\right] \tag{2.54}$$

and

$$\phi = \pm\phi_r \coth\left[\phi_r \sqrt{-\frac{B}{2}}\xi\right]. \tag{2.55}$$

Observing that $\phi = \phi(\xi)$ and $\xi = kx + \omega t + \xi_0$ and together with eqs (2.1), (2.7) and (2.8), we get two kink-shaped solutions

$$|u_7(x,t)| = \left|\phi_r \tanh\left[\phi_r \sqrt{-\frac{B}{2}}(kx + \omega t + \xi_0)\right]\right|,$$

$$|n_7(x,t)| = \left|\frac{\beta\omega^2}{k^2 - \omega^2}\phi_r^2 \tanh^2\left[\phi_r \sqrt{-\frac{B}{2}}(kx + \omega t + \xi_0)\right]\right| \tag{2.56}$$

and two blow-up solutions

$$|u_8(x,t)| = \left|\phi_r \coth\left[\phi_r \sqrt{-\frac{B}{2}}(kx + \omega t + \xi_0)\right]\right|,$$

$$|n_8(x,t)| = \left|\frac{\beta\omega^2}{k^2 - \omega^2}\phi_r^2 \coth^2\left[\phi_r \sqrt{-\frac{B}{2}}(kx + \omega t + \xi_0)\right]\right|. \tag{2.57}$$

Setting $k = \sqrt{2}$, $\omega = 1$, $\xi_0 = 0$, $A = -1$, $B = -1$, $\beta = 1$, $h = -3/4$, we obtain the solution of (2.56) and (2.57), i.e.

$$|u_7(x,t)| = \left|\tanh\left[\sqrt{\frac{1}{2}}(\sqrt{2}x + t)\right]\right|,$$

$$|n_7(x,t)| = \left|\tanh^2\left[\sqrt{\frac{1}{2}}(\sqrt{2}x + t)\right]\right|, \tag{2.58}$$

and

$$|u_8(x,t)| = \left|\coth\left[\sqrt{\frac{1}{2}}(\sqrt{2}x + t)\right]\right|,$$

$$|n_8(x,t)| = \left|\coth^2\left[\sqrt{\frac{1}{2}}(\sqrt{2}x + t)\right]\right| \tag{2.59}$$

respectively. Figure 11 shows the graphics of eq. (2.58) and figure 12 shows the graphics of eq. (2.59).

Case 8. If $A > 0$, $B < 0$ and $h < H(\phi_r, 0)$, from the phase portrait in figure 4, the orbits $\Gamma_4^{\pm}$, and periodic orbit points Γ_5, are given as

$$y = \pm\sqrt{\frac{-B}{2}(\phi_0^2 - \phi^2)(\bar{\phi}_0 - \phi^2)}. \tag{2.60}$$

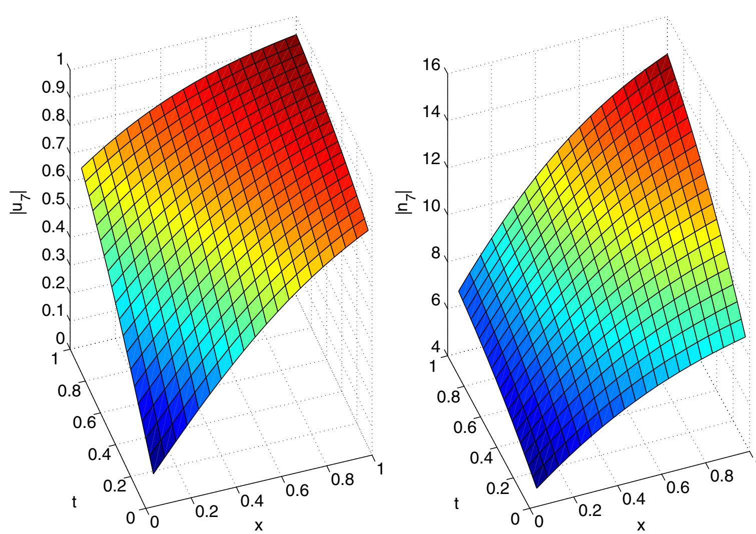

Figure 11. The graphics of solution (2.58).

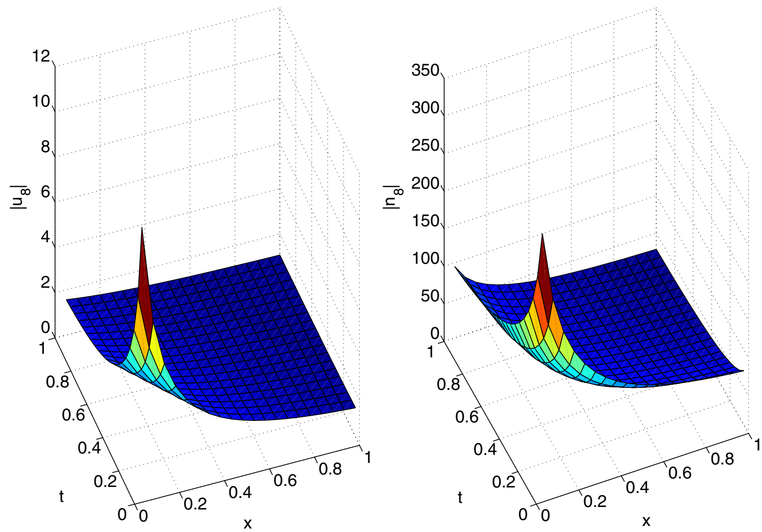

Figure 12. The graphics of solution (2.59).

Substituting eq. (2.60) into $y = \phi'(\xi)$, and integrating them along the orbits $\Gamma_4^{\pm}$, and periodic orbit points Γ_5, it follows that

$$\int_0^{\phi} \frac{1}{\sqrt{(\phi_0^2 - \phi^2)(\bar{\phi}_0 - \phi^2)}} \mathrm{d}s = \sqrt{-\frac{B}{2}}|\xi| \tag{2.61}$$

and

$$\int_{\phi}^{+\infty} \frac{1}{\sqrt{(\phi_0^2 - \phi^2)(\bar{\phi}_0 - \phi^2)}} \mathrm{d}s = \sqrt{-\frac{B}{2}}|\xi|. \tag{2.62}$$

From eqs (2.61) and (2.62), we have

$$\phi = \phi_0 \,\mathrm{sn}\left[\bar{\phi}_0 \sqrt{-\frac{B}{2}}\xi\right] \tag{2.63}$$

and

$$\phi = \pm \frac{\bar{\phi}_0}{\mathrm{sn}\left[\bar{\phi}_0\sqrt{-(B/2)}\xi\right]}. \tag{2.64}$$

Observing that $\phi = \phi(\xi)$ and $\xi = kx + \omega t + \xi_0$ and together with eqs (2.1), (2.7) and (2.8), we get one periodic solution

$$|u_9(x,t)| = \left|\phi_0 \mathrm{sn}\left[\bar{\phi}_0\sqrt{-\frac{B}{2}}(kx + \omega t + \xi_0)\right]\right|,$$

$$|n_9(x,t)| = \left|\frac{\beta\omega^2}{k^2 - \omega^2}\phi_0^2 \mathrm{sn}^2\left[\bar{\phi}_0\sqrt{-\frac{B}{2}}(kx + \omega t + \xi_0)\right]\right| \tag{2.65}$$

and two blow-up solutions

$$|u_{10}(x,t)| = \left| \frac{\bar{\phi}_0}{\mathrm{sn}\left[\bar{\phi}_0\sqrt{-(B/2)}(kx+\omega t+\xi_0)\right]} \right|,$$

$$|n_{10}(x,t)| = \left| \frac{\beta\omega^2}{k^2-\omega^2} \frac{\bar{\phi}_0^2}{\mathrm{sn}^2\left[\bar{\phi}_0\sqrt{-(B/2)}(kx+\omega t+\xi_0)\right]} \right|. \tag{2.66}$$

Setting $k=\sqrt{2}$, $\omega=1$, $\xi_0=0$, $A=1$, $B=-1$, $\beta=1$ and $\phi_0=1$, we obtain the solution of eqs (2.65) and (2.66), i.e.

$$|u_9(x,t)| = \left| \mathrm{sn}\left[\sqrt{\frac{1}{2}}(\sqrt{2}x+t)\right] \right|,$$

$$|n_9(x,t)| = \left| \mathrm{sn}^2\left[\sqrt{\frac{1}{2}}(\sqrt{2}x+t)\right] \right|, \tag{2.67}$$

$$|u_{10}(x,t)| = \left| \frac{1}{\mathrm{sn}\left[\sqrt{\frac{1}{2}}(\sqrt{2}x+t)\right]} \right|,$$

$$|n_{10}(x,t)| = \left| \frac{1}{\mathrm{sn}^2\left[\sqrt{\frac{1}{2}}(\sqrt{2}x+t)\right]} \right|, \tag{2.68}$$

respectively. Figure 13 shows the graphics of eq. (2.67) and figure 14 shows the graphics of eq. (2.68).

Remark 2.1. Recently, Zhang *et al* [16] investigated the travelling wave solutions of the Klein–Gordon–Zakharov equations by using the modified trigonometric function series method according to the ideas of Zhang *et al* [15]. Exact travelling wave solutions are obtained.

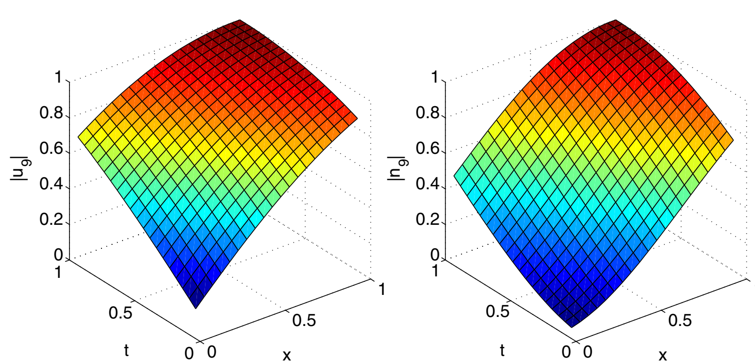

Figure 13. The graphics of solution (2.67).

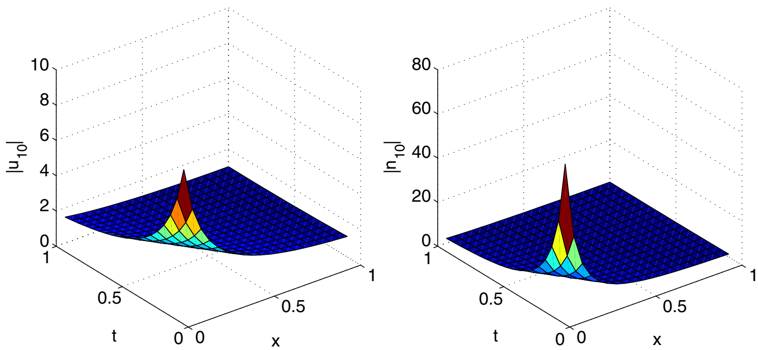

Figure 14. The graphics of solution (2.68).

3. Conclusion and discussion

In our contribution, we have employed bifurcation method of dynamical systems to investigate the travelling wave solutions of the KGZ equations. Under different parameter conditions, some exact explicit parametric representations of travelling wave solutions are obtained. Indeed, the ODE (2.10) is the well-known Duffing equation. It is well known that the Duffing equation is the equation governing the oscillations of a mass attached to the end of a spring whose tension (or compression) can be obtained from ref. [31]. Finally, it is worthwhile to mention that the method can also be applied to solve many other NPDEs in mathematical physics which will be investigated in our future research.

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References

[1] Y D Shang, Y Huang and W J Yuan, *Comput. Math. Appl.* **56**, 1441 (2008)
[2] Z Y Zhang, *Turk. J. Phys.* **32**, 235 (2008)
[3] J L Zhang, M L Wang, D M Chen and Z D Fang, *Commun. Theor. Phys.* **40**, 129 (2003)
[4] Naranmandula and K X Wang, *Phys. Lett.* **A336**, 112 (2005)
[5] C H Zhao and Z M Sheng, *Acta Phys. Sinica* **53**, 1629 (2004)
[6] D J Huang and H Q Zhang, *Acta Phys. Sinica* **53**, 2434 (2004)
[7] Y Chen and B Li, *Commun. Theor. Phys.* **41**, 1 (2004)
[8] B Li, Y Chen and H Q Zhang, *Appl. Math. Comput.* **146**, 653 (2003)
[9] Y Chen, B Li and H Q Zhang, *Commun. Theor. Phys.* **39**, 135 (2003)
[10] O P Layeni, *Appl. Math. Comput.* **215**, 2901 (2009)
[11] M Javidi and A Golbabai, *Comput. Math. Appl.* **54**, 1003 (2007)
[12] X L Yang and J S Tang, *Comput. Math. Appl.* **57**, 1622 (2009)
[13] Y D Shang, *Choas, Solitons and Fractals* **36**, 762 (2008)

[14] J B Li, *Choas, Solitons and Fractals* **34**, 867 (2007)
[15] Z Y Zhang, Y X Li, Z H Liu and X J Miao, *Commun. Nonlin. Sci. Numer. Simulat.* **16**, 3097 (2011)
[16] Z Y Zhang, Y H Zhang, X Y Gan and D M Yu, *Zeitschrift für Naturforschung* **A67**, 167 (2012)
[17] A Yıldırım and Z Pınar, *Comput. Math. Appl.* **60**, 1873 (2010)
[18] L Bergé, B Bidégaray and T Colin, *Phys.* **D95**, 351 (1996)
[19] R O Dendy, *Plasma dynamics* (Oxford University Press, Oxford, 1990)
[20] P M Bellan, *Fundamentals of plasmas physics* (Cambridge University Press, Cambridge, 2006)
[21] A Biswas and S Konar, *Introduction to non-Kerr law optical solitons* (CRC Press, Boca Raton, FL, USA, 2007)
[22] A Biswas, *Chaos, Solitons and Fractals* **13**, 815 (2002)
[23] A Biswas, *Appl. Math. Comput.* **153**, 369 (2004)
[24] A Biswas, *Appl. Math. Comput.* **151**, 41 (2004)
[25] Z Y Zhang, Z H Liu, X J Miao and Y Z Chen, *Appl. Math. Comput.* **216**, 3064 (2010)
[26] Z Y Zhang, Z H Liu, X J Miao and Y Z Chen, *Phys. Lett.* **A375**, 1275 (2011)
[27] X J Miao and Z Y Zhang, *Commun. Nonlin. Sci. Numer. Simulat.* **16**, 4259 (2011)
[28] Z Y Zhang, X Y Gan and D M Yu, *Zeitschrift für Naturforschung* **A66**, 721 (2011)
[29] S N Chow and J K Hale, *Method of bifurcation theory* (Springer-Verlag, New York, 1981)
[30] D H Feng, J L Lü, J B Li and T L He, *Appl. Math. Comput.* **189**, 271 (2007)
[31] Z S Feng, *Phys. Lett.* **A317**, 115 (2003)