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Exact solutions of Feinberg–Horodecki equation for time-dependent anharmonic oscillator

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Abstract. In this work, an alternative treatment known as Nikiforov–Uvarov (NU) method is proposed to find the exact solutions of the Feinberg–Horodecki equation for the time-dependent potentials. The present procedure is illustrated with two examples: (1) time-dependent Wei Hua oscillator, (2) time-dependent Manning–Rosen potential.

Keywords. Feinberg–Horodecki equation; Nikiforov–Uvarov (NU) method; exact solutions of time-dependent anharmonic oscillators.

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1. Introduction

Many of the physical phenomena of nature are characterized by some basic differential equations. For example, quantum-mechanical phenomena are described by time-independent Schrödinger's equation, which dictates the dynamics of some quantum systems. One is primarily interested in finding all eigenvalues and eigenstates of such quantum systems by solving them by different methods. But the time-dependent Schrödinger equations are not generally solved in closed forms. Only a few systems are analytically solved whose time-dependent potentials are constant, linear and quadratic functions of the coordinates [1–6].

In an interesting work, Molski [7] has demonstrated the possibility of describing the biological systems in terms of the time-like supersymmetric quantum mechanics [8] to include space-like quantum states, which are solutions of the space-like counterpart of the Schrödinger equation as

$$-\frac{\hbar^2}{2mc^2}\frac{d^2}{dt^2}\psi_n(t) + V(t)\psi_n(t) = cP_n\psi_n(t) \tag{1}$$

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derived by Horodecki [9] from the relativistic Feinberg equation [10] by non-relativistic approximation. Here, $V(t)$ denotes the vector potential, m is the mass of a particle, c is the velocity of light whereas P_n is the quantized momentum according to the quantum number $n = 0, 1, 2, 3....$. The bound states of eq. (1) have not been considered yet, as they are difficult to interpret in terms of temporal vibrational motion. However, in the case of anharmonic vector potential, there are no bound states in the dissociation limit and the direction of temporal motion is consistent with the arrow of time (is not of the oscillatory type). In such circumstances, the space-like solution of eq. (1) can be employed to test their relevance in different areas of science including physics, biology and medicine [11].

Recently, there is a renewed interest in solving simple quantum-mechanical systems within the framework of the Nikiforov–Uvarov (NU) method [12]. This algebraic technique is based on solving the second-order linear differential equations, which has been used successfully to solve Schrödinger, Dirac, Klein–Gordon and Duffin–Kemmer–Petiau wave equations in the presence of some well-known central and non-central potentials [13–23]. In the present work, we focus our attention to solve Feinberg–Horodecki (FH) equation by Nikiforov–Uvarov (NU) [12] method for systems whose potentials have a certain time-dependence like time-dependent Wei Hua oscillator and Manning–Rosen potential.

This paper is organized as follows: After a brief introductory discussion of the NU method in §2, we obtain the eigenvalues and eigenfunctions for time-dependent potentials in §3 and finally conclusions are drawn in §4.

2. Basic equations of Nikiforov–Uvarov method

The Nikiforov–Uvarov (NU) [12] method is based on reducing the second-order differential equation to a generalized equation of hypergeometric type. In this sense, the Schrödinger equation, after employing an appropriate coordinate transformation $s = s(r)$, transforms to the following form:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{2}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To find the particular solution of eq. (2), one can use the following transformation as

$$\psi_n(s) = \phi_n(s)y_n(s) \tag{3}$$

leading to a hypergeometric-type equation like

$$\sigma(s)y_n''(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0, \tag{4}$$

where

$$\sigma(s) = \pi(s)\frac{\phi_n(s)}{\phi_n'(s)}, \tag{5}$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \qquad \tau'(s) < 0. \tag{6}$$

The most significant point at this stage is that the prime factor of $\tau(s)$ shows the differentials at first degree and must be negative to reproduce physically acceptable λ-values which are defined as

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \quad n = 0, 1, 2, 3, \ldots . \tag{7}$$

It is to be noted that λ or λ_n is obtained from a particular solution of the form $y_n(s)$ which is a polynomial of degree n and satisfies the Rodrigues relation [24]

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{\mathrm{d}^n}{\mathrm{d}s^n}\big(\sigma^n(s)\rho(s)\big). \tag{8}$$

In this equation, B_n is the normalization constant and the weight function $\rho(s)$ satisfies the condition

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\sigma(s)\rho(s)\big) = \tau(s)\rho(s). \tag{9}$$

Here, the function $\pi(s)$ and the parameter λ are defined as

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}, \tag{10}$$

$$\lambda = \lambda_n = k + \pi'(s), \tag{11}$$

where $\pi(s)$ obviously is a polynomial depending on the transformation function $s(r)$ and the determination of k is the essential point in the calculation of $\pi(s)$, for which the discriminant of the square root in eq. (10) is set to zero so that, the expression of $\pi(s)$ becomes the square of a polynomial of first degree. In this case, an equation for k is obtained. After solving this equation of k, the obtained values of k are substituted in eq. (11) to find the values of λ. Then, by comparing eqs (7) and (11), one can obtain the values of λ_n.

It is well known that many special functions of mathematics represent solutions to differential equations of the form in eq. (2) where the function $\tilde{\tau}/\sigma$ and $\tilde{\sigma}/\sigma^2$ are well defined for any particular function [25]. Bearing this in mind, we proceed first with a transformation of Schrödinger equation to the one similar to eq. (2).

3. Nikiforov–Uvarov method for exact solutions of the Feinberg–Horodecki equation for the time-dependent potentials

Now we shall apply the NU method to find the exact solutions of Feinberg–Horodecki equation for the following time-dependent anharmonic oscillators.

3.1 *Wei Hua Oscillator*

The time-dependent Wei Hua oscillator [26] takes the form

$$V(t) = D\left(\frac{1 - \exp(-c_1(t - t_e))}{1 - a\exp(-c_1(t - t_e))}\right)^2,$$

which gives the description of anharmonic vibrations of diatomic molecules. Here D represents the dissociation energy of the system. The Feinberg–Horodecki eq. (1) takes the following form for this potential:

$$-\frac{\hbar^2}{2mc^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\psi_n(t) + D\left(\frac{1-\exp(-c_1(t-t_e))}{1-a\,\exp(-c_1(t-t_e))}\right)^2\psi_n(t) = cP_n\psi_n(t). \quad (12)$$

With the dimensionless coordinate $s = a\,\exp(-c_1(t-t_e))$, one can rewrite the more general quantal Feinberg–Horodecki eq. (1) for the time-dependent Wei Hua oscillator as

$$\frac{\mathrm{d}^2\psi_n(s)}{\mathrm{d}s^2} + \frac{(1-s)}{s(1-s)}\frac{\mathrm{d}\psi_n(s)}{\mathrm{d}s} + \frac{1}{s^2(1-s)^2}(-c_0^2 + c_2s^2 - c_3s)\psi_n(s) = 0, \quad (13)$$

where

$$-c_0^2 = 2mc^2\frac{(cP_n - D)}{\hbar^2c_1^2}, \quad (13a)$$

$$c_2 = 2mc^2\frac{(cP_n - D/a^2)}{\hbar^2c_1^2} \quad (13b)$$

and

$$c_3 = 2mc^2\frac{(2cP_n - 2D/a)}{\hbar^2c_1^2}. \quad (13c)$$

After comparing eq. (13) with eq. (2), we obtain

$$\tilde{\tau}(s) = 1-s, \quad \sigma(s) = s(1-s), \quad \tilde{\sigma}(s) = -c_0^2 - c_3s + c_2s^2. \quad (14)$$

Substituting these values into eq. (10), we obtain $\pi(s)$ as

$$\pi(s) = -\frac{s}{2} \pm \sqrt{s^2\left(\frac{1}{4} - c_2 - k\right) + (k+c_3)s + c_0^2}. \quad (15)$$

The discriminant of expression (15) under the square root has to be zero, so that the expression becomes the square of a polynomial of first degree. So, we write

$$s^2\left(\frac{1}{4} - c_2 - k\right) + (k+c_3)s + c_0^2 = 0 \quad (16)$$

Solving it we get

$$s = \frac{-(k+c_3) \pm \sqrt{(k+c_3)^2 - 4c_0^2(\frac{1}{4} - c_2 - k)}}{2(\frac{1}{4} - c_2 - k)}. \quad (17)$$

Then, we put for our purpose

$$(k+c_3)^2 - 4c_0^2\left(\frac{1}{4} - c_2 - k\right) = 0. \quad (18)$$

When the required arrangements are done with respect to the constant k, its values are derived as

$$k_{\pm} = -c_3 - 2c_0^2 \pm 2c_0\left(\frac{1}{A} - \frac{1}{2}\right), \tag{19}$$

where

$$\frac{1}{A} - \frac{1}{2} = \sqrt{\frac{2mc^2}{\hbar^2 c_1^2} D\left(\frac{1}{a} - 1\right)^2 + \frac{1}{4}}. \tag{20}$$

Substituting k_- into eq. (15), the following possible solution is obtained for $\pi(s)$:

$$\pi_-(s) = c_0(1-s) - \frac{s}{A}. \tag{21}$$

Now, we select $\pi_-(s)$ for which the function $\tau(s)$ in eq. (6) has a negative derivative. Therefore, the function $\tau(s)$ satisfies these requirements, with

$$\tau(s) = 1 - s - 2\left(c_0 + \frac{1}{A}\right)s + 2c_0. \tag{22}$$

Putting the values of $\tau_-'(s)$, $\sigma''(s)$, $\pi_-'(s)$ and k_- into eqs (7) and (11), we get

$$\lambda = \frac{2mc^2}{c_1^2\hbar^2} 2D\left(\frac{1}{a} - 1\right) - \frac{2c_0}{A} - \frac{1}{A} \tag{23}$$

and also

$$\lambda = \lambda_n = n\left(n + \frac{2}{A}\right) + 2nc_0. \tag{24}$$

From eqs (23) and (24), we get quantized momentum eigenvalue as

$$P_n = \frac{1}{c}\left(D - \frac{c_1^2\hbar^2}{2mc^2}\left(\frac{n(n+\frac{2}{A}) + \frac{1}{A} - \frac{2mc^2}{c_1^2\hbar^2}2D\left(\frac{1}{a}-1\right)}{2\left(n + \frac{1}{A}\right)}\right)^2\right). \tag{25}$$

Let us now find the corresponding eigenfunctions for this potential. Due to the NU method, the polynomial solutions of the hypergeometric function $y(s)$ depend on the determination of the weight function $\rho(s)$ which is obtained by solving eq. (9)

$$\rho(s) = s^{2c_0}(1-s)^{(2/A)-1}. \tag{26}$$

Substituting into the Rodrigues relation given in eq. (8), the eigenfunctions are obtained in the following form:

$$y_n(s) = B_n s^{-2c_0}(1-s)^{-((2/A)-1)} \frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(s^{n+2c_0}(1-s)^{n+(2/A)-1}\right), \tag{27}$$

where B_n is the normalization constant. The polynomial solutions of $y_n(s)$ in eq. (27) are expressed in terms of the Jacobi polynomials as

$$y_n(s) = B_n n! P_n^{2c_0,(2/A)-1}(1-2s). \tag{28}$$

By substituting $\pi_-(s)$ and $\sigma(s)$ into the eq. (5) and solving it we get

$$\phi_n(s) = s^{c_0}(1-s)^{1/A}. \tag{29}$$

Combining the Jacobi polynomials $y_n(s)$ and $\phi_n(s)$, we get wave functions as

$$\psi_n(s) = A_n s^{c_0}(1-s)^{(1/A)} P_n^{2c_0,(2/A)-1}(1-2s), \tag{30}$$

where A_n is the normalization constant.

In the ground state ($n = 0$), we have $c_0 = 0$, and hence the quantal function (30) reduces to the West, Brown and Enquist (WBE) [7,27] function of growth

$$\psi_n(t) = (1 - a\exp(-c_1(t-t_e))^{1/A} = (1 - \tilde{c}_3\exp(-c_1 t))^{1/A} = y(t) \tag{31}$$

in which

$$t_e = \frac{1}{c_1}\ln\left(\frac{\tilde{c}_3(2-A)}{A}\right), \qquad a = \frac{A}{2-A}$$

and t_e is an equilibrium time point at which the potential attains a minimum equal to $V(t = t_e) = 0$ and

$$y(t) = \frac{m(t)}{M}, \quad \tilde{c}_3 = 1 - \left(\frac{m_0}{M}\right)^A, \qquad c_1 = \frac{n_1}{4M^A}, \quad A = 1 - p. \tag{32}$$

Here, $m_0 = m(t = 0)$ is the initial mass of the system whereas $M = m(t \to \infty) = (n_1/n_2)^{1/A}$ is the maximum body size reached. The WBE [27] function (31) fits the data very well for a variety of different species from protozoa to mammalians, and parameters fitted can be related to the biological characteristics of the system under consideration. The solution $\psi(t)$ or $y(t)$ is the solution of the first-order equation [27]

$$\frac{\mathrm{d}m}{\mathrm{d}t} = n_1 m^p - n_2 m, \quad p = \frac{3}{4} \tag{33}$$

which has been derived on the basis of the conservation of metabolic energy, the allometric scaling of metabolic rate, and energetic costs of producing and maintaining biomass. Here, $n_1 = B_0 m_c/E_c$ and $n_2 = B_c/E_c$, where B_0 is the initial ($t = 0$) average resting metabolic rate of the whole organism, B_c is the metabolic rate of a single cell, E_c is the metabolic energy required to form a cell and m_c is the mass of a cell.

3.2 *Time-dependent Manning–Rosen potential*

The time-dependent Manning–Rosen potential [28] is given by

$$V(t) = c_1^2\left(\frac{\alpha(\alpha-1)\exp(-2c_1(t-t_e))}{(1-\exp(-c_1(t-t_e)))^2} - V_0\frac{\exp(-c_1(t-t_e))}{1-\exp(-c_1(t-t_e))}\right). \tag{34}$$

With the dimensionless coordinate $\tau = c_1(t - t_e)$, one can rewrite the more general quantal Feinberg–Horodecki eq. (1) for the time-dependent Manning–Rosen potential as

$$\frac{\mathrm{d}^2\psi_n(\tau)}{\mathrm{d}\tau^2} + \frac{2mc^2}{\hbar^2 c_1^2}\left(-\alpha(\alpha-1)\frac{\mathrm{e}^{-2\tau}}{(1-\mathrm{e}^{-\tau})^2} + \beta\frac{\mathrm{e}^{-\tau}}{1-\mathrm{e}^{-\tau}} - \epsilon^2\right)\psi_n(\tau) = 0 \tag{35}$$

in which P_n is the quantized momentum eigenvalue and we have introduced the following dimensional parameters:

$$\gamma = \frac{2mc^2}{\hbar^2}, \quad \beta = \frac{2mc^2 V_0}{\hbar^2}, \quad \epsilon^2 = -\frac{\gamma c P_n}{c_1^2}.$$

In order to apply the NU method, we rewrite eq. (35) by using a new variable of the form $s = \mathrm{e}^{-\tau}$ like

$$\frac{\mathrm{d}^2\psi_n(s)}{\mathrm{d}s^2} + \frac{1-s}{s(1-s)}\frac{\mathrm{d}\psi_n(s)}{\mathrm{d}s} + \frac{1}{s^2(1-s)^2}(-\epsilon^2 + (\beta + 2\epsilon^2)s - (\beta + \epsilon^2 + \gamma\alpha(\alpha-1))s^2)\psi_n(s) = 0 \quad (36)$$

which leads to a hypergeometric-type equation. After comparing eq. (36) with eq. (1), we obtain the corresponding polynomials as

$$\tilde{\tau}(s) = 1 - s, \quad \sigma(s) = s(1-s),$$
$$\tilde{\sigma}(s) = -\epsilon^2 + (\beta + 2\epsilon^2)s - (\beta + \epsilon^2 + \gamma\alpha(\alpha-1))s^2. \quad (37)$$

Substituting these polynomials into eq. (10), we obtain π function as

$$\pi(s) = -\frac{s}{2} \pm \sqrt{s^2\left(\frac{1}{4} + \beta + \epsilon^2 + \gamma\alpha(\alpha-1) - k\right) + s(k - \beta - 2\epsilon^2) + \epsilon^2}. \quad (38)$$

The discriminant of expression (38) under the square root has to be zero, i.e.,

$$(k - \beta - 2\epsilon^2)^2 - 4\epsilon^2\left(\frac{1}{4} + \beta + \epsilon^2 + \gamma\alpha(\alpha-1) - k\right) = 0. \quad (39)$$

When the required arrangements are done with respect to the constant k, its double roots are derived as

$$k_{\pm} = \beta \pm \epsilon\sqrt{1 + 4\gamma\alpha(\alpha-1)}. \quad (40)$$

We take $k_- = \beta - \epsilon\Delta$, where $\Delta = \sqrt{1 + 4\gamma\alpha(\alpha-1)}$. Substituting k_- into eq. (38), the following possible solution is obtained for $\pi(s)$:

$$\pi_-(s) = -\frac{s}{2} - \left(\epsilon + \frac{\Delta}{2}\right)s + \epsilon. \quad (41)$$

We select the polynomial $\pi_-(s)$ for which the function $\tau(s)$ in eq. (6) has a negative derivative. Therefore, the function $\tau(s)$ satisfies these requirements, with

$$\tau(s) = 1 - 2s - 2\left(\epsilon + \frac{\Delta}{2}\right)s + 2\epsilon, \qquad \tau'(s) = -2 - 2\left(\epsilon + \frac{\Delta}{2}\right). \quad (42)$$

It is clearly seen that the eigenvalues are found by comparing eqs (7) and (11). So, from eqs (7) and (11), we get

$$\lambda = \beta - (1 + \Delta)\left(\epsilon + \frac{1}{2}\right) \quad (43)$$

and also

$$\lambda_n = n^2 + n + 2n\left(\epsilon + \frac{\Delta}{2}\right). \tag{44}$$

From eqs (43) and (44), we get quantized momentum eigenvalue as

$$P_n = -\frac{\hbar^2 c_1^2}{2mc^3}\left(\frac{(\beta - \frac{1}{4}(1-\Delta^2)) - (n + \frac{1}{2}(1+\Delta))^2}{2(n+\frac{1}{2}(1+\Delta))}\right)^2, \quad n = 0, 1, 2, 3\ldots . \tag{45}$$

Let us now find the corresponding eigenfunctions for this potential. Due to the NU method, the polynomial solutions of the hypergeometric function $y(s)$ depend on the determination of the weight function $\rho(s)$ which satisfies the differential equation (9). Solving, we get weight function $\rho(s)$ as

$$\rho(s) = s^{2\epsilon}(1-s)^{\Delta}. \tag{46}$$

Substituting into the Rodrigues relation given in eq. (8), the eigenfunctions are obtained in the following form:

$$y_n(s) = B_n n! s^{-2\epsilon}(1-s)^{-\Delta}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(s^{n+2\epsilon}(1-s)^{n+\Delta}\right), \tag{47}$$

where B_n is the normalization constant. The polynomial solutions of $y_n(s)$ in eq. (47) are expressed in terms of the Jacobi polynomials, which is one of the Jacobi polynomials, that is

$$y_n(s) = B_n n! P_n^{2\epsilon,\Delta}(1-2s). \tag{48}$$

By substituting $\pi_-(s)$ and $\sigma(s)$ into eq. (5) and solving it, we obtain

$$\phi_n(s) = s^{\epsilon}(1-s)^{(1+\Delta)/2}. \tag{49}$$

Combining the Jacobi polynomials $y_n(s)$ and $\phi_n(s)$, we get wave functions as

$$\psi_n(s) = A_n s^{\epsilon}(1-s)^{(1+\Delta)/2} P_n^{2\epsilon,\Delta}(1-2s), \tag{50}$$

where A_n is a normalization constant. More explicitly, one can write eq. (50) as

$$\begin{aligned}\psi_n(t) &= A_n \exp(-\epsilon c_1(t-t_e)(1-\exp(-c_1(t-t_e))^{(1+\Delta)/2} P_n^{2\epsilon,\Delta} \\ &\quad \times (1 - 2\ \exp(-c_1(t-t_e))).\end{aligned} \tag{51}$$

Now, if we take $\alpha = 0$ and/or $\alpha = 1$, and hence $\Delta \to 1$, we shall obtain the results of the time-dependent Hulthen potential.

4. Conclusions

This paper has presented a different approach, the Nikiforov–Uvarov method, to solve the Feinberg–Horodecki equation. Exact eigenvalues and eigenfunctions for the Feinberg–Horodecki equation in the presence of time-dependent Wei Hua oscillator and Manning–Rosen potential are derived. We conclude by noting that the NU method is an elegant and powerful technique and if the problem is analytically solved, it provides closed forms for the energy eigenvalues as well as the corresponding eigenfunctions. Here, the formalism systematically recovers known results in a natural way and allows one to extend certain results in particular cases.

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