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Solutions of several coupled discrete models in terms of Lamé polynomials of arbitrary order

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Abstract. Coupled discrete models are ubiquitous in a variety of physical contexts. We provide an extensive set of exact quasiperiodic solutions of a number of coupled discrete models in terms of Lamé polynomials of arbitrary order. The models discussed are: (i) coupled Salerno model, (ii) coupled Ablowitz–Ladik model, (iii) coupled ϕ^4 model and (iv) coupled ϕ^6 model. In all these cases we show that the coefficients of the Lamé polynomials are such that the Lamé polynomials can be re-expressed in terms of Chebyshev polynomials of the relevant Jacobi elliptic function.

Keywords. Solitons; Jacobi elliptic functions; field theories; phase transitions.

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1. Introduction

In a previous paper [1] we have obtained solutions of a number of coupled discrete models in terms of Lamé polynomials of order one and two. The purpose of the present paper is to show that in the same models one can in fact obtain solutions in terms of Lamé polynomials [2] (and hence trigonometric and hyperbolic polynomials) of arbitrary order. In particular, we obtain solutions of (i) coupled Salerno model, (ii) coupled Ablowitz–Ladik model, (iii) coupled ϕ^6 model and (iv) coupled ϕ^4 model, in terms of Lamé (and hence trigonometric and hyperbolic) polynomials of arbitrary order. As an illustration, we confine our discussion to coupled Salerno model and then show how similar solutions also exist in coupled Ablowitz–Ladik (AL), coupled ϕ^4 and coupled ϕ^6 models. Quite remarkably, we find that the coefficients of the Lamé polynomials are such that the Lamé polynomials can be re-expressed as Chebyshev polynomials [2] of the relevant Jacobi elliptic function.

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As indicated in [1], the motivation for this work comes from the fact that there are many physical situations where a discrete field theory is appropriate to model the phenomena of interest with a specific coupling between the two fields. An example of current intense interest is the coexistence of magnetism and ferroelectricity (i.e., magnetoelectricity) in multiferroic materials [3,4]. Some of the multiferroics can be modelled by a coupled ϕ^4 model [5] in the presence of a magnetic field. Coupled ϕ^4 models also arise in the context of many ferroelectric and other second-order phase transitions with bilinear, biquadratic or other couplings. There are examples of coupled discrete AL, coupled discrete Salerno and coupled saturated discrete nonlinear Schrödinger (DNLS) models known in the literature [6–8]. Analogous coupled models also exist in field theory [9,10]. Several related models have been discussed in the literature and their soliton solutions have been found [11–18] including periodic ones [19–21].

The paper is organized as follows: In §2 we provide a family of solutions for the coupled Salerno model in terms of Lamé polynomials of order three and four. Based on these results as well as those obtained in [1], in §3 we generalize these results and propose solutions in terms of Lamé (and hence trigonometric and hyperbolic) polynomials of arbitrary order. In §4 we show that these Lamé polynomials can be re-expressed as Chebyshev polynomials of the relevant Jacobi elliptic function. This also proves that our proposed Lamé polynomials of arbitrary order are indeed solutions of the coupled equations. In §5 we show how the coupled Ablowitz–Ladik, coupled discrete ϕ^6 [22], and coupled ϕ^4 [23] models also admit solutions in terms of Lamé polynomials of arbitrary order. Section 6 contains the summary of main results.

2. Coupled Salerno model

As discussed in [1], the field equations of the coupled Salerno model are given by

$$\frac{i\mathrm{d}u_n}{\mathrm{d}t} + [u_{n+1} + u_{n-1} - 2u_n] + (\mu_1|u_n|^2 + \mu_2|v_n|^2) \times \left[u_{n+1} + u_{n-1} + \frac{\nu_1 - 2\mu_1}{\mu_1}u_n\right] = 0, \tag{1}$$

$$\frac{i\mathrm{d}v_n}{\mathrm{d}t} + \left[v_{n+1} + v_{n-1} - \left(2 + \frac{\nu_1\mu_2}{\mu_1^2} - \frac{\nu_2}{\mu_2}\right)v_n\right] + (\mu_1|u_n|^2 + \mu_2|v_n|^2)\left[v_{n+1} + v_{n-1} + \frac{\nu_2 - 2\mu_2}{\mu_2}v_n\right] = 0. \tag{2}$$

Based on various properties of Jacobi elliptic functions and their relation to nonlinear soliton-bearing equations, in [1] we have already obtained solutions of these coupled equations in terms of Lamé polynomials of order one and two. We now show that the same model also admits Lamé polynomial solutions of arbitrary order.

As in [1], we start with the ansatz

$$u_n = f_n \exp[-i(\omega_1 t + \delta_1)], \quad v_n = g_n \exp[-i(\omega_2 t + \delta_2)], \tag{3}$$

then it is easily shown that the above coupled equations take the form

$$(\omega_1 - 2) f_n + (f_{n+1} + f_{n-1})(\mu_1 f_n^2 + \mu_2 g_n^2 + 1) + (\mu_1 f_n^2 + \mu_2 g_n^2)\frac{(\nu_1 - 2\mu_1)}{\mu_1} f_n = 0, \tag{4}$$

$$\left(\omega_2 - 2 + \frac{\nu_1\mu_2}{\mu_1^2} - \frac{\nu_2}{\mu_2}\right) g_n + (\mu_1 f_n^2 + \mu_2 g_n^2 + 1)(g_{n+1} + g_{n-1}) + (\mu_1 f_n^2 + \mu_2 g_n^2)\frac{(\nu_2 - 2\mu_2)}{\mu_2} g_n = 0. \tag{5}$$

It is clear from eqs (4) and (5) that in general these coupled equations will have exact solutions if

$$1 + \mu_1 f_n^2 + \mu_2 g_n^2 = 0, \tag{6}$$

and in that case

$$\omega_1 = \frac{\nu_1}{\mu_1}, \quad \omega_2 = \frac{\nu_1\mu_2}{\mu_1^2}. \tag{7}$$

We now show that eq. (6) has solutions in terms of Lamé polynomials of arbitrary order. In particular, we show that at every order there are three distinct solutions satisfying eq. (6). As a first step, let us explicitly obtain solutions of eq. (6) in terms of Lamé polynomials of order three and four. (It may be noted that we have already obtained such solutions before in terms of Lamé polynomials of order one and two [1].)

2.1 *Lamé polynomial solutions of order three*

It is easily checked that one of the exact solution to eq. (6) is

$$f_n = \mathrm{dn}[\beta(n + c_2), m](A\mathrm{dn}^2[\beta(n + c_2), m] + B),$$
$$g_n = \sqrt{m}\mathrm{sn}[\beta(n + c_2), m](C\mathrm{dn}^2[\beta(n + c_2), m] + D), \tag{8}$$

provided

$$\mu_1, \mu_2 < 0, \quad |\mu_1|A^2 = |\mu_2|C^2, \quad |\mu_2|D^2 = 1,$$
$$|\mu_1|(A + B)^2 = 1, \quad |\mu_1|(A^2 + 2AB) = 2CD|\mu_2|. \tag{9}$$

Here c_2 is an arbitrary constant signifying discrete translation invariance. Also, notice that for this solution, the (inverse) width β is also completely arbitrary. We shall see that all the solutions discussed in this paper are valid for arbitrary c_2 and width β. On solving, we find that

$$\sqrt{|\mu_1|}A = 4, \quad B = -\frac{3}{4}A, \quad \sqrt{|\mu_2|}C = 4, \quad C = -4D. \tag{10}$$

Since the field eqs (4) and (5) are invariant under $f_n \to \pm f_n$, $g_n \to \pm g_n$, one can trivially write three other solutions from here. Note that β is completely arbitrary. Using the fact that $\mathrm{dn}(x, m)$ has period $2K(m)$ while $\mathrm{cn}(x, m)$ and $\mathrm{sn}(x, m)$ are periodic functions with period $4K(m)$, it follows that for the solution (8), f_n, g_n and hence u_n, v_n satisfy the boundary condition

$$u_{n+(2K(m)/\beta)} = u_n, \quad v_{n+(4K(m)/\beta)} = v_n. \tag{11}$$

Here $K(m)$ is the complete elliptic integral of the first kind.

Another solution to eq. (6) is

$$\begin{aligned} f_n &= \mathrm{cn}[\beta(n+c_2), m](A\mathrm{cn}^2[\beta(n+c_2), m] + B), \\ g_n &= \mathrm{sn}[\beta(n+c_2), m](C\mathrm{cn}^2[\beta(n+c_2), m] + D), \end{aligned} \tag{12}$$

provided A, B, C, D are again given by eq. (10). Note however that the solution (12) is distinct from the solution (8). In particular, while for the solution (8), u_n, v_n satisfy the boundary condition (11), for the solution (12), f_n, g_n and hence u_n, v_n satisfy the boundary condition

$$u_{n+(4K(m)/\beta)} = u_n, \quad v_{n+(4K(m)/\beta)} = v_n. \tag{13}$$

However, in the limit $m = 1$, both the solutions (8) and (12) go over to the hyperbolic solution

$$\begin{aligned} f_n &= A\,\mathrm{sech}^3[\beta(n+c_2)] + B\,\mathrm{sech}[\beta(n+c_2)], \\ g_n &= \tanh[\beta(n+c_2)](C\,\mathrm{sech}^2[\beta(n+c_2)] + D). \end{aligned} \tag{14}$$

Further, in the limit $m = 0$, the solution (12) goes over to the trigonometric solution

$$\begin{aligned} f_n &= \cos[\beta(n+c_2)](A\cos^2[\beta(n+c_2)] + B), \\ g_n &= \sin[\beta(n+c_2)](C\cos^2[\beta(n+c_2)] + D). \end{aligned} \tag{15}$$

All the four solutions discussed above are valid only if $\mu_1, \mu_2 < 0$. If, instead μ_1, μ_2 have opposite signs, then also one has solutions to eq. (6). One such solution is

$$\begin{aligned} f_n &= \frac{1}{\mathrm{dn}[\beta(n+c_2), m]}\left(\frac{A}{\mathrm{dn}^2[\beta(n+c_2), m]} + B\right), \\ g_n &= \frac{\sqrt{m}\,\mathrm{sn}[\beta(n+c_2), m]}{\mathrm{dn}[\beta(n+c_2), m]}\left(\frac{C}{\mathrm{dn}^2[\beta(n+c_2), m]} + D\right), \end{aligned} \tag{16}$$

provided $\mu_1 < 0, \mu_2 > 0$ while A, B, C, D are still given by eq. (10). Note that for this solution f_n, g_n and hence u_n, v_n satisfy the boundary condition (11). Further, if we interchange f_n and g_n, then $\mu_1 > 0, \mu_2 < 0$. In the limit $m = 1$, this solution goes over to the hyperbolic solution

$$\begin{aligned} f_n &= A\cosh^3[\beta(n+c_2)] + B\cosh[\beta(n+c_2)], \\ g_n &= \sinh[\beta(n+c_2)](C\cosh^2[\beta(n+c_2)] + D)\,. \end{aligned} \tag{17}$$

While obtaining these solutions and the ones given below, we have made use of several identities for the Jacobi elliptic functions [24].

2.2 *Lamé polynomial solutions of order four*

It is easily checked that one of the exact solution to eq. (6) is

$$f_n = A\mathrm{dn}^4[\beta(n+c_2),m] + B\mathrm{dn}^2[\beta(n+c_2),m] + C,$$
$$g_n = \sqrt{m}\,\mathrm{sn}[\beta(n+c_2),m]\mathrm{dn}[\beta(n+c_2),m](D\,\mathrm{dn}^2[\beta(n+c_2),m]+E)\,, \tag{18}$$

provided

$$\mu_1,\mu_2<0\,,\quad |\mu_1|A^2=|\mu_2|D^2\,,\quad |\mu_1|C^2=1\,,$$
$$A(A+2B)|\mu_1| = 2DE|\mu_2|\,,$$
$$|\mu_1|(A+B)^2+2AC|\mu_1| = |\mu_2|E^2,$$
$$2BC|\mu_1| = -|\mu_2|E^2\,. \tag{19}$$

On solving, we find that

$$\sqrt{|\mu_1|}C=1,\quad A=-B=8C,\quad \sqrt{|\mu_2|}E=-4,\quad D=-2E. \tag{20}$$

For the solution (18), f_n, g_n and hence u_n, v_n satisfy the boundary condition (11).

Another solution to eq. (6) is

$$f_n = A\,\mathrm{cn}^4[\beta(n+c_2),m] + B\,\mathrm{cn}^2[\beta(n+c_2),m] + C,$$
$$g_n = \mathrm{sn}[\beta(n+c_2),m]\,\mathrm{cn}[\beta(n+c_2),m](D\,\mathrm{cn}^2[\beta(n+c_2),m]+E), \tag{21}$$

provided A, B, C, D, E are again given by eq. (20). Note however that the solution (21) is distinct from the solution (18). In particular, while for the solution (18), u_n, v_n satisfy the boundary condition (11), for the solution (21), f_n, g_n and hence u_n, v_n satisfy the boundary condition (13).

However, in the limit $m=1$, both the solutions (18) and (21) go over to the hyperbolic solution

$$f_n = A\,\mathrm{sech}^4[\beta(n+c_2)] + B\,\mathrm{sech}^2[\beta(n+c_2)] + C,$$
$$g_n = \tanh[\beta(n+c_2)]\,\mathrm{sech}[\beta(n+c_2)](D\,\mathrm{sech}^2[\beta(n+c_2)]+E)\,. \tag{22}$$

Further, in the limit $m=0$, the solution (21) goes over to the trigonometric solution

$$f_n = A\cos^4[\beta(n+c_2)] + B\cos^2[\beta(n+c_2)] + C,$$
$$g_n = \sin[\beta(n+c_2)]\cos[\beta(n+c_2)](D\cos^2[\beta(n+c_2)]+E)\,. \tag{23}$$

All the four solutions discussed above are valid only if $\mu_1,\mu_2<0$. If instead μ_1,μ_2 have opposite signs then also one has solutions to eq. (6). One such solution is

$$f_n = \frac{A}{\mathrm{dn}^4[\beta(n+c_2),m]} + \frac{B}{\mathrm{dn}^2[\beta(n+c_2),m]} + C,$$
$$g_n = \sqrt{m}\,\mathrm{sn}[\beta(n+c_2),m]\left(\frac{D}{\mathrm{dn}^4[\beta(n+c_2),m]} + \frac{E}{\mathrm{dn}^2[\beta(n+c_2),m]}\right), \tag{24}$$

provided $\mu_1 < 0, \mu_2 > 0$ while A, B, C, D, E are still given by eq. (20). For this solution, f_n, g_n and hence u_n, v_n satisfy the boundary condition (11). Note that if we interchange f_n and g_n, then $\mu_1 > 0, \mu_2 < 0$. In the limit $m = 1$, this solution goes over to the hyperbolic solution

$$f_n = A\cosh^4[\beta(n+c_2)] + B\cosh^2[\beta(n+c_2)] + C,$$
$$g_n = \sinh[\beta(n+c_2)](D\cosh^3[\beta(n+c_2)] + E\cosh[\beta(n+c_2)])\,. \tag{25}$$

3. General results

Looking at the structure of the solutions in terms of Lamé polynomials of order one to four, it is easy to generalize and write down the solutions of eq. (6) in terms of Lamé polynomials of arbitrary order. For this, we need to divide the discussion into two parts depending on whether we are considering Lamé polynomials of odd or even order.

Case I: *Lamé polynomials of odd order*

One of the solutions can be written in the form (note n is an odd integer)

$$f_n = \sum_{k=1}^{(n+1)/2} A_k(\mathrm{dn}[\beta(n+c_2), m])^{2k-1},$$
$$g_n = \sqrt{m}\,\mathrm{sn}[\beta(n+c_2), m] \sum_{k=1}^{(n+1)/2} B_k(\mathrm{dn}[\beta(n+c_2), m])^{2k-2}. \tag{26}$$

For this solution, f_n, g_n and hence u_n, v_n satisfy the boundary condition (11). Note that there are $(n+1)/2$ number of terms in both f_n and g_n.

Another solution is given by

$$f_n = \sum_{k=1}^{(n+1)/2} A_k(\mathrm{cn}[\beta(n+c_2), m])^{2k-1},$$
$$g_n = \mathrm{sn}[\beta(n+c_2), m] \sum_{k=1}^{(n+1)/2} B_k(\mathrm{cn}[\beta(n+c_2), m])^{2k-2}. \tag{27}$$

For this solution, f_n, g_n and hence u_n, v_n satisfy the boundary condition (13).

In the limit $m = 1$, both the solutions (26) and (27) go over to the hyperbolic solution

$$f_n = \sum_{k=1}^{(n+1)/2} A_k(\mathrm{sech}[\beta(n+c_2)])^{2k-1},$$
$$g_n = \tanh[\beta(n+c_2)] \sum_{k=1}^{(n+1)/2} B_k(\mathrm{sech}[\beta(n+c_2)])^{2k-2}. \tag{28}$$

Further, in the limit $m = 0$, the solution (27) goes over to the trigonometric solution

$$f_n = \sum_{k=1}^{(n+1)/2} A_k (\cos[\beta(n+c_2)])^{2k-1},$$
$$g_n = \sin[\beta(n+c_2)] \sum_{k=1}^{(n+1)/2} B_k (\cos[\beta(n+c_2)])^{2k-2}. \quad (29)$$

Note that these four solutions are valid when $\mu_1, \mu_2 < 0$. However, if μ_1 and μ_2 have opposite signs, say $\mu_1 < 0$, $\mu_2 > 0$, then the solution is given by

$$f_n = \sum_{k=1}^{(n+1)/2} \frac{A_k}{(\mathrm{dn}[\beta(n+c_2), m])^{2k-1}},$$
$$g_n = \sqrt{m}\,\mathrm{sn}[\beta(n+c_2), m] \sum_{k=1}^{(n+1)/2} \frac{B_k}{(\mathrm{dn}[\beta(n+c_2), m])^{2k-1}}. \quad (30)$$

For this solution, f_n, g_n and hence u_n, v_n satisfy the boundary condition (11). In the limit $m = 1$, this solution goes over to the hyperbolic solution

$$f_n = \sum_{k=1}^{(n+1)/2} A_k (\cosh[\beta(n+c_2)])^{2k-1},$$
$$g_n = \tanh[\beta(n+c_2)] \sum_{k=1}^{(n+1)/2} B_k (\cosh[\beta(n+c_2)])^{2k-1}. \quad (31)$$

On substituting any of the expressions for f_n, g_n given by eqs (26) to (31) in eq. (6), we obtain $n+1$ equations which determine the $n+1$ parameters A_k, B_k. While all A_k's are numbers in units of $1/\sqrt{|\mu_1|}$, all B_k's are numbers in units of $1/\sqrt{|\mu_2|}$. For simplicity, from now onwards we shall merely give the numerical values of A_k, B_k and it is understood that they are in units of $1/\sqrt{|\mu_1|}$ and $1/\sqrt{|\mu_2|}$, respectively. Some of the relations are

$$A^2_{(n+1)/2} = B^2_{(n+1)/2},$$
$$2A_{(n+1)/2}A_{(n-1)/2} + B^2_{(n+1)/2} = 2B_{(n+1)/2}B_{(n-1)/2},$$
$$A^2_{(n-1)/2} + 2A_{(n+1)/2}A_{(n-3)/2} + 2B_{(n+1)/2}B_{(n-1)/2}$$
$$= B^2_{(n-1)/2} + 2B_{(n+1)/2}B_{(n-3)/2},$$
$$B^2_{(n-1)/2} + 2A_{(n+1)/2}A_{(n-5)/2} + 2B_{(n+1)/2}B_{(n-3)/2} + 2A_{(n-1)/2}A_{(n-3)/2}$$
$$= 2B_{(n-1)/2}B_{(n-3)/2} + 2B_{(n+1)/2}B_{(n-5)/2},$$
$$B_1^2 = 1, \quad A_1^2 + 2B_1B_2 = B_1^2, \quad 2A_1A_2 + B_2^2 + 2B_1B_3 = 2B_1B_2,$$
$$A_2^2 + 2A_1A_3 - B_2^2 + 2B_2B_3 + 2B_1B_4 - 2B_1B_3 = 0. \quad (32)$$

On comparing these results with the exact expressions for $n = 1, 3$, we obtain the following general results for arbitrary odd n:

$$
\begin{aligned}
&A_{(n+1)/2} = B_{(n+1)/2} = 2^{n-1}, \quad A_{(n-1)/2} = -n2^{n-3},\\
&B_{(n-1)/2} = -(n-2)2^{n-3},\\
&A_{(n-3)/2} = \frac{n(n-3)}{2!}2^{n-5}, \quad B_{(n-3)/2} = \frac{(n-3)(n-4)}{2!}2^{n-5},\\
&A_{(n-5)/2} = -\frac{n(n-4)(n-5)}{3!}2^{n-7},\\
&B_{(n-5)/2} = -\frac{(n-4)(n-5)(n-6)}{3!}2^{n-7},\\
&\sum_{i=1}^{(n+1)/2} A_i = 1, \quad \sum_{i=1}^{(n+1)/2} B_i = n.
\end{aligned} \tag{33}
$$

Further, depending on whether $n = 4k + 3$ or $4k + 1$ we have the following results:

$$A_1 = -n, \quad B_1 = -1, \quad \text{if} \quad n = 4k+3, \quad k = 0, 1, 2, ... \tag{34}$$

$$A_1 = +n, \quad B_1 = +1, \quad \text{if} \quad n = 4k+5, \quad k = 0, 1, 2, ... \tag{35}$$

In fact, looking at these general results, we obtain the following expressions for various A_i, B_i:

$$
\begin{aligned}
A_{\frac{n-(2k-1)}{2}} &= (-1)^k \frac{n(n-k-1)(n-k-2)\cdots(n-2k+1)}{k!}2^{n-2k-1},\\
k &= 1, 2, \ldots, \frac{n-1}{2},
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
B_{\frac{n-(2k-1)}{2}} &= (-1)^k \frac{(n-k-1)(n-k-2)\cdots(n-2k)}{k!}2^{n-2k-1},\\
k &= 0, 1, 2, \ldots, \frac{n-1}{2}.
\end{aligned} \tag{37}
$$

It can be verified that for $n = 1, 3$, A_i and B_i which follow from eqs (33)–(35) agree with the values obtained by us above as well as in [1]. Further, for low values of (odd) n, it can be numerically checked that A_i and B_i which follow from eqs (36) and (37) indeed satisfy the various relations which follow by demanding the validity of the constraint relation (6).

In the next section, we shall prove that the above identities hold for all n. But first, using eqs (33), (36) and (37), we predict that the following identities hold for all odd n:

$$2^{n-1} + \sum_{k=1}^{(n-1)/2} (-1)^k \frac{n(n-k-1)(n-k-2)\cdots(n-2k+1)}{k!}2^{n-2k-1} = 1. \tag{38}$$

$$\sum_{k=0}^{(n-1)/2} (-1)^k \frac{(n-k-1)(n-k-2)\cdots(n-2k)}{k!}2^{n-2k-1} = n. \tag{39}$$

We shall see that these identities naturally lead to the proof given in §4.

Case II: *Lamé polynomials of even order*

In this case there are $(n/2)+1$ number of terms in f_n and $n/2$ number of terms in g_n. Here, one of the solutions can be written in the form (note n is an even integer)

$$
\begin{aligned}
f_n &= \sum_{k=1}^{(n/2)+1} A_k (\mathrm{dn}[\beta(n+c_2), m])^{2(k-1)}, \\
g_n &= \sqrt{m}\,\mathrm{sn}[\beta(n+c_2), m]\,\mathrm{dn}[\beta(n+c_2), m] \\
&\quad \times \sum_{k=1}^{n/2} B_k (\mathrm{dn}[\beta(n+c_2), m])^{2(k-1)}.
\end{aligned} \tag{40}
$$

For this solution, f_n, g_n and hence u_n, v_n satisfy the boundary condition (11).

Another solution is given by

$$
\begin{aligned}
f_n &= \sum_{k=1}^{(n/2)+1} A_k (\mathrm{cn}[\beta(n+c_2), m])^{2(k-1)}, \\
g_n &= \mathrm{sn}[\beta(n+c_2), m]\,\mathrm{cn}[\beta(n+c_2), m] \\
&\quad \times \sum_{k=1}^{n/2} B_k (\mathrm{cn}[\beta(n+c_2), m])^{2(k-1)}.
\end{aligned} \tag{41}
$$

For this solution, f_n, g_n and hence u_n, v_n satisfy the boundary condition (13).

In the limit $m = 1$, both the solutions (40) and (41) go over to the hyperbolic solution

$$
\begin{aligned}
f_n &= \sum_{k=1}^{(n/2)+1} A_k (\mathrm{sech}[\beta(n+c_2)])^{2(k-1)}, \\
g_n &= \tanh[\beta(n+c_2)]\,\mathrm{sech}[\beta(n+c_2)] \\
&\quad \times \sum_{k=1}^{n/2} B_k (\mathrm{sech}[\beta(n+c_2)])^{2(k-1)}.
\end{aligned} \tag{42}
$$

Further, in the limit $m = 0$, the solution (41) goes over to the trigonometric solution

$$
\begin{aligned}
f_n &= \sum_{k=1}^{(n/2)+1} A_k (\cos[\beta(n+c_2)])^{2(k-1)}, \\
g_n &= \sin[\beta(n+c_2)]\cos[\beta(n+c_2)] \sum_{k=1}^{n/2} B_k (\cos[\beta(n+c_2)])^{2(k-1)}.
\end{aligned} \tag{43}
$$

Note that these four solutions are valid when $\mu_1, \mu_2 < 0$. However, if μ_1 and μ_2 have opposite signs, say $\mu_1 < 0$, $\mu_2 > 0$, then the solution is given by

$$
\begin{aligned}
f_n &= \sum_{k=1}^{(n/2)+1} \frac{A_k}{(\mathrm{dn}[\beta(n+c_2), m])^{2(k-1)}}, \\
g_n &= \sqrt{m}\,\mathrm{sn}[\beta(n+c_2), m] \sum_{k=2}^{(n/2)+1} \frac{B_k}{(\mathrm{dn}[\beta(n+c_2), m])^{2(k-1)}}.
\end{aligned} \tag{44}
$$

For this solution, f_n, g_n and hence u_n, v_n satisfy the boundary condition (11). In the limit $m = 1$, this solution goes over to the hyperbolic solution

$$f_n = \sum_{k=1}^{(n/2)+1} A_k(\cosh[\beta(n+c_2)])^{2(k-1)},$$
$$g_n = \tanh[\beta(n+c_2)] \sum_{k=2}^{(n/2)+1} B_k(\cosh[\beta(n+c_2)])^{2(k-1)}. \tag{45}$$

On substituting any of the expressions for f_n, g_n as given by eqs (40)–(45) in eq. (6) we again obtain $n+1$ equations which determine the $n+1$ parameters A_k, B_k. Some of these relations are

$$\begin{aligned}
&A^2_{(n/2)+1} = B^2_{n/2},\\
&2A_{(n/2)+1}A_{n/2} + B^2_{n/2} = 2B_{n/2}B_{(n/2)-1},\\
&A^2_{n/2} + 2A_{(n/2)+1}A_{(n/2)-1} + 2B_{n/2}B_{(n/2)-1} = B^2_{(n/2)-1} + 2B_{n/2}B_{(n/2)-2},\\
&B^2_{(n/2)-1} + 2A_{(n/2)+1}A_{(n/2)-2} + 2B_{n/2}B_{(n/2)-2}\\
&\quad + 2A_{n/2}A_{(n/2)-1} = 2B_{(n/2)-1}B_{(n/2)-2} + 2B_{n/2}B_{(n/2)-3},\\
&A_1^2 = 1, \quad B_1^2 + 2A_1A_2 = 0, \quad 2A_1A_3 + A_2^2 + 2B_1B_2 = B_1^2,\\
&B_2^2 + 2A_2A_3 + 2A_1A_4 + 2B_2B_3 = 2B_1B_2.
\end{aligned} \tag{46}$$

On comparing these results with the exact expressions for $n = 2, 4$, we obtain the following general results for arbitrary even n:

$$\begin{aligned}
&A_{(n/2)+1} = B_{n/2} = 2^{n-1}, \quad A_{n/2} = -n2^{n-3}, \quad B_{n/2-1} = -(n-2)2^{n-3},\\
&A_{(n/2)-1} = \frac{n(n-3)}{2!}2^{n-5}, \quad B_{(n/2)-2} = \frac{(n-3)(n-4)}{2!}2^{n-5},\\
&A_{(n/2)-2} = -\frac{n(n-4)(n-5)}{3!}2^{n-7},\\
&B_{(n/2)-3} = -\frac{(n-4)(n-5)(n-6)}{3!}2^{n-7},\\
&\sum_{i=1}^{(n/2)+1} A_i = 1, \quad \sum_{i=1}^{n/2} B_i = n.
\end{aligned} \tag{47}$$

Further, depending on whether $n = 4k+4$ or $4k+2$ we have the following results ($k = 0, 1, 2, ...$):

$$A_1 = -1, \quad B_1 = n, \quad A_2 = \frac{n^2}{2}, \quad \text{if} \quad n = 4k+2, \tag{48}$$

$$A_1 = +1, \quad B_1 = -n, \quad A_2 = -\frac{n^2}{2}, \quad \text{if} \quad n = 4k+4. \tag{49}$$

In fact, looking at these general results, we obtain the following expressions for various A_i, B_i:

$$A_{(n/2)-k+1} = (-1)^k \frac{n(n-k-1)(n-k-2)...(n-2k+1)}{k!} 2^{n-2k-1},$$
$$k = 1, 2, ..., n/2, \tag{50}$$

$$B_{(n/2)-k} = (-1)^k \frac{(n-k-1)(n-k-2)...(n-2k)}{k!} 2^{n-2k-1},$$
$$k = 0, 1, 2, ..., (n/2) - 1. \tag{51}$$

It can be verified that for $n = 2, 4$, A_i and B_i which follow from eqs (46)–(49) agree with the values obtained independently by us above and in [1]. Further, for low values of (even) n, it can be numerically checked that A_i and B_i which follow from eqs (50) and (51) indeed satisfy the various relations which follow by demanding the validity of the constraint relation (6).

In the next section, we shall prove that various identities obtained above hold for all n. But first, using eqs (49)–(51), we predict that the following identities hold for all even n:

$$2^{n-1} + \sum_{k=1}^{n/2} (-1)^k \frac{n(n-k-1)(n-k-2)\cdots(n-2k+1)}{k!} 2^{n-2k-1} = 1. \tag{52}$$

$$\sum_{k=0}^{(n/2)-1} (-1)^k \frac{(n-k-1)(n-k-2)\cdots(n-2k)}{k!} 2^{n-2k-1} = n. \tag{53}$$

As mentioned earlier, since the field eqs (4) and (5) are invariant under $f_n \to \pm f_n$, $g_n \to \pm g_n$, one can trivially write down three other solutions in both odd and even n cases.

4. Connection to Chebyshev polynomials

In the previous section, we have defined two families f_n, g_n and stated that they lead to the solutions of the constraint relation (6). Here, f_n is a polynomial, and g_n equals a polynomial times an extra factor, and we took their common argument to be a Jacobi elliptic function. We now prove that these proposed functions do indeed give rise to solutions for all n. To do so, we take a closer look at the identities stated in eqs (38), (39), (52) and (53). If we look at eqs (38) and (52), then we notice that in both the cases, the summation goes from $k = 0$ to $k = m$, where $n = 2m + 1$ in eq. (38) and $n = 2m$ in eq. (52). In other words, in both cases the summation goes from $k = 0$ to $\lfloor n/2 \rfloor$. Thus, eqs (38) and (52) change to the odd and even cases of

$$\sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n-l}{l} \frac{(-1)^l n 2^{n-2l}}{2(n-l)} = 1. \tag{54}$$

It turns out that replacing the 2 in the numerator by other bases yields other similar identities as well, that can be numerically verified for small n. To simplify the notation, define

$$f_n(x) := \sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n-l}{l} \frac{(-1)^l n (2x)^{n-2l}}{2(n-l)}, \tag{55}$$

where $x \in \mathbb{R}$. Our identities (38) and (52) say that $f_n(1) = 1$. But before we prove this, we remark that we can also compute $f_n(x)$ for other values of x and low values of n. We omit writing the details down, but doing so leads to the following identities:

$$f_n(1) = 1 = \cos(2n\pi)\,, \quad f_n(1/2) = \cos(n\pi/3)\,, \quad f_n(0) = \cos(n\pi/2). \tag{56}$$

Similarly, it is possible to propose closed-form expressions for $f_n(-1/2)$ and $f_n(-1)$ as well. The connection between the terms on both sides of eq. (56) is made by noting that

$$1 = \cos(2\pi), \quad 1/2 = \cos(\pi/3), \quad 0 = \cos(\pi/2). \tag{57}$$

This leads to the following result for all x.

Theorem 1. *For any* $\theta \in [0, \pi]$,

$$f_n(\cos(\theta)) = \cos(n\theta). \tag{58}$$

More generally, for any $x \in \mathbb{R}$, $f_n(x) = T_n(x)$, *where* T_n *is the* nth *Chebyshev polynomial of the first kind.*

Proof. We make use of an ‘explicit formula’ on Chebyshev polynomials:

$$T_n(x) = \sum_{l=0}^{\lfloor n/2 \rfloor} \frac{(-1)^l n}{2} \frac{(n-l-1)!}{l!(n-2l)!} (2x)^{n-2l}. \tag{59}$$

Simplifying the right-hand side, we obtain the desired result. Note that for $x \in [-1, 1]$, $x = \cos(\theta)$ for a unique $\theta \in [0, \pi]$, and then the definition of T_n proves that $f_n(\cos(\theta)) = \cos(n\theta)$. Thus, $f_n \equiv T_n$ on $[-1,1]$. Since both f_n and T_n are polynomials that agree on an infinite set of points, they are equal at every $x \in \mathbb{R}$. □

In particular, when $x = 1, 1/2$, or 0, $T_n(x) = 1$, $\cos(n\pi/3)$, or $\cos(n\pi/2)$ respectively, as claimed in eq. (56).

Similarly, if we look at eqs (39) and (53), then we observe that both these cases change to the odd and even cases of

$$\frac{1}{2} \sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n-l-1}{l} (-1)^l 2^{n-2l} = n\,. \tag{60}$$

It remains to compute the above series. More generally, we prove:

Theorem 2. *Let*

$$g_n(x) = \frac{1}{2} \sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n-l-1}{l} (-1)^l (2x)^{n-2l}. \tag{61}$$

Then for all $x \in \mathbb{R}$, $g_n(x) = xU_{n-1}(x)$, *where* $U_n(x)$ *are the Chebyshev polynomials of the second kind.*

Proof. We start from eq. (59). On differentiating both sides, we have

$$T_n'(x) = \sum_{l=0}^{\lfloor n/2 \rfloor} (-1)^l \frac{n(n-l-1)!}{l!(n-2l)!}(n-2l)(2x)^{n-2l-1} \tag{62}$$

$$= \frac{n}{x} g_n(x). \tag{63}$$

Simplifying, we get

$$g_n(x) = \frac{x}{n} \cdot T_n'(x) = x U_{n-1}(x), \tag{64}$$

where the second equality is standard. □

It remains to prove the original identity. But the summation is simply $g_n(1)$, and it is well known that $U_n(1) = n + 1$ for all n. Hence

$$g_n(1) = U_{n-1}(1) = n. \tag{65}$$

In fact it is now clear that for arbitrary n, f_n and g_n (as given by eqs (55) and (61) respectively, for any even or odd integer n) are simply

$$f_n = T_n(y), \quad g_n = (1-y^2)^{1/2} U_{n-1}(y), \tag{66}$$

where y is one of the following:

$$\begin{aligned} &\mathrm{dn}[\beta(n+c_2), m], \quad \mathrm{cn}[\beta(n+c_2), m], \quad \mathrm{sech}[\beta(n+c_2)], \\ &\cos[\beta(n+c_2)], \quad \frac{1}{\mathrm{dn}[\beta(n+c_2), m]}, \quad \cosh[\beta(n+c_2)]. \end{aligned} \tag{67}$$

Note that

$$f_n^2(y) + g_n^2(y) = T_n^2(y) + (1-y^2) U_{n-1}(y) = 1. \tag{68}$$

In other words, the solutions of the coupled equations are simply Chebyshev polynomials with argument in terms of Jacobi elliptic functions. Actually, once one realizes this, then the structure of other solutions is also simplified. In particular, it is known that

$$T_n(\cos(\theta)) = \cos(n\theta), \quad T_n(\cosh(x)) = \cosh(nx). \tag{69}$$

Thus, $f_n(x) = \cos(n\theta)$ or $\cosh(nx)$, which makes $g_n(x) = \sin(n\theta)$ or $\sinh(nx)$ respectively. Thus, one now has a better understanding of the solutions (31), (45), (29) and (43).

5. Other coupled models

We now consider the coupled Ablowitz–Ladik (AL), coupled ϕ^6 and coupled ϕ^4 models as discussed in [1] and show that all these models also admit solutions in terms of Lamé polynomials of arbitrary order.

5.1 *Solutions of a coupled AL model*

As shown in [1], in the special case when $\nu_1 = 2\mu_1$ and $\nu_2 = 2\mu_2$, the coupled Salerno model given by eqs (1) and (2) reduces to the coupled AL model with the field equations

$$\frac{i\mathrm{d}u_n}{\mathrm{d}t} + [u_{n+1} + u_{n-1} - 2u_n] + (\mu_1|u_n|^2 + \mu_2|v_n|^2)[u_{n+1} + u_{n-1}] = 0\,, \quad (70)$$

$$\frac{i\mathrm{d}v_n}{\mathrm{d}t} + \left[v_{n+1} + v_{n-1} - \frac{2\mu_2}{\mu_1}v_n\right] + (\mu_1|u_n|^2 + \mu_2|v_n|^2)[v_{n+1} + v_{n-1}] = 0. \quad (71)$$

It is then clear that all the solutions of the coupled Salerno model in terms of Lamé polynomials of arbitrary order, are automatically the solutions of the coupled AL model and further in this case, $\omega_1 = 2$, $\omega_2 = 2\mu_1/\mu_2$.

5.2 *Solutions of a coupled discrete ϕ^6 model*

The field equations of the coupled discrete ϕ^6 model discussed in our recent paper [1] are

$$\frac{1}{h^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) = a_1\phi_n - b_1\phi_n^3 + d\psi_n^2\phi_n + [c_1\phi_n^4 + e\phi_n^2\psi_n^2 + f\psi_n^4][\phi_{n+1} + \phi_{n-1}]\,, \quad (72)$$

$$\frac{1}{h^2}(\psi_{n+1} + \psi_{n-1} - 2\psi_n) = a_2\psi_n - b_2\psi_n^3 + d\phi_n^2\psi_n + \left[c_2\psi_n^4 + \frac{e}{2}\phi_n^4 + 2f\phi_n^2\psi_n^2\right][\psi_{n+1} + \psi_{n-1}]\,. \quad (73)$$

As shown in [1], solutions to these coupled equations in terms of Lamé polynomials of order one and two are obtained provided

$$\phi_n^2 + \psi_n^2 = \sqrt{\frac{1}{c_1h^2}}\,, \quad (74)$$

and further if

$$c_1 = c_2 = f = \frac{e}{2}, \quad b_1 = b_2 = -d, \quad a_1 = a_2, \quad c_1h^2b^2 = 1,$$
$$a_1 + \frac{2}{h^2} = \frac{b_1}{\sqrt{h^2c_1}}\,. \quad (75)$$

It is then clear that the solutions (26)–(28) and (40)–(42) in terms of Lamé polynomials of arbitrary order obtained in the case of the coupled Salerno model will also be the solutions of this coupled model.

5.3 *Solutions for a coupled discrete ϕ^4 model*

In our recent publication [1], we considered the following coupled ϕ^4 model:

$$\frac{1}{h^2}(\phi_{n+1}+\phi_{n-1}-2\phi_n)-2\alpha_1\phi_n-[2\beta_1\phi_n^2+\gamma\psi_n^2][\phi_{n+1}+\phi_{n-1}]=0, \quad (76)$$

$$\frac{1}{h^2}(\psi_{n+1}+\psi_{n-1}-2\psi_n)-2\alpha_2\psi_n-[2\beta_2\psi_n^2+\gamma\phi_n^2][\psi_{n+1}+\psi_{n-1}]=0, \quad (77)$$

and showed that solutions to these coupled equations can be obtained in terms of Lamé polynomials of order one and two provided

$$\phi_n^2+\psi_n^2=\frac{1}{2\beta_1 h^2}, \quad (78)$$

and further if

$$2\beta_1=2\beta_2=\gamma, \quad \alpha_1=\alpha_2=-\frac{1}{h^2}\,. \quad (79)$$

It is then clear that the solutions (26)–(29) and (40)–(43) in terms of Lamé polynomials (or the corresponding hyperbolic and trigonometric polynomials) of arbitrary order obtained in the case of the coupled Salerno model will also be the solutions of this coupled model.

6. Summary

In this paper we have shown that for a number of coupled discrete models, e.g., coupled Salerno, coupled Ablowitz–Ladik, coupled ϕ^6 and coupled ϕ^4, there are solutions in terms of Lamé polynomials of arbitrary order while the uncoupled equations do not admit solutions in terms of Lamé polynomials of order two and higher. In particular, we showed that the Lamé polynomials can be re-expressed as Chebyshev polynomials of the relevant Jacobi elliptic function. These solutions have relevance in physical contexts ranging from multiferroic [3–5] materials to ferroelectrics to the models in field theory [9] as well as for a variety of discrete contexts [6–8].

There are several open issues that need to be explored, e.g. the stability of various solutions found here can be studied numerically, particularly since the soliton solutions obtained here are of arbitrary width. Similarly, the scattering of solitons of various discrete models is an important issue with these static solutions boosted with a certain velocity. Finally, the Peierls–Nabarro discreteness barrier for the solutions remains to be explored. However, since all our solutions have discrete translation invariance (i.e., they are valid for arbitrary c_2), it is likely that for all our solutions this barrier may be zero.

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