Dephasing of a qubit due to quantum and classical noise

EBAD KAMIL[1,2,*] and SUSHANTA DATTAGUPTA[1,3]

[1]Indian Institute of Science Education and Research-Kolkata, Mohanpur 741 252, India
[2]Permanent address: Institute for Theoretical Physics, Friedrich Hund Platz 1,
University of Goettingen, Germany
[3]Present address: Visva-Bharati, Santiniketan 731 235, India
*Corresponding author. E-mail: kamil@theorie.physik.uni-goettingen.de

Abstract. The qubit (or a system of two quantum dots) has become a standard paradigm for studying quantum information processes. Our focus is decoherence due to interaction of the qubit with its environment, leading to noise. We consider quantum noise generated by a dissipative quantum bath. A detailed comparative study with the results for a classical noise source such as generated by a telegraph process, enables us to set limits on the applicability of this process *vis à vis* its quantum counterpart, as well as lend handle on the parameters that can be tuned for analysing decoherence. Both Ohmic and non-Ohmic dissipations are treated and appropriate limits are analysed for facilitating comparison with the telegraph process.

1. Introduction

Environment-induced decoherence is a vexed issue in confronting the challenges to quantum information processes. Simple model calculations thus assume significance, to grasp the underlying parameter-regimes that can be manipulated, in order to minimize the effect of the environment. In quantum optics and solid-state physics, the device that has gained popularity in recent times is a qubit (which can be approximately represented by a pair of two quantum dots) [1,2]. This system is equivalent to a superconducting Josephson junction [3] or a spin-1/2 NMR nucleus [4]. A qubit can be described by a two-level Hamiltonian, the quantum mechanics of which is rather straightforward. Environmental influences can also be easily incorporated in this Hamiltonian. There have been two distinctive attempts in modelling the environment, either in terms of (i) a classical stochastic process, such as a Gaussian or a telegraph one, or (ii) an explicit quantum collection of bosonic oscillators. One of the main objectives in this paper is to seek a unification of these two apparently disparate approaches.

Ebad Kamil and Sushanta Dattagupta

We consider an electron hopping between two single-level quantum dots, coupled via a tunnelling coefficient Δ and an asymmetric bias ϵ. Denoting by $|L\rangle$ and $|R\rangle$ the left and right dot states, the qubit Hamiltonian, indicated by the subscript q, can be written as

$$\mathcal{H}_q = \Delta \left(|L\rangle\langle R| + |R\rangle\langle L|\right) + \epsilon \left(|L\rangle\langle L| - |R\rangle\langle R|\right). \tag{1}$$

If we introduce the so-called bonding and antibonding states as the spin 'up' and spin 'down' states of the Pauli matrix σ_z, denoted by $|+\rangle$ and $|-\rangle$ respectively, as

$$
\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{2}} \left(|L\rangle + |R\rangle\right), \\
|-\rangle &= \frac{1}{\sqrt{2}} \left(|L\rangle - |R\rangle\right).
\end{aligned}
\tag{2}
$$

$\mathcal{H}_q$ can be cast into more familiar notation:

$$\mathcal{H}_q = \Delta\sigma_z + \epsilon\sigma_x, \tag{3}$$

where σ_x is the other Pauli matrix that is off-diagonal in the representation in which σ_z is diagonal.

In order to incorporate the effect of the environment on the qubit, our strategy is to expand the Hilbert space to re-express the system Hamiltonian as

$$\mathcal{H}_s = \mathcal{H}_q + \tau_z(\zeta_\Delta\sigma_z + \zeta_\epsilon\sigma_x), \tag{4}$$

where τ_z is yet another Pauli matrix, which could represent a spin-1/2 impurity, for instance. The idea is to couple the τ_z-system to a bath of bosonic oscillators, thereby causing fluctuations in τ_z. The resulting full Hamiltonian can be written as

$$\mathcal{H} = \mathcal{H}_s + \tau_x \sum_k g_k(b_k + b_k^\dagger) + \sum_k \omega_k b_k^\dagger b_k. \tag{5}$$

Here $b_k^\dagger (b_k)$ are boson creation (annihilation) operators, g_k are coupling constants and ω_k is the harmonic oscillator frequency of the kth mode.

It is interesting to physically assess the effect of the coupling term (proportional to g_k) on the system Hamiltonian $\mathcal{H}_s$. Because τ_x is purely off-diagonal in the τ_z-representation, it would cause 'spin-flips' in τ_z between two allowed values $+1$ and -1, as in the celebrated Glauber model of Ising kinetics [5]. These flips would occur with an amplitude field that would be proportional to g_k and time-varying bosonic operators $b_k^\dagger(t)(b_k(t))$, in the interaction representation of the last term in eq. (5). The net effect on $\mathcal{H}_s$ is a 'quantum noise', encapsulated by $\tau_z(t)$. In a suitable limit, when the latter, i.e. $\tau_z(t)$, could be replaced by a classical noise $\eta(t)$ that jumps at random between ± 1, the system Hamiltonian would be stochastic, given by

$$\mathcal{H}_s(t) = \mathcal{H}_q + \eta(t)(\zeta_\Delta\sigma_z + \zeta_\epsilon\sigma_x), \tag{6}$$

where $\eta(t)$ is a two-state jump process or a telegraph process. While we shall present below a fully quantum mechanical treatment of eq. (5), our aim will also be to set conditions under which a classical stochastic description via a telegraph process of $\eta(t)$, as expounded in detail by Tokura and Itakura [6], would ensue from the quantum formulation.

The form of the Hamiltonian in eq. (5) falls under the general scheme of a system-plus-bath approach to non-equilibrium statistical mechanics, in which the full Hamiltonian is written as

$$\mathcal{H} = \mathcal{H}_s + \mathcal{H}_I + \mathcal{H}_B, \tag{7}$$

where $\mathcal{H}_I$ is the interaction Hamiltonian and $\mathcal{H}_B$ is the bath Hamiltonian. In the present instance,

$$\begin{aligned}\mathcal{H}_I &= \tau_x \sum_k g_k (b_k + b_k^\dagger), \\ \mathcal{H}_B &= \sum_k \omega_k b_k^\dagger b_k.\end{aligned} \tag{8}$$

As we shall be employing a Liouvillean approach to the dynamics governed by $\mathcal{H}$, it is useful to explain the notation. The Liouville operator $\mathcal{L}$, associated with $\mathcal{H}$, is defined by

$$\mathcal{L}A = \frac{1}{\hbar}\left[\mathcal{H}, A\right], \tag{9}$$

where A is an arbitrary but ordinary operator. Thus a Liouville operator yields an ordinary operator when it operates on an operator (as in the right-hand side of eq. (9)), just as an operator, operating on a wavefunction, yields another wavefunction. We shall designate the 'states' of a Liouville operator by round brackets: $|)$, in analogy with the Dirac ket vectors: $|\rangle$ for the wavefunctions. The decomposition in eq. (7) further allows $\mathcal{L}$ to be split as

$$\mathcal{L} = \mathcal{L}_s + \mathcal{L}_I + \mathcal{L}_B. \tag{10}$$

The Liouville dynamics effected by $\mathcal{L}$ on the density operator ρ, leading to a non-Markovian master equation, in the context of the decoherence of a qubit (as well as a collection of qubits) have been extensively studied by Kurizki and collaborators [7]. They have also considered the additional influence of external time-varying fields for controlling decoherence, dynamically. While we shall be employing a very similar approach, our treatment will differ from Gordon *et al* [7]. The latter utilize the Born approximation, valid for weak coupling constants g_k, whereas we shall consider strong coupling in which g_k will be treated to all orders. Also the dissipation in the more general quantum system has been thoroughly studied in the context of semiclassical theory for many-body systems [8] and the important case of qubit relaxation, while being resonantly coupled to forcing has been treated in ref. [9].

Such strong coupling considerations are the hallmark of functional integral treatments of quantum dissipative systems, e.g. that of a spin-boson Hamiltonian [10]. Furthermore, our focus in the classical limit of the quantum noise will be a telegraph process as opposed to the much-studied Gaussian stochastic processes. Similar spin-boson Hamiltonians, akin to eq. (5), have been considered in the past [11], though the comparison has not been dealt with, to the best of our knowledge.

With these preliminaries the paper is organized as follows: In §2 we outline the mathematical steps for a resolvent expansion of the averaged time-development operator in the Laplace transform space, the average being carried out over the Hilbert spaces of τ_z and $\mathcal{H}_B$. The latter is expressed in terms of a 'self-energy' whose form is provided.

In §3 explicit results are presented for the much explored Ohmic-dissipation model and comparisons are drawn with the classical telegraph process, results of which are given in Appendix. For facilitating comparison we discuss fluctuation in bias and hopping separately. In a subsection of §3, we indicate results for non-Ohmic situations as well. Finally in §4, we discuss the issue of partial decoherence, recently studied by us in the context of the telegraph process [12], to put it under the perspective of quantum noise. Section 5 contains a few concluding remarks.

2. Theoretical formulation

In order to incorporate strong-coupling effects, it is convenient to transform the Hamiltonian $\mathcal{H}$ in eq. (5) with the aid of a unitary transformation defined by the operator:

$$S = \exp\left[-\sum_q \frac{g_q}{2\omega_q}(b_q - b_q^\dagger)\tau_z\right]\exp\left[-i\frac{\pi}{2}\tau_y\right]. \tag{11}$$

The transformed Hamiltonian is given by

$$\tilde{\mathcal{H}} = S\mathcal{H}S^{-1}. \tag{12}$$

What the second term in S does is to cause a rotation in the τ-space by an angle of $\pi/2$ about the y-axis, in the anticlockwise direction such that $\tau_x \to \tau_z$ and $\tau_z \to -\tau_x$. The first term in S then eliminates the interaction term $\mathcal{H}_I$ but puts the onus of coupling on $\mathcal{H}_s$ itself (besides generating an innocuous 'counter term' that is constant, and can be dropped). The net result is

$$\tilde{\mathcal{H}} = \mathcal{H}_q - \frac{1}{2}(\tau^- B_+ + \tau^+ B_-)\left(\zeta_\Delta\sigma_z + \zeta_\epsilon\sigma_x\right) + \sum_k \omega_k b_k^\dagger b_k, \tag{13}$$

where

$$f\tau_\pm = \tau_x \pm i\tau_y,$$
$$B_\pm = \exp\left[\pm\sum_q \frac{g_q}{2\omega_q}(b_q - b_q^\dagger)\right]. \tag{14}$$

The transformed Hamiltonian $\mathcal{H}$ is now endowed with a new interaction term that may be written as

$$\tilde{\mathcal{H}}_I = -\frac{1}{2}(\tau^- B_+ + \tau^+ B_-)\left(\zeta_\Delta\sigma_z + \zeta_\epsilon\sigma_x\right). \tag{15}$$

The important point however is that any perturbation treatment of $\tilde{\mathcal{H}}_I$ is tantamount to treating the coupling to all orders as g_q now occurs in the exponent as is evident from the structure of $B_\pm$ (see eq. (14)). Equation (13) is again of the generic form of eq. (7) except that the 'system' is now the qubit itself as $\mathcal{H}_s$ is replaced by $\mathcal{H}_q$!

With reference to the Liouvillean in eq. (10) the equation of motion for the density operator ρ reads as

$$\frac{\partial}{\partial t}\rho(t) = -i\mathcal{L}\rho(t). \tag{16}$$

Unlike other approaches [6,7] we find it convenient to work with the Laplace transforms, defined by

$$\rho(z) = \int_0^\infty dt\, e^{-zt} \rho(t).$$ (17)

From eq. (16) then

$$\rho(z) = (U(z))\,\rho(t=0),$$
$$U(z) = (z + i\mathcal{L})^{-1}.$$ (18)

We employ the usual factorization approximation that at $t = 0$ the qubit and the bath are decoupled [13], so that

$$\rho(t=0) = \rho_q(0) \otimes \rho_B,$$ (19)

where $\rho_q(0)$ is the qubit density operator at time $t = 0$, $\rho_B = \exp(-\beta\mathcal{H}_B)/Z_B$, is the canonical density operator for the bath and Z_B is the corresponding partition function. Underlying the prescription in eq. (19) is the assumption that the bath always remains in equilibrium at a fixed temperature T while the qubit evolves from an arbitrary state. For most of our derived expressions we shall assume that the electron is on the left dot at $t = 0$, i.e.

$$\rho_q(0) = |L\rangle\langle L|.$$ (20)

Translated to the 'bonding' and 'antibonding' basis, this implies

$$\rho_q(0) = \frac{1}{2}(1 + \sigma_x).$$ (21)

(We shall return in §4 to a more general initial condition.) The Laplace transform of the so-called reduced density operator can be written as

$$\rho_{\mathcal{R}}(z) = \text{Tr}_\tau \otimes \text{Tr}_B\,[(U(z))\rho_B]\rho_q(0),$$ (22)

where $\text{Tr}_\tau(\cdots)$ denotes trace over the eigenstates of τ_z whereas $\text{Tr}_B(\cdots)$ specifies trace over the bath (i.e. eigenstates of $\mathcal{H}_B$). Because the trace is invariant under the unitary transformations (e.g. in eq. (11)), Eq. (22) can be equivalently expressed as

$$\rho_{\mathcal{R}}(z) = \text{Tr}_\tau \otimes \text{Tr}_B[(\tilde{U}(z))\rho_B]\rho_q(0),$$ (23)

where the tilde on $U(z)$ implies that the corresponding time-evolution is now governed by $\tilde{U}$ of eq. (13).

Denoting the Liouville operator associated with $\tilde{\mathcal{H}}_I$ as $\tilde{\mathcal{L}}_I$, we have upto second order in $\tilde{\mathcal{H}}_I^8$,

$$\rho_R(z) = \left[\left(z + i\mathcal{L}_q + \tilde{\sum}(z)\right)^{-1}\rho_q(0)\right],$$ (24)

where the self-energy $\tilde{\sum}(z)$ is given as

$$\tilde{\sum}(z) = \text{Tr}_\tau \otimes \text{Tr}_B[\tilde{\mathcal{L}}_I\left(z + i(\mathcal{L}_q + \mathcal{L}_B)\right)^{-1}\tilde{\mathcal{L}}_I\rho_B].$$ (25)

The z-dependence (or frequency dependence) of $\tilde{\sum}(z)$ implies that the underlying dynamics is non-Markovian, in general. We emphasize once more that though the self-energy is computed to second order in $\tilde{\mathcal{L}}_I$, the theory is valid to arbitrary orders in the original coupling constants g_k. Hence, the resultant master equation in the time domain that emanates from eq. (24) is of more general validity than the one obtained in the Born-approximation [7].

Our next step is to write the matrix elements of the self-energy $\tilde{\sum}(z)$. It is clear that $\tilde{\sum}(z)$ is a Liouvillean in the σ-subspace alone because τ and the bath variables are averaged over, in eq. (25). Hence, $\tilde{\sum}(z)$ has a 4×4 matrix representation in the 'eigenbasis' $|\mu v)$ (note the round brackets used for 'states' of the Liouvillean), wherein $|\mu\rangle$, $|v\rangle$,... are the eigenkets of σ_z. Clearly then, the rows and columns of the 4×4 matrix would be labelled by $|++\rangle$, $|--\rangle$, $|+-\rangle$ and $|-+\rangle$, respectively. Additionally, while performing the traces in eq. (25), we would need to sum over the eigenstates of τ_z and $\mathcal{H}_B$, as follows:

$$\tau_z |s_1\rangle = s_1 |s_1\rangle,$$
$$\mathcal{H}_B |n_b\rangle = E_{n_b} |n_b\rangle, \tag{26}$$

etc. With these explanatory remarks about the notation, the matrix elements of $\tilde{\sum}(z)$ turn out to be generalized forms of those given in ref. [14]:

$$\left(\mu v \left| \tilde{\sum}(z) \right| \mu' v'\right) = \sum_{n_b n_b^1 s_1 s_2} \left[\delta_{vv'} \sum_\eta \frac{\langle \mu n_b s_1 | H_I | \eta n_b^1 s_2\rangle \langle \eta n_b^1 s_2 | H_I | \mu' n_b s_1\rangle}{z + i(E_\eta - E_v) + i(E_{n_b^1} - E_{n_b})} \right.$$

$$+ \delta_{\mu'\mu} \sum_\eta \frac{\langle v' n_b s_1 | H_I | \eta n_b^1 s_2\rangle \langle \eta n_b^1 s_2 | H_I | v n_b s_1\rangle}{z + i(E_\mu - E_\eta) + i(E_{n_b} - E_{n_b^1})}$$

$$- \frac{\langle \mu n_b s_1 | H_I | \mu' n_b^1 s_2\rangle \langle v' n_b^1 s_2 | H_I | v n_b s_1\rangle}{z + i(E_{\mu'} - E_v) + i(E_{n_b^1} - E_{n_b})}$$

$$- \left. \frac{\langle \mu n_b s_1 | H_I | \mu' n_b^1 s_2\rangle \langle v' n_b^1 s_2 | H_I | v n_b s_1\rangle}{z + i\left(E_\mu - E_{v'}\right) + i(E_{n_b} - E_{n_b^1})} \right]$$

$$\times \langle n_b | \rho_B | n_b\rangle. \tag{27}$$

From the structure of $\tilde{\mathcal{H}}_I$ in eq. (15) it is evident that the bath variables enter only through the operators $B_\pm$. Therefore, when we sum over the bath indices n_b, n_b', etc., with the Boltzmann weight $\langle n_b | \rho_B | n_b\rangle$, we end up with correlation fucntions of $B_\pm$, as follows:

$$\Phi_{ij}(t) = \text{Tr}_B \left(\rho_B B_i(0) B_j(t) \right) = \langle B_i(0) B_j(t)\rangle, \quad i, j = + \text{ or } - . \tag{28}$$

As it happens, the only non-zero components of the correlation function are

$$\Phi_{+-}(t) = \Phi_{-+}(t) = \langle B_\pm(0) B_\mp(t)\rangle$$
$$= \exp\left\{ -\sum_q \frac{4g_q^2}{\omega_q^2} \left[\coth\left(\frac{\beta\omega_q}{2}\right) (1 - \cos\omega_q t) + i \sin\omega_q t \right] \right\}. \tag{29}$$

An essential attribute of quantum dissipative system is that the system $\mathcal{H}_B$ is taken to the limit of an infinitely large number of bosonic modes. This implies that the discrete

sum over q has to be replaced by an integral over continously varying frequencies ω with an appropriate weight called the 'spectral density' $J(\omega)$. Thus

$$\phi(\tau) = \exp\left\{-2\int_0^\infty d\omega \frac{J(\omega)}{\omega^2}\left[\coth\left(\frac{\beta\omega}{2}\right)(1-\cos\omega\tau) + i\sin\omega\tau\right]\right\},$$
(30)

where

$$J(\omega) = 2\sum_q g_q^2 \delta(\omega - \omega_q).$$
(31)

3. Explicit results and comparison with telegraph noise

3.1 *Ohmic dissipation*

In the literature on dissipative quantum systems, one model for the spectral density $J(\omega)$ that has received the maximum amount of attention is what gives rise to Ohmic dissipation [15]. In this, $J(\omega)$ is assumed to have a linear frequency dependence with an exponential cut-off ω_c

$$J(\omega) = K\omega \exp\left(-\frac{\omega}{\omega_c}\right),$$
(32)

where K is a phenomenological damping parameter that measures the strength of the coupling with the heat bath. This form allows for an analytic expression for the Laplace transform as defined in eq. (17), for $\beta\omega_c \gg 1$ as

$$\tilde{\phi}(z) = \frac{e^{i\pi K}}{\omega_c}\left(\frac{2\pi}{\beta\omega_c}\right)^{(2K-1)}\frac{\Gamma(1-2K)\Gamma(K+z\beta/2\pi)}{\Gamma(1-K+z\beta/2\pi)},$$
(33)

where $\Gamma(\cdot\cdot)$ denotes gamma functions.

The significance of the condition $\beta\omega_c \gg 1$ can be ascertained from the fact that the quantity $\hbar\tau_q = \beta$ (Note: $\hbar$ has been set to unity, hitherto) sets the time-scale over which quantum coherence is maintained. Thus, for all frequencies greater than the cut-off ω_c, quantum fluctuations remain strictly coherent. Before we take up the comparison with the telegraph noise it is pertinent to point out that it is meaningful to think of a quantum bath as classical only in the incoherent regime, i.e. over time-scales larger than τ_q. In other words, the classical limit of the bath is obtained for frequencies $\omega \ll 1/\tau_q$, i.e. the temperature is significantly large, but not so large as to violate the condition $\omega_c\tau_q \gg 1$, if we want to restrict our discussion within the domain of validity of the analytical result of eq. (33). If the restriction $\omega\tau_q \ll 1$ does apply, the frequency or z-dependence in the arguments of the gamma functions in eq. (33) can be dropped yielding

$$\lambda = \tilde{\phi}(z=0) = \frac{e^{i\pi K}}{\omega_c}\left(\frac{2\pi}{\beta\omega_c}\right)^{(2K-1)}\frac{\Gamma(1-2K)\Gamma(K)}{\Gamma(1-K)}.$$
(34)

The parameter λ will turn out later to be related to the jump rate of the telegraph process.

With this background we are ready to provide explicit results for the different elements of the density matrix in order to examine decoherence as well as to draw comparison

with the results for the telegraph noise. Because the latter have been extensively studied recently by Itakura and Tokura [6], we shall follow their lead in ignoring the bias term in the original qubit Hamiltonian $\mathcal{H}_q$, i.e. set $\epsilon = 0$ though it should be evidently clear that the formalism can easily embrace asymmetric cases ($\epsilon \neq 0$) as well. With this, the relevant Hamiltonian for further considerations can then be written as (see eqs (3) and (13))

$$\tilde{\mathcal{H}} = \Delta\sigma_z - \frac{1}{2}(\tau^- B_+ + \tau^+ B_-)\,(\zeta_\Delta\sigma_z + \zeta_\epsilon\sigma_x) + \sum_k \omega_k b_k^\dagger b_k. \tag{35}$$

For ease of discussion and for remaining close to the treatment of Itakura and Tokura we take up the cases of fluctuation in bias ($\zeta_\Delta = 0$) and fluctuation in hopping ($\zeta_\epsilon = 0$) separately below.

3.2 *Fluctuation in bias* ($\zeta_\Delta = 0$)

The appropriate interaction Hamiltonian that has to be substituted in the expression for the self-energy $\sum(z)$ in eq. (25) is now given by

$$\tilde{H}_I = -\frac{1}{2}\zeta_\epsilon\sigma_x(\tau^- B_+ + \tau^+ B_-). \tag{36}$$

Carrying out the summations implied in eq. (25) we arrive at

$$\tilde{\sum}(z) = \begin{bmatrix} a_1(z) & -a_1(z) & 0 & 0 \\ -a_2(z) & a_2(z) & 0 & 0 \\ 0 & 0 & a_3(z) & -a_3(z) \\ 0 & 0 & -a_3(z) & a_3(z) \end{bmatrix}, \tag{37}$$

where

$$a_1(z) = 2\zeta_\epsilon^2[\tilde{\phi}(z_+) + \tilde{\phi}'(z_-)], \quad a_2(z) = 2\zeta_\epsilon^2[\tilde{\phi}(z_-) + \tilde{\phi}'(z_+)]$$

$$a_3(z) = 2\zeta_\epsilon^2[\tilde{\phi}(z) + \tilde{\phi}'(z)], \quad z_\pm = z \pm 2i\Delta. \tag{38}$$

Here $\tilde{\phi}(z)$ and $\tilde{\phi}'(z)$ are the Laplace transforms of the correlation functions $\phi(t)$ and $\phi(-t)$, respectively.

The matrix element of $(z + i\mathcal{L}_q + \tilde{\sum}(z))$, required in eq. (24), is obtained from a combination of eq. (27) and the definition of the matrix elements of the Liouville operator $\mathcal{L}_q$ *a la* eq. (A.7). The resultant matrix is of a block-diagonal form and can be easily inverted to yield

$$(\rho_\mathcal{R}(z))_{++} = (\rho_\mathcal{R}(z))_{--} = \frac{1}{2z},$$

$$(\rho_\mathcal{R}(z))_{+-} = \frac{z + 2i\Delta + 4\zeta_\epsilon^2(\tilde{\phi}(z) + \tilde{\phi}'(z))}{2[z(z + 4\zeta_\epsilon^2(\tilde{\phi}(z) + \tilde{\phi}'(z))) + 4\Delta^2]}. \tag{39}$$
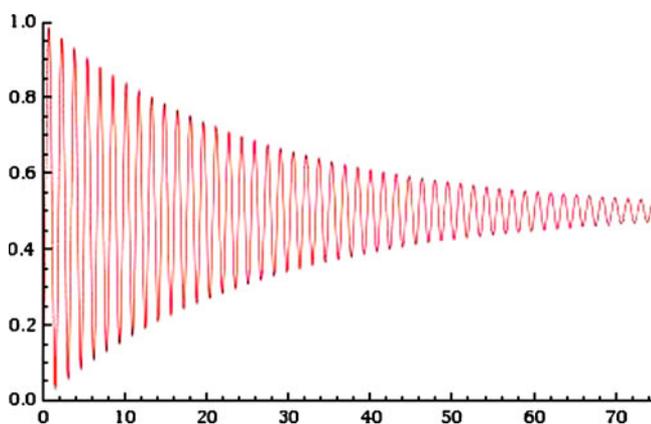
It is of course possible to go back to the dot-basis with the aid of eq. (2). Results are shown in figures 1, 2 and 3 in which $\text{Im}[\rho_{LR}(t)]$, $\rho_{LL}(t)$ and $\rho_{RR}(t)$ are plotted vs. time t for a certain choice of parameters. While $\text{Im}[\rho_{LR}(t)]$ oscillates and decays to zero as
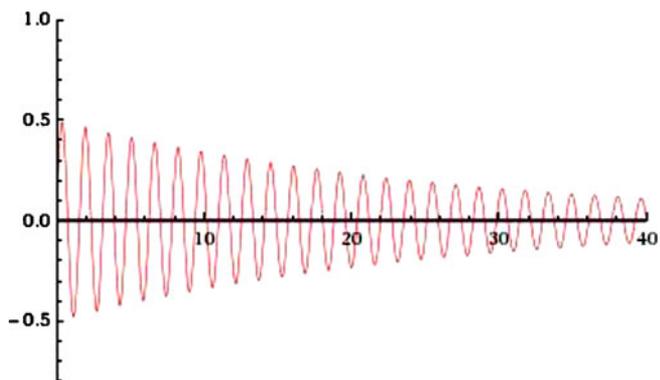
Figure 1. $\mathrm{Im}[(\rho_{\mathcal{R}})_{LR}(t)]$ is plotted for $\beta = 100$, $k = 0.45$ and $\Delta = 2$.
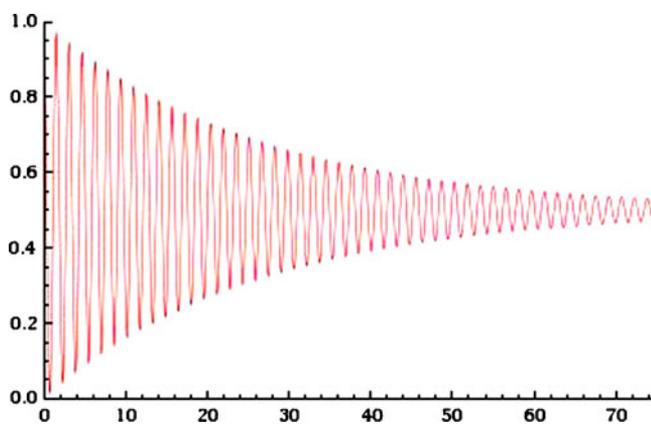
Figure 2. $(\rho_{\mathcal{R}})_{LL}(t)$ is plotted for $\beta = 100$, $k = 0.45$ and $\Delta = 2$.

Figure 3. $(\rho_{\mathcal{R}})_{RR}(t)$ is plotted for $\beta = 100$, $k = 0.45$ and $\Delta = 2$.

$t \rightarrow \infty$, the diagonal elements, proportional to the populations of the two dot states, oscillate and settle to the common value of 0.5, as we expect for full decoherence. If we compare figures 1–3 with figure A.1 for the telegraph process below, it is evident that the quantum case exhibits persistence of oscillations over longer time-scales.

3.3 *Fluctuation in hopping* ($\zeta_\epsilon = 0$)

The appropriate interaction Hamiltonian that has to be substituted in the expression for the self-energy $\tilde{\Sigma}(z)$ in eq. (25) is now given by

$$\tilde{H}_I = -\frac{1}{2}\zeta_\epsilon \sigma_z(\tau^- B_+ + \tau^+ B_-). \tag{40}$$

Following the formalism developed in §2 and using eq. (25), the self-energy matrix $\tilde{\Sigma}(z)$ is given by

$$\tilde{\Sigma}(z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_3(z) & 0 \\ 0 & 0 & 0 & b_4(z) \end{bmatrix}, \tag{41}$$

where

$$\begin{aligned} b_3(z) &= 4\zeta_\Delta^2[\phi'(z_+) + \phi(z_+)], \\ b_4(z) &= 4\zeta_\Delta^2[\phi'(z_-) + \phi(z_-)], \quad z_\pm = z \pm 2i\Delta. \end{aligned} \tag{42}$$

$\phi(z)$ and $\phi'(z)$ have already been defined in the earlier section.

Using the form $\tilde{U}(z) = (z + i\mathcal{L}_s + \tilde{\Sigma}(z))^{-1}$, it is evident from the diagonal form of the matrix that the population of the bonding and antibonding state remains constant for all $t > 0$. Using $\tilde{U}(z)$ the matrix elements are given by

$$\begin{aligned} (\rho_\mathcal{R})_{++}(z) &= (\rho_R)_{--}(z) = \frac{1}{2z}, \\ (\rho_\mathcal{R})_{+-}(z) &= \frac{1}{2}\left[\frac{1}{z + 2i\Delta + b_3}\right]. \end{aligned} \tag{43}$$

However, in the dot basis we expect the usual saturation to the common value of 0.5 for both left and right dot populations. Figure 4 shows the variation of left and right dot populations with time and also the loss of coherence over time from the decay of $\text{Im}(\rho_{LR}(t))$.

We are now in a position to make a connection between the results of quantum and classical regimes considered separately in §3 and Appendix A. There are two critical distinctions between the two regimes. At the outset, the quantum behaviour is non-Markovian, reflected in the z (or frequency)-dependence of the self-energy $\tilde{\Sigma}(z)$ in eq. (25). On the other hand, in the classical stochastic case, the Markovian assumption is invoked, to begin with, as seen in eq. (A.2). The second crucial ingredient in the quantum situation, even after the Markovian approximation is incorporated, is in the temperature dependence of the relaxation rate λ. While the latter, in the classical domain, is usually endowed with an exponential dependence on temperature, with a barrier activation in mind, it has a richer structure in the quantum regime, as discussed below.
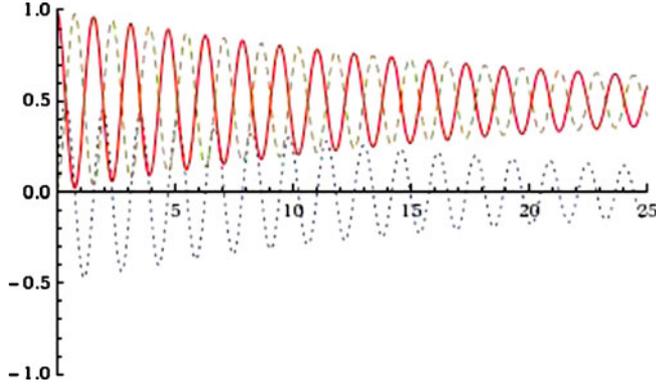
Figure 4. $(\rho_\mathcal{R})_{LL}(t)$ (solid line), $(\rho_\mathcal{R})_{RR}(t)$ (dashed line) and $\mathrm{Im}(\rho_\mathcal{R})_{RR}(t)$ (dotted line) are plotted for $\beta = 100$, $k = 0.45$ and $\Delta = 2$.

We mention in Appendix A.2 that in the long-time region, the off-diagonal element of the system density operator follows an exponential decay with time constant $T_2 = 1/\sqrt{|\lambda^2 - 16\zeta_\Delta^2|}$. From eq. (43), as can be clearly seen from the Laplace transform of ρ_{+-}, the decay is exponential only if we assume $b_3(z)$ to be independent of z, wherein the time constant is just the inverse of $\mathrm{Re}[b_3]$. Thus, the comparison with the classical telegraph process is meaningful only if one takes the Markovian limit of heat bath-induced relaxation, at the outset.

At sufficiently high temperatures given by $k_\mathrm{B}T \gg \zeta_\Delta/K$ we can see from eq. (33) that $\tilde{\phi}(z)$ can be approximated as

$$\tilde{\phi}(z) = \frac{e^{i\pi K}}{\omega_c} \left(\frac{2\pi}{\beta \omega_c} \right)^{2K-1} \frac{\Gamma(1-2K)\Gamma(K)}{\Gamma(1-K)}$$

which leads to a rather simple expression for b_3 (for $z = 0$) in eq. (42) as

$$b_3 = \frac{8\zeta_\Delta^2}{\omega_c} \cos(\pi K) \left(\frac{2\pi}{\beta \omega_c} \right)^{2K-1} \frac{\Gamma(1-2K)\Gamma(K)}{\Gamma(1-K)},$$

that is real. Hence λ, which appears as the jump rate in the case of telegraph noise, discussed in the Appendix in the sequel, can be compared with the relaxation rate $\mathrm{Re}[b_3]$ in the case of quantum noise, is now a function of K and β. Also, it is obvious from the expression of the relaxation rate that the dependence on temperature is not an exponential, rather it shows a power-law behaviour. However, it is pertinent to emphasize that these conclusions hinge on our assumption that quantum dissipation is Ohmic. In the more general case, discussed in §3.4, we shall see that the temperature dependence of the relaxation rate can in fact be exponential, under certain special situations.

At low temperatures and weak damping, i.e. 'small' K, $\tilde{\phi}(z)$ is given by

$$\tilde{\phi}(z) = \frac{\mu^2}{z + 2\pi/\beta},$$

where μ is a renormalized form of tunnelling frequency given by

$$\mu^2 = 4\zeta_\Delta^2 \left(\frac{2\pi}{\beta \omega_c} \right)^{2K-1} \Gamma(1-2K).$$

Using this form of $\tilde{\phi}(z)$, and eq. (43) we can deduce,

$$(\rho_{\mathcal{R}})_{+-}(z) = \frac{1}{2}\left[\frac{1}{z + 2i\Delta - \frac{4i\Delta\cos(\pi K)\mu^2}{(2\pi/\beta)^2 + 4\Delta^2} + \frac{4\mu^2(\pi/\beta)\cos(\pi K)}{(2\pi/\beta)^2 + 4\Delta^2}}\right]. \qquad (44)$$

From eq. (44), it is evident that the relaxation rate becomes $\frac{4\mu^2(\pi/\beta)\cos(\pi K)}{(2\pi/\beta)^2 + 4\Delta^2}$, which is far from an exponential in temperature. Thus, it may be stressed that even in the Markovian limit the temperature dependence of both the relaxation rate as well as the Rabi frequency is much more complex than an exponential, as is usually assumed for classical activated processes.

These conclusions are similar to those obtained earlier in the context of spin relaxation of a muon, tunnelling in a double well [16].

3.4 *Non-Ohmic case*

We had in the text presented results based on the assumption of Ohmic dissipation, i.e. choosing $J(\omega) = K\omega\exp(-\omega/\omega_c)$. The Ohmic model is a rather limited one, not applicable to phonons but is relevant to a bath consisting of electron–hole excitations near the Fermi surface, as in metals [15,16]. In this subsection we shall allow for a general spectral density function, and shall be analysing the relaxation rate in the case of quantum noise. We shall try to make a connection with the classical telegraph process and show that in the high-temperature limit, the relaxation rate indeed follows an exponential dependence on temperature with however quantum corrections, which was not the case in the Ohmic limit. Our starting point is eq. (30) which upon clubbing the oscillating terms together, can be rewritten as

$$\phi(\tau) = \exp\left[-2\int_0^\infty d\omega\frac{J(\omega)}{\omega^2}\left[\coth\left(\frac{\beta\omega}{2}\right) - \frac{\cos(\omega(\tau - i\beta/2))}{\sinh(\beta\omega/2)}\right]\right]. \qquad (45)$$

By making a change of variable from $\tau - i\beta/2$ to t in eq. (45), $\phi(z)$ can be written as

$$\phi(z) = \exp\left(-\frac{iz\beta}{2}\right)\int_{-i\beta/2}^\infty dt\, e^{-zt}$$
$$\times \exp\left[-2\int_0^\infty d\omega\frac{J(\omega)}{\omega^2}\left[\coth\left(\frac{\beta\omega}{2}\right) - \frac{\cos(\omega t)}{\sinh(\beta\omega/2)}\right]\right]. \qquad (46)$$

In the high-temperature regime [17], the integrand in the second term in eq. (46) can be replaced by its short-time limit as

$$\phi(z) = \exp\left(-\frac{iz\beta}{2}\right)\int_{-i\beta/2}^\infty dt\, e^{-zt}$$
$$\times \exp\left[-2\int_0^\infty d\omega\frac{J(\omega)}{\omega^2}\left[\tanh\left(\frac{\beta\omega}{4}\right) + \frac{\omega^2 t^2}{2\sinh(\beta\omega/2)}\right]\right]. \qquad (47)$$

Also in the high-temperature regime, i.e. low β, $\tanh(x)$ and $\sinh(x)$ can be replaced by the argument, i.e. x, and hence the integrations in eq. (47) can be carried out easily to yield,

$$\phi(z) = \exp(-\beta(a + iz/2))\int_{-i\beta/2}^\infty dt\, e^{-zt}e^{4at^2/\beta}, \qquad (48)$$

where $a = \frac{1}{2} \int_0^\infty d\omega (J(\omega)/\omega)$. Also $\phi'(z)$ is obtained by replacing i by $-i$ in eq. (48). Thus the relaxation rate which is given by Re(b_3), for the quantum case in §3.3, is obtained by imposing the Markovian limit, i.e. putting $z = 0$, leading to

$$b_3 = 4\zeta_\Delta^2 \left\{ \exp\left(-\beta\left(a - \Delta - \frac{\Delta^2}{4a}\right)\right) \int_{-\frac{i\beta}{2}+\frac{i\beta\Delta}{4a}}^{\infty} du \exp(-4au^2/\beta) \right.$$
$$\left. + \exp\left(-\beta\left(a + \Delta - \frac{\Delta^2}{4a}\right)\right) \int_{\frac{i\beta}{2}+\frac{i\beta\Delta}{4a}}^{\infty} du \exp(-4au^2/\beta) \right\}. \tag{49}$$

Finally, the lower limit in eq. (49) can be set to zero and thus the integral is just a function of temperature which follows a power law. Thus, the dominant behaviour of the relaxation rate is an exponential dependence on temperature, which is what we expect at high temperatures. This analysis therefore provides a regime where we can more meaningfully compare the relaxation rate λ for classical telegraph process with the Laplace transform of the bath correlation function in the quantum case. Note that the present treatment does not depend on any specific assumption for the spectral density $J(\omega)$ but is more generally couched, irrespective of whether the phonons are acoustic or optic.

4. The issue of partial decoherence

In the context of fluctuation in hopping, it is evident that the system Hamiltonian commutes with the interaction Hamiltonian, which means that there is no energy exchange between the system and the bath. Such cases are important for studying partial decoherence [12]. Thus, we generalize our discussion to a more general initial state of a qubit (than of eq. (20)) given by

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|L\rangle + e^{i\gamma} \sin\left(\frac{\theta}{2}\right)|R\rangle. \tag{50}$$

In such a scenario it can be established that the density matrix does not evolve to a completely mixed state, rather it approches a limiting value, which does contain the off-diagonal components. In this situation coherence is not completely lost and the off-diagonal terms give information about the initial state of the system. We note that the initial qubit information can be retrieved, either from experiments related to persistent current or by measuring the population of states. The former path is not applicable in our case as we have not allowed for the existence of an Aharonov–Bohm flux [12].

The density matrix (in $|L\rangle$ and $|R\rangle$ basis) at time $t = 0$ is given from eq. (50) by,

$$\tilde{\rho}_{\mathcal{R}}(0) = \begin{bmatrix} \cos^2(\theta/2) & \cos(\theta/2)\sin(\theta/2)e^{-i\gamma} \\ \cos(\theta/2)\sin(\theta/2)e^{i\gamma} & \sin^2(\theta/2) \end{bmatrix}, \tag{51}$$

which can be translated to the bonding–antibonding basis, via the transformation given by eq. (2). The off-diagonal term $(\rho_{\mathcal{R}})_{LR} (t \to \infty)$, which measures coherence, does not go to zero unlike in the cases discussed in §3.3 and A.2. These particular symmetric cases of $\epsilon = 0$ and $\zeta_\epsilon = 0$ show partial coherence which is important for decoherence-free quantum computation protocols.

The above formalism is general and is valid for any kind of environment, as long as the commutator of the system Hamiltonian and the interaction Hamiltonian is zero. As
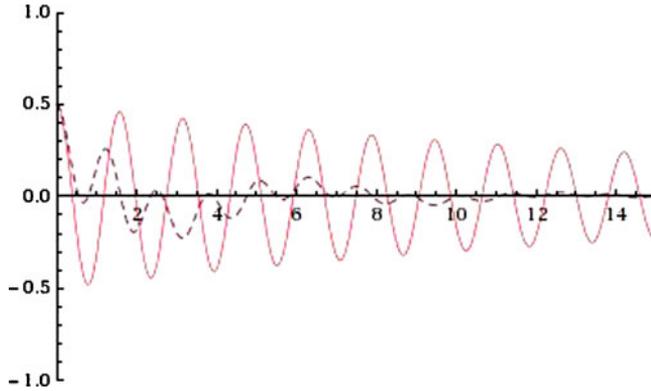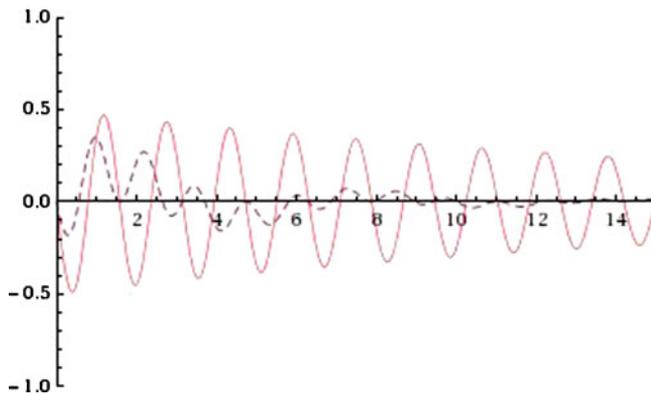
Figure 5. $\text{Re}[(\rho_{\mathcal{R}})_{+-}(t)]$ for quantum (solid line) and classical (dashed line) cases are plotted vs. time t.

shown in classical and quantum noises (figures 5 and 6), the imaginary and real parts of $(\rho_{\mathcal{R}})_{+-}(t) \to 0$ as $t \to \infty$, whereas the populations of bonding and antibonding states stay constant, as is evident from eq. (43). Thus, in general, the off-diagonal elements of the density matrix (in $|L\rangle$ and $|R\rangle$ basis) approach limiting values, emphasizing that the asymptotic state is not a fully mixed state:

$$\hat{\rho}_{\mathcal{R}}(t \to \infty) = \begin{bmatrix} 1/2 & \text{Re}((\rho_{\mathcal{R}})_{LR}(0)) \\ \text{Re}((\rho_{\mathcal{R}})_{RL}(0)) & 1/2 \end{bmatrix}. \tag{52}$$

At this stage it is also worthwhile to underscore the importance of time-scales involved in attaining partial decoherence. For this we define a function $C(t)$ as

$$C(t) = \frac{\text{Im}((\rho_{\mathcal{R}})_{LR}(\infty)) - \text{Im}((\rho_{\mathcal{R}})_{LR}(t))}{\text{Im}((\rho_{\mathcal{R}})_{LR}(\infty)) - \text{Im}((\rho_{\mathcal{R}})_{LR}(0))}. \tag{53}$$

Figure 6. $\text{Im}[(\rho_{\mathcal{R}})_{+-}(t)]$ for quantum (solid line) and classical (dashed line) cases are plotted vs. time t.

This function is 1 at time $t = 0$ and 0 at time $t = \infty$. When integrated over all times, we get a 'relaxation' time for sustenance of decoherence as

$$\tau = \int_0^\infty \mathrm{d}t\, C(t). \tag{54}$$

We carry out an explicit calculation for τ in both classical and quantum regimes. Using eqs (A.14) and (43) and the initial density matrix given by eq. (45), we can calculate $(\rho_\mathcal{R})_{LR}(t)$ to obtain

$$\tau = \frac{\frac{1}{2}I_1 + I_2 \operatorname{Im}((\rho_\mathcal{R})_{LR}(0))}{\operatorname{Im}((\rho_\mathcal{R})_{LR}(0))}, \tag{55}$$

where, for the quantum case,

$$I_1 = \frac{2\Delta}{(2\Delta)^2 + b_3^2},$$
$$I_2 = \frac{b_3}{(2\Delta)^2 + b_3^2}. \tag{56}$$

On the other hand, for the classical telegraph process,

$$I_1 = \frac{(2\Delta)(-\lambda + k)}{(2\Delta)^2 + \left(\frac{\lambda+k}{2}\right)^2} + \frac{(2\Delta)(\lambda + k)}{(2\Delta)^2 + \left(\frac{\lambda-k}{2}\right)^2},$$
$$I_2 = \frac{\left(\frac{\lambda+k}{2}\right)(-\lambda + k)}{(2\Delta)^2 + \left(\frac{\lambda+k}{2}\right)^2} + \frac{\left(\frac{\lambda-k}{2}\right)(-\lambda + k)}{(2\Delta)^2 + \left(\frac{\lambda-k}{2}\right)^2},$$
$$k = \sqrt{(\lambda^2 - 16\zeta_\Delta^2)}, \tag{57}$$

where b_3 is given by the $z = 0$ limit of eq. (42). It is evident from figure 6 that the partial decoherence is attained more rapidly in the case of classical noise.

5. Concluding remarks

Our main emphasis in this paper has been a relative assessment of quantum and classical noise sources attached to a pair of quantum dots or a qubit. The analysis has been made with the aid of a unified formalism, comprising a resolvant expansion of an averaged time-development operator. The quantum formalism enabled us to provide a microscopic meaning and detailed temperature dependence to the phenomenologically introduced parameter of the relaxation rate, that appears in the classical case of a telegraphic noise. Our treatment of the quantum noise has included the much-studied Ohmic model of dissipation that characterizes electron–hole excitations off the Fermi surface, in a metal, as well as non-Ohmic dissipation which covers both acoustic and optic phonons [15]. It is eventually in the phonon model of dissipation that the usually assumed exponential temperature dependence of the relaxation rate (of a telegraph process) is realized, making the comparison between the classical and quantum cases more direct.

In the last subsection of the paper (§4) we focussed on the important issue of partial decoherence that can be utilized for quantum computation, which has received recent

attention [12]. This case was analysed when fluctuation is ascribed only to the hopping term, yielding a situation in which the system Hamiltonian commutes with the coupling to the heat bath. Here, the comparison between classical and quantum noises is quite striking – coherence persists over longer time-scales for the quantum case. This attribute can be effectively exploited in the context of quantum computation, in which it is essential to be able to retrieve information about the initial quantum state, notwithstanding heat bath-induced effects.

Acknowledgements

Appendix A. Results for telegraph process

A.1 *Fluctuation in bias ($\zeta_\Delta = 0$) under telegraph process*

The stochastic Hamiltonian appropriate to eqs (35) and (36) is given by (see eq. (6))

$$\mathcal{H}_s = \Delta\sigma_z + \eta(t)\zeta_\epsilon\sigma_x, \tag{A.1}$$

where $\eta(t)$ is a telegraph process. Concomitantly eq. (24) is replaced by [18]

$$\rho_\mathcal{R}(z) = \left[\sum_{a,b} \langle a| \left(z + i\mathcal{L}_q + i\sum_{j=\pm} \mathcal{L}_j F_j - W \right)^{-1} |b\rangle p_b \right] \rho_q(0), \tag{A.2}$$

where the 'stochastic states' $|a\rangle$, $|b\rangle$... are associated with the two possible values ± 1 of $\eta(t)$ (over which the summations in eq. (A.2) are performed), p_b is the *a priori* probability of the occurrence of the state $|b\rangle$, $\mathcal{L}_q$ is the Liouville operator accompanying $\mathcal{H}_q = \Delta\sigma_z$ and $\mathcal{L}_j$ is the Liouville operator associated with an ordinary operator V_j ($j = +1$ or -1) such that

$$V_\pm = \pm\zeta_\epsilon\sigma_x. \tag{A.3}$$

It remains then to define the 'stochastic matrices' F_j and W. The matrix F_j is a projection governed by

$$\langle a|F_j|b\rangle = \delta_{ab}\delta_{aj}, \tag{A.4}$$

whereas W is a jump matrix whose element $\langle a|W|b\rangle$, for instance, is the rate of jump of the process $\eta(t)$ from the state $|b\rangle$ to the state $|a\rangle$. The underlying Markovian assumption is manifest in the frequency independence of W.

The properties of the telegraph process allow us to calculate the average over the stochastic indices $a, b,...$ in eq. (A.2) in closed form, leading to

$$\rho_\mathcal{R}(z) = \left[\frac{\bar{U}_0(z + \lambda)}{1 - \lambda\bar{U}_0(z + \lambda)} \right] \rho_q(0), \tag{A.5}$$

where

$$(\tilde{U}_0(z + \lambda)) = \sum_j p_j (z + \lambda + i\mathcal{L}_q + i\mathcal{L}_j)^{-1}, \tag{A.6}$$

λ being the jump rate.

In calculating the matrix elements of $\tilde{U}_0(z + \lambda)$ we need to use the properties of Liouville operators given by [19]

$$(\mu\nu|\mathcal{L}_j|\mu'\nu') = \langle\mu|V_j|\mu'\rangle\delta_{\nu\nu'} - \langle\nu'|V_j|\nu\rangle\delta_{\mu\mu'}. \tag{A.7}$$

Using (A.7), it is straightforward to obtain

$$(\bar{z} + i\mathcal{L}_q + i\mathcal{L}_\pm) \equiv \begin{bmatrix} \bar{z} & \mp i\zeta_\epsilon & \pm i\zeta_\epsilon & 0 \\ \mp i\zeta_\epsilon & \bar{z} + 2i\Delta & 0 & \pm i\zeta_\epsilon \\ \pm i\zeta_\epsilon & 0 & \bar{z} - 2i\Delta & \mp i\zeta_\epsilon \\ 0 & \pm i\zeta_\epsilon & \mp i\zeta_\epsilon & \bar{z} \end{bmatrix}, \tag{A.8}$$

where

$$\bar{z} = z + \lambda. \tag{A.9}$$

The matrix of $\bar{U}_0(\bar{z})$ is therefore of a simplified block-diagonal form from which matrix under the square brackets in eq. (A.5) can be evaluated easily. Combined with the initial condition in eq. (21) we find, after some straightforward algebra,

$$(\rho_\mathcal{R})_{++}(z) = \left[\frac{2(a + b) - \lambda(a^2 - b^2)}{D_1}\right](\rho_\mathcal{R})_{++}(0)$$

$$(\rho_\mathcal{R})_{--}(z) = \left[\frac{2(a + b) - \lambda(a^2 - b^2)}{D_1}\right](\rho_\mathcal{R})_{--}(0) \tag{A.10}$$

and

$$(\rho_\mathcal{R})_{+-}(z) = \left[\frac{2(c + b) - \lambda(dd^* - b^2)}{D_2}\right](\rho_\mathcal{R})_{+-}(0), \tag{A.11}$$

where

$$a = \frac{2(\bar{z}^2 + 2\zeta_\epsilon^2 + 4\Delta^2)}{\bar{z}[\bar{z}^2 + 4(\zeta_\epsilon^2 + \Delta^2)]},$$

$$b = \frac{4\zeta_\epsilon^2}{\bar{z}[\bar{z}^2 + 4(\zeta_\epsilon^2 + \Delta^2)]},$$

$$c = \frac{2[\zeta_\epsilon^2 + \bar{z}(\bar{z} - 2i\Delta)]}{\bar{z}[\bar{z}^2 + 4(\zeta_\epsilon^2 + \Delta^2)]},$$

$$D_1 = (2 - \lambda a)^2 - (\lambda b)^2,$$

$$D_2 = (2 - \lambda c)(2 - \lambda c^*) - (\lambda b)^2. \tag{A.12}$$

With the chosen initial condition in eq. (21), eq. (A.10) simplifies to $(\rho_R(z))_{++} = (\rho_R(z))_{--} = 1/2z$. It is of course a matter of ease to go from eqs (A.10) and (A.11) to the dot basis, with the aid of the transformation in eq. (2) to rederive the elements of the density operator, results of which are presented in figure A.1. The latter are in complete conformity with those of ref. [6] and display full decoherence at infinite times, i.e. $(\rho_\mathcal{R})_{LL} = (\rho_\mathcal{R})_{RR} = \frac{1}{2}$ and $(\rho_\mathcal{R})_{LR} = 0$ as expected.
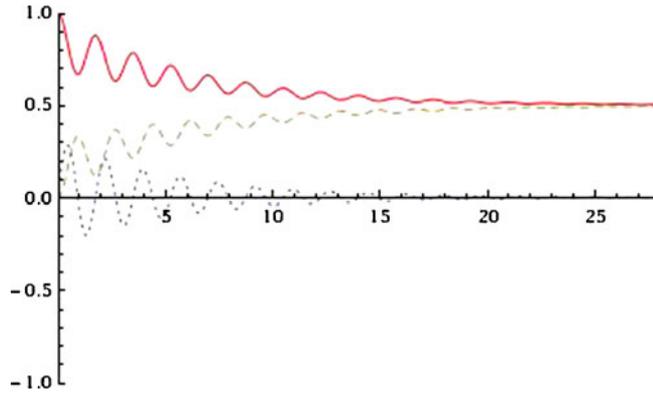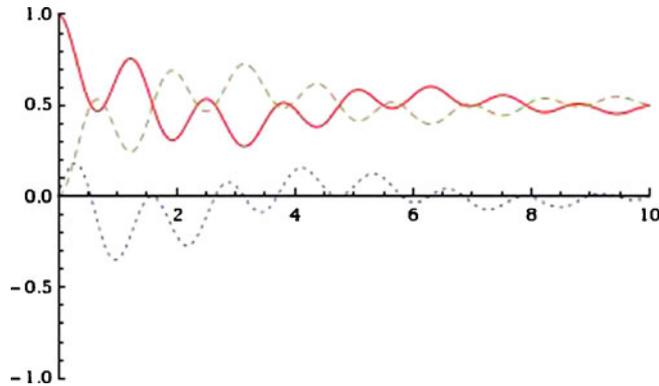
Figure A.1. $(\rho_{\mathcal{R}})_{LL}(t)$ (solid line), $(\rho_{\mathcal{R}})_{RR}(t)$ (dashed line) and $\text{Im}[(\rho_{\mathcal{R}})_{LR}(t)]$ (dotted line) are plotted vs. time t for $\Delta = 1$, $\zeta_\epsilon = 1.5$ and $\lambda = 0.5$.

A.2 *Fluctuation in hopping ($\zeta_\epsilon = 0$) under telegraph process*

The relevant stochastic Hamiltonian is now given by (see eq. (6))

$$\mathcal{H}_s(t) = \Delta\sigma_z + \eta(t)\zeta_\Delta\sigma_z. \tag{A.13}$$

In this case the fluctuating part of the Hamiltonian commutes with the system part of the Hamiltonian, a case which has received attention recently [12]. In this case now there is no energy transfer between the system and the bath, allowing for partial decoherence (see §4). In this case the calculations are simpler than in §A.1 as the matrix of $\bar{U}_0(\bar{z})$ is diagonal in the σ_z representation. Therefore, the diagonal elements of $\rho_R(z)$ do not evolve at all from their initial values of one-half whereas the off-diagonal element becomes zero. The above formalism is general and is valid for any kind of environment, as long as the commutator of the system Hamiltonian and the interaction Hamiltonian are zero. As shown in the cases of classical and quantum noises (figures 5 and 6), the

Figure A.2. $(\rho_{\mathcal{R}}(z))_{LL}(t)$ (solid line), $(\rho_{\mathcal{R}}(z))_{RR}(t)$ (dashed line) and $[(\rho_{\mathcal{R}}(z))_{LR}(t)]$ (dotted line) are plotted as a function of time t for $\Delta = 1$, $\zeta_\Delta = 1.5$ and $\lambda = 0.5$.

imaginary and real parts of $(\rho_{\mathcal{R}})_{+-}(t) \to 0$ as $t \to \infty$, whereas the populations of bonding and antibonding states stay constant, as is evident from eq. (43). Thus, in general, the off-diagonal elements of the density matrix (in $|L\rangle$ and $|R\rangle$ basis) approach limiting values, emphasizing that the asymptotic state is not a fully mixed state:

$$(\rho_{\mathcal{R}}(z))_{+-} = \frac{1}{2}\left[z + 2i\Delta + \frac{4\zeta_\Delta^2}{\bar{z} + 2i\Delta} \right]^{-1}. \tag{A.14}$$

The time dependence of the off-diagonal term $(\rho_{\mathcal{R}}(z))_{+-}(t)$ is obtained from the inverse Laplace transform of $(\rho_{\mathcal{R}}(z))_{+-}(z)$ to yield, for $\lambda > 4\zeta_\Delta$,

$$|(\rho_{\mathcal{R}}(z))_{+-}(t)| = \frac{1}{4\sqrt{\lambda^2 - 16\zeta_\Delta^2}}\left[\left(\sqrt{\lambda^2 - 16\zeta_\Delta^2} - \lambda \right) e^{\left(-\frac{1}{2}(\sqrt{\lambda^2 - 16\zeta_\Delta^2} + \lambda)t\right)} \right.$$
$$\left. + \left(\sqrt{\lambda^2 - 16\zeta_\Delta^2} + \lambda \right) e^{\left(-\frac{1}{2}(\lambda - \sqrt{\lambda^2 - 16\zeta_\Delta^2})t\right)} \right], \tag{A.15}$$

This expression coincides with the one derived by Itakura and Tokura [6] following an elaborate Dyson series-type time-domain treatment. The off-diagonal term initially follows a Gaussian decay, $|(\rho_{\mathcal{R}}(z))_{+-}(t)| \sim \frac{1}{2}\exp\{-2\zeta_\Delta^2 t^2\}$, discussed in [6], and it becomes exponential in the long-time regime. The time constant appearing in the exponential decay is referred to as T_2 following the NMR literature [4], turning out to be $T_2 = 1/\sqrt{|\lambda^2 - 16\zeta_\Delta^2|}$ for $\zeta_\Delta \ll \lambda$. Transforming to the dot basis the behaviour of the elements of the density operator is exhibited in figure A.2.

References

[1] C Bennett, *Phys. Today* **48**, 24 (1995)
     D P DiVincenzo, *Science* **270**, 255 (1995)
     A M Steane, *Rep. Prog. Phys.* **61**, 117 (1998)
[2] D Loss and D P DiVincenzo, *Phys. Rev.* **A57**, 120 (1998)
     T Tanamoto, *Phys. Rev.* **A61**, 022305 (2000)
[3] Y Nakamura, Yu A Pashkin and J S Tsai, *Nature (London)* **398**, 786 (1999)
[4] C P Slichter, *Principles of magnetic resonance* (Harper and Row, New York, 1963)
     A Abragam, *Principles of nuclear magnetism* (Oxford University Press, New York, 1961)
     L Allen and J H Eberly, *Optical resonance and two level atoms* (Dover Publications, New York, 1987)
[5] R J Glauber, *J. Math. Phys.* **4**, 294 (1963)
[6] T Itakura and Y Tokura, *Phys. Rev.* **B67**, 195320 (2003)
[7] G Gordon, *J. Phys. B: At. Mol. Opt. Phys.* **42**, 223001 (2009)
     G Gordon, N Erez and G Kurizki, *J. Phys. B: At. Mol. Opt. Phys.* **40**, S75 (2007)
     G Gordon, G Kurizki and D A Lidar, *Phys. Rev. Lett.* **101**, 010403 (2008)
[8] P Gaspard and S Jain, *Pramana – J. Phys.* **48**, 503 (1997)
     S Jain, *Nucl. Phys.* **A673**, 423 (2000)
[9] I Serban, M I Dykman and F K Wilhelm, *Phys. Rev.* **A81**, 022305 (2010)
[10] A J Leggett, S Chakravarty, A T Dorsey, M P A Fischer, A Garg and W Zwerger, *Rev. Mod. Phys.* **59**, 1 (1987)

[11] A Shnirman, G Schon and Z Hermon, *Phys. Rev. Lett.* **79**, 2371 (1997)
     Y M Galperin, B L Altshuler and D V Shanter, in: *Fundamental problems of mesoscopic physics* edited by I V Lerner, B L Altshuler and Y Gefen (Kluwer, Dordrecht, 2004) pp. 141–165
[12] A Aharony, S Gurvitz, O Entin-Wohlman and S Dattagupta, *Phys. Rev.* **B82**, 245417 (2010)
[13] S Dattagupta and S Puri, *Dissipative phenomena in condensed matter* (Springer Verlag, Heidelberg, 2004)
[14] S Dattagupta, H Grabert and R Jung, *J. Phys. Condens. Matter* **1**, 1405 (1989)
[15] U Weiss, *Quantum dissipative systems* (World Scientific, Singapore, 1993)
[16] T Qureshi and S Dattagupta, *Phys. Rev.* **B47**, 1092 (1993)
[17] S Dattagupta and H R Schober, *Phys. Rev.* **B57**, 7606 (1998)
[18] M Blume, *Phys. Rev.* **174**, 351 (1968)
[19] S Dattagupta, *Relaxation phenomena in condensed matter physics* (Academic Press, Orlando, 1987)