PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 79, No. 3
September 2012
pp. 345–356

Approximate solution of Schrödinger equation in D dimensions for inverted generalized hyperbolic potential

OLADUNJOYE A AWOGA and AKPAN N IKOT*

Theoretical Physics Group, Department of Physics, University of Uyo, Uyo, Nigeria
*Corresponding author. E-mail: ndemikot2005@yahoo.com

MS received 11 February 2012; revised 3 April 2012; accepted 18 April 2012

Abstract. The Nikiforov–Uvarov method is used to investigate the bound state solutions of Schrödinger equation with a generalized inverted hyperbolic potential in D-space. We obtain the energy spectrum and eigenfunction of this potential for arbitrary l-state in D dimensions. We show that the potential reduces to special cases such as Rosen–Morse, Poschl–Teller and Scarf potentials. The energy spectra and wave functions of these special cases are also discussed. The numerical results of these potentials are presented.

Keywords. Nikiforov–Uvarov method; Schrödinger equation; generalized inverted hyperbolic potential.

PACS Nos 03.65.Ge; 03.65.Fd; 03.65.Ca

1. Introduction

The analytical solution of Schrödinger equation (SE) is of great interest in quantum mechanics and different analytical methods have been used to solve it either exactly or approximately depending on the potential under consideration since their solutions contain all the necessary information regarding the quantum system. Among such methods are supersymmetric method [1], variational method [2], factorization method [3], shape invariance [4], asymptotic interaction method (AIM) [5,6] and others [7–9]. Recently, the study of exponential-type potentials have attracted the attention of many authors. Only few potentials can be solved exactly, for example the Coulomb potential and harmonic potential [10,11]. Many of the potentials for the description of physical system are not exactly solvable for non-vanishing angular momentum and thus, the centrifugal term is often approximated. In literatures, many authors have solved the Schrödinger equation for s-wave case. Nonetheless, approximations to the centrifugal term have been applied by different authors to obtain analytical approximation to the l-wave solutions of the Schrödinger equation with various potentials. Though the problems considered earlier are three-dimensional problems, the SE in generalized D dimensions for different potentials is getting the attention of researchers [12–28]. The multidimensional space analysis of the Klein–Gordon equation has also been investigated for different potentials [29–31].

In this paper, we consider the inverted generalized hyperbolic potential and solve the D-dimensional SE for arbitrary l using an approximate scheme via Nikiforov–Uvarov (NU) method [32–41]. Hyperbolic potentials such as Manning–Rosen, Scarf, Poschl–Teller and Rosen–Morse potentials are useful in PT-symmetry and are also used to model interatomic and intermolecular forces [42,43].

This paper is organized as follows: In §2 we give a brief review of the SE in D dimension. In §3 we take a cursory look at the NU method. In §4 we calculate the bound states of the generalized hyperbolic potential. In §5 and 6 we give a brief discussion and conclusion respectively.

2. Schrödinger equation in D dimensions

In analogy to the well-known 3D SE, the SE in D-dimensional space is [44]

$$-\frac{\hbar^2}{2\mu}[\nabla_D^2 + V(r)]\Psi_{nlm}(r,\Omega_D) = E_{nl}\Psi_{nlm}, \tag{1}$$

where the Laplacian operator is

$$\nabla_D^2 = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) - \frac{\Lambda_D^2(\Omega_D)}{r^2}. \tag{2}$$

The second term of (2) is the multidimensional space centrifugal term. Ω_D represents the angular coordinates. In this line, the operator Λ_D^2 yields hyperspherical harmonics as its eigenfunction. This helps us to write the wave function as

$$\Psi_{nlm}(r,\Omega_D) = R_{nl}(r)Y_l^m(\Omega_D), \tag{3}$$

where R_{nl} is the radial part of the equation and $Y_l^m(\Omega_D)$ is the angular part called hyperspherical harmonics. $Y_l^m(\Omega_D)$ obeys the eigenvalue equation

$$\Lambda_D^2 Y_l^m(\Omega_D) = l(l+D-2)Y_l^m(\Omega_D). \tag{4}$$

Substituting (3) into (1) and making use of (4), we obtain

$$\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial R_{nl}(r)}{\partial r}\right) + \frac{2\mu}{\hbar^2}\left[E_{nl} + \frac{l(l+D-2)}{r^2} - V(r)\right]R_{nl}(r) = 0. \tag{5}$$

Equation (5) is the SE in D-dimensional space.

3. Review of Nikiforov–Uvarov method

A complete detail of the NU method can be seen in [45]. We present here a brief description of the method. The NU method comes in handy in solving second-order differential equation by reducing it to a generalized equation of hypergeometric type. This method has been used to solve the SE, Dirac equation and Klein–Gordon wave equation for certain kind of potentials. In this method, the second-order differential equation can be written in the following form:

$$\Psi''(s)+\frac{\bar{\tau}(s)}{\sigma(s)}\Psi'(s)+\frac{\bar{\sigma}(s)}{\sigma^2(s)}\varphi(s)=0, \tag{6}$$

where $\sigma(s)$ and $\bar{\sigma}(s)$ are polynomials of at most of second degree and $\bar{\tau}(s)$ is at most first degree polynomial. To solve eq. (6), we invoke a common ansatz for the wave function as

$$\Psi(s)=\varphi(s)\chi(s). \tag{7}$$

Substituting eq. (7) into eq. (6), reduces eq. (6) to the form

$$\sigma(s)\chi''(s)+\bar{\tau}(s)\chi'(s)+\lambda\chi(s)=0, \tag{8}$$

where $\varphi(s)$ is defined as a logarithm derivative

$$\frac{\varphi'(s)}{\varphi(s)}=\frac{\pi(s)}{\sigma(s)} \tag{9}$$

and $\chi(s)$ is the hypergeometric-type function whose polynomials are given by the Rodrigues relation

$$\chi_n(s)=\frac{B_n}{\rho(s)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}[\sigma^n(s)\rho(s)], \tag{10}$$

where B_n is a normalization constant and the weight function $\rho(s)$ must satisfy the condition

$$\frac{\mathrm{d}}{\mathrm{d}s}(\sigma\rho)=\tau\rho. \tag{11}$$

The function $\pi(s)$ and the parameter λ required for NU method are defined as follows:

$$\pi(s)=\frac{\sigma'(s)-\tau(s)}{2}\pm\sqrt{\left(\frac{\sigma'(s)-\tau(s)}{2}\right)^2-\bar{\sigma}(s)+k\sigma(s)} \tag{12}$$

$$\lambda=k+\pi'(s). \tag{13}$$

On the other hand, in order to find the value of k in eq. (12), the expression under the square root must be the square of a polynomial. Thus, a new eigenvalue for the second-order differential equation becomes

$$\lambda=\lambda_n=-n\tau'(s)-\frac{n(n-1)}{2}\sigma''(s), \tag{14}$$

where

$$\tau(s)=\bar{\tau}(s)+2\pi(s). \tag{15}$$

The derivative of eq. (15) must be negative. By comparing eqs (13) and (14), we obtain the energy eigenvalues.

4. Bound states

The inverted generalized hyperbolic potential under consideration is given by [46]

$$V(r) = -aV_0 \coth(\alpha r) + bV_1 \coth^2(\alpha r) - cV_2 \operatorname{cosech}^2(\alpha r) + d, \tag{16}$$

where V_0, V_1 and V_2 are potential depths and a, b, c and d are real constants.

Equation (16) yields some special cases by varying the values of the real constants a, b, c, and d. The following special cases can be obtained from eq. (16) by making adjustment to the parameters as

$$\text{(i)}\ \ V_{-a,0,c,0}(r) = aV_0 \coth(\alpha r) - cV_2 \operatorname{cosech}^2(\alpha r), \tag{17}$$

$$\text{(ii)}\ \ V_{0,0,c,0}(r) = -cV_2 \operatorname{cosech}^2(\alpha r), \tag{18}$$

$$\text{(iii)}\ \ V_{0,b,0,0}(r) = bV_1 \coth^2(\alpha r), \tag{19}$$

$$\text{(iv)}\ \ V_{0,0,0,d}(r) = d. \tag{20}$$

The potentials in eqs (17)–(20) are the Rosen–Morse potential, Poschl–Teller potential, Scarf potential and constant potential respectively. The behaviour of the potential of eq. (16) is shown in figure 1 and the behaviour of potentials in eqs (17)–(19) are depicted in figures 2–4. The form of the SE in eq. (5) is not suitable for our quest since we require bound state solution. Hence, we choose the radial wave function in the form

$$R_{nl}(r) = r^{-(D-1)/2} U_{nl}(r). \tag{21}$$

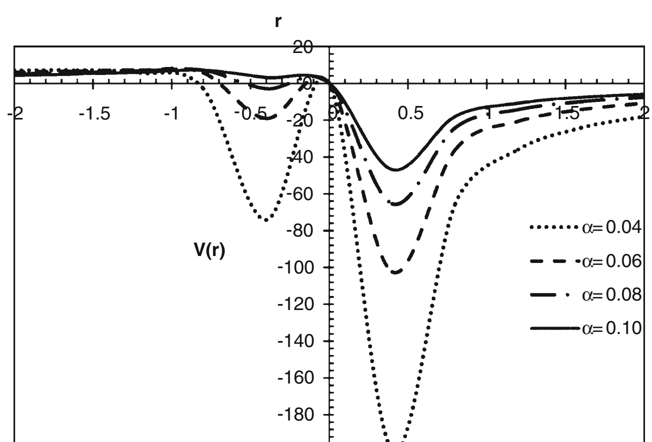

Figure 1. A plot of inverted generalized hyperbolic potential with r for $a = 1$, $b = 0.01$, $c = 2.00$, $d = 0.02$, $V_0 = 1.00$ MeV, $V_1 = 0.5$ MeV, $V_2 = 0.02$ MeV and $\alpha = 0.04$, 0.06, 0.08 and 0.10.

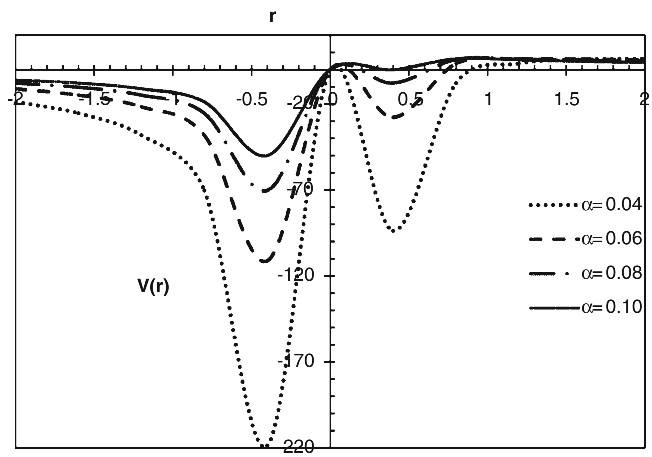

Figure 2. Variation of Rosen–Morse potential with r for $a=-1$, $b=0$, $c=2$, $d=0$, $V_0=1.00$ MeV, $V_1=0.5$ MeV, $V_2=0.02$ MeV for $\alpha=0.04$, 0.06, 0.08 and 0.10.

Substituting eqs (21) and (16) in eq. (5), we obtain

$$U''_{nl}(r)+\frac{2\mu}{\hbar^2}\left[E_{nl}+aV_0\coth(\alpha r)-bV_1\coth^2(\alpha r)+cV_2\,\mathrm{cosech}^2(\alpha r)-d\right]$$
$$-\left[\left(\frac{(D-1)(D-3)}{4}+l(l+D-2)\right)\frac{1}{r^2}\right]U_{nl}(r)=0. \tag{22}$$

We now introduce the approximation for the centrifugal term as [47]

$$\frac{1}{r^2}\approx\alpha^2\,\mathrm{cosech}^2(\alpha r). \tag{23}$$

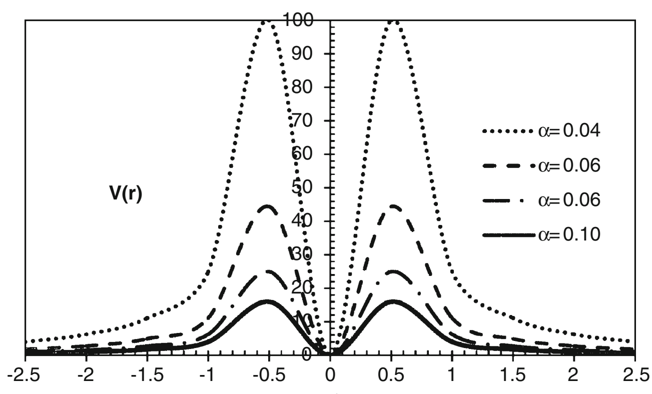

Figure 3. A plot of Poschl–Teller potential with r for $a=0$, $b=0$, $c=-2$, $d=0$, $V_0=1.00$ MeV, $V_1=0.5$ MeV, $V_2=0.02$ MeV for $\alpha=0.04$, 0.06, 0.08 and 0.1.

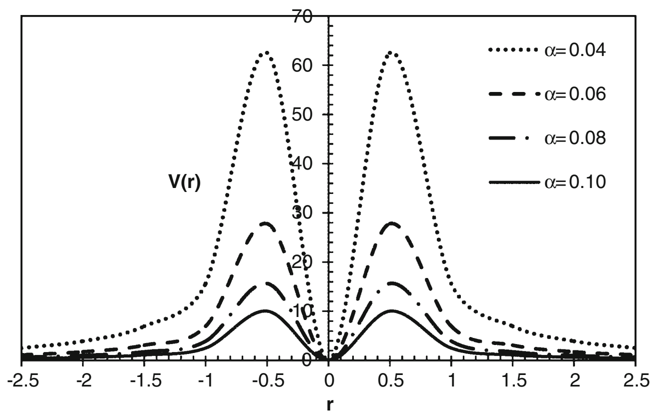

Figure 4. A plot of Scarf potential with r for $a = 0$, $b = 0.05$, $c = 0$, $d = 0$, $V_0 = 1.00$ MeV, $V_1 = 0.5$ MeV, $V_2 = 0.02$ MeV for $\alpha = 0.04$, 0.06, 0.08 and 0.1.

With this, eq. (22) becomes

$$U''(r)+\frac{2\mu}{\hbar^2}\left\{E_{nl}+aV_0\coth(\alpha r)-bV_1\coth^2(\alpha r)+\left[cV_2-\frac{\hbar^2\alpha^2}{2\mu}\right.\right.$$
$$\left.\left.\times\left(\frac{(D-1)(D-3)}{4}+l(l+d-2)\right)\right]\operatorname{cosech}^2(\alpha r)-d\right\}U(r)=0. \tag{24}$$

Introducing the transformation,

$$s=\coth(\alpha r) \tag{25}$$

leads eq. (24) into the form,

$$U''_{nl}(s)+\frac{2s}{s^2-1}U'_{nl}(s)+\frac{1}{(s^2-1)^2}\left[\gamma^2 s^2+\beta^2 s-\varepsilon^2\right]U_{nl}(s)=0, \tag{26}$$

with

$$-\varepsilon^2=\frac{2\mu E}{\hbar^2\alpha^2}-\frac{2\mu d}{\hbar^2\alpha^2}-\frac{2\mu cV_2}{\hbar^2\alpha^2}+\left[\frac{(D-1)(D-3)}{4}+l(l+D-2)\right], \tag{27}$$

$$\beta^2=\frac{2\mu aV_0}{\hbar^2\alpha^2}, \tag{28}$$

$$\gamma^2=\frac{2\mu cV_2}{\hbar^2\alpha^2}-\frac{2\mu bV_1}{\hbar^2\alpha^2}-\left[\frac{(D-1)(D-3)}{4}+l(l+D-2)\right]. \tag{29}$$

Comparing eqs (26) and (6), we obtain the following polynomials:

$$\bar{\tau}(s)=2s,\qquad \sigma(s)=s^2-1\qquad\text{and}\qquad \bar{\sigma}(s)=\gamma^2 s^2+\beta^2 s-\varepsilon^2. \tag{30}$$

Substituting these polynomials into eq. (13), we get

$$\pi(s) = \sqrt{(k-\gamma^2)s^2 - \beta^2 s - (k-\varepsilon^2)}. \tag{31}$$

The expression within the square root must be square of a polynomial according to the NU method. Therefore, we obtain $\pi(s)$ as

$$\pi\,(s) = \pm \begin{cases} \omega s - \eta, & \text{for } k = \varepsilon^2 + \gamma^2 - \sqrt{(\varepsilon^2-\gamma^2)^2 - \beta^4} \\ \omega s + \eta, & \text{for } k = \varepsilon^2 + \gamma^2 + \sqrt{(\varepsilon^2-\gamma^2)^2 - \beta^4} \end{cases}, \tag{32}$$

where

$$\omega = \sqrt{\frac{(\varepsilon^2-\gamma^2) - \sqrt{(\varepsilon^2-\gamma^2)^2-\beta^4}}{2}} \tag{33}$$

and

$$\eta = \frac{\beta^2}{\sqrt{2[(\varepsilon^2-\gamma^2) - \sqrt{(\varepsilon^2-\gamma^2)^2-\beta^4}]}}. \tag{34}$$

$\pi(s)$ and k values that will give negative derivatives are given as

$$\pi(s) = -\omega s + \eta, \tag{35}$$

$$k = \varepsilon^2 + \gamma^2 - \sqrt{(\varepsilon^2-\gamma^2)^2-\beta^4}. \tag{36}$$

Hence, from eq. (15), we get

$$\tau(s) = 2(1-\omega)s + 2\eta \tag{37}$$

and

$$\tau'(s) = 2(1-\omega). \tag{38}$$

The derivative will be negative for $\omega > 1$ and thus the bound state conditions are satisfied.

From eqs (13) and (14), we obtain

$$\lambda = (\varepsilon^2+\gamma^2) - \sqrt{(\varepsilon^2-\gamma^2)^2-\beta^4} - \omega \tag{39}$$

and

$$\lambda_n = (2\omega-1)\,n - n^2. \tag{40}$$

Solving eqs (39) and (40) while keeping eq. (33) in mind, we arrived at

$$\varepsilon^2 = 2\left[\frac{\beta^2}{2\,(n+\sigma_\pm)} - \frac{n+\sigma_\pm}{2}\right]^2 + \beta^2 + \gamma^2, \tag{41}$$

where

$$\sigma_\pm = \frac{1}{2}\left[1 \pm \sqrt{1-4\gamma^2}\right]. \tag{42}$$

Substituting eq. (27) in eq. (41), we obtain the energy eigenvalues as

$$E_{nl} = -\frac{\hbar^2\alpha^2}{\mu}\left[\frac{\mu a V_0}{\hbar^2\alpha^2\,(n+\sigma_\pm)} - \frac{n+\sigma_\pm}{2}\right]^2 - aV_0 + bV_1 + d. \tag{43}$$

Special cases

In this subsection, we consider the special cases of our potential discussed earlier.

(I) *Rosen–Morse*: To reduce our potential to the Rosen–Morse potential we make the adjustments $a = -a$, $b = d = 0$, $c = c$ and $D = 3$ in eq. (43). With these we get

$$E_{nl}^{\mathrm{RM}} = -\frac{\hbar^2\alpha^2}{\mu}\left[\frac{-\mu a V_0}{\hbar^2\alpha^2(n+\upsilon_\pm)} - \frac{n+\upsilon_\pm}{2}\right]^2 + aV_0, \tag{44}$$

where

$$\upsilon_\pm = \frac{1}{2}\left[1 \pm \sqrt{1 + 4\left[l(l+1) - \frac{2\mu c V_2}{\hbar^2\alpha^2}\right]}\right]. \tag{45}$$

Choosing υ_+ gives the well-known eigenvalues of the Rosen–Morse potential.

(II) *Poschl–Teller*: When we make the adjustments $a = b = d = 0$, $c = c$ and $D = 3$, we obtain

$$E_{nl}^{\mathrm{PT}} = -\frac{\hbar^2\alpha^2}{\mu}\left[\frac{n+Q_\pm}{2}\right]^2 \tag{46}$$

as the energy spectrum of the Poschl–Teller potential, where

$$Q_\pm = \frac{1}{2}\left[1 \pm \sqrt{1 + 4\left[l(l+1) - \frac{2\mu c V_2}{\hbar^2\alpha^2}\right]}\right]. \tag{47}$$

(III) *Scarf*: Setting $a = c = d = 0$, $b = b$ and $D = 3$, we obtain the eigenvalue of the Scarf potential as

$$E_{nl}^{\mathrm{S}} = -\frac{\hbar^2\alpha^2}{\mu}\left[\frac{n+\rho_\pm}{2}\right]^2 + bV_1, \tag{48}$$

where

$$\rho_\pm = \frac{1}{2}\left[1 \pm \sqrt{1 + 4\left[l(l+1) + \frac{2\mu b V_1}{\hbar^2\alpha^2}\right]}\right]. \tag{49}$$

(IV) *Constant potential*: Finally, we set $a = b = c = 0$, $\alpha = a/2$ and $D = 3$ and use σ_- in eq. (43) to obtain the energy of the constant potential as

$$E_n^{\mathrm{C}} = -\frac{\hbar^2 a^2 n^2}{2\mu} + d. \tag{50}$$

In eqs (46) and (48), Q_+ and ρ_+ are the values that will yield the eigenvalues of the Poschl–Teller potential and Scarf potential respectively.

Having obtained the eigenvalues of the generalized inverted hyperbolic potential, we turn our attention to its corresponding wave function. Using eqs (30) and (35) in eq. (9) and carrying out the simple integration involved, we obtain the first part of the wave function as

$$\varphi(s) = A\frac{(s-1)^{\eta/2}}{(s^2-1)^{\omega/2}(s+1)^{\eta/2}}. \tag{51}$$

Also using the value of $\sigma(s)$ and $\tau(s)$ in eq. (11), we obtain the weight function as

$$\rho(s) = \frac{(s-1)^{\eta}}{(s^2-1)^{\omega}(s+1)^{\eta}}. \tag{52}$$

Finally, substituting eq. (52) and $\sigma(s)$ into eq. (10) we get

$$\chi_n(s) = B_n(s^2-1)^{\omega}(s-1)^{-\eta}(s+1)^{\eta}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left[(s^2-1)^{n-\omega}(s-1)^{\eta}(s+1)^{-\eta}\right]. \tag{53}$$

This can be further written as

$$\chi_n(s) = B_n(1-s)^{\omega-\eta}(1+s)^{\omega+\eta}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left[(1-s)^{n-\omega+\eta}(1+s)^{n-\omega-\eta}\right]. \tag{54}$$

Equation (54) is in a form reminiscent of the Jacobi polynomials. The polynomials are documented in literatures, whose standard form is [48–50]

$$P_n^{\alpha,\beta}(s) = \frac{(-1)^n}{2^n n}(1-s)^{n+\alpha}(1+s)^{n+\beta}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left[(1-s)^{n+\alpha}(1+s)^{n+\beta}\right]. \tag{55}$$

Comparing eqs (54) and (55),

$$\chi_n(s) = N_n P_n^{(\eta-\omega,-\eta-\omega)}(s). \tag{56}$$

The total wave function is obtained by substituting eqs (51) and (56) into eq. (7) as

$$U(s) = N_n(1-s)^{(\eta-\omega)/2}(1+s)^{(\eta+\omega)/2}P_n^{(\eta-\omega,-\eta-\omega)}(s). \tag{57}$$

Again using eq. (25) in eq. (57), we get

$$U_{nl} = N_n[1-\coth(\alpha r)]^{(\eta-\omega)/2}[1+\coth(\alpha r)]^{(\eta+\omega)/2}P_n^{(\eta-\omega,-\eta-\omega)}(\coth(\alpha r)). \tag{58}$$

Finally, the total wave function becomes

$$\Psi_{nlm}(r) = N_n[1-\coth(\alpha r)]^{(\eta-\omega)/2} \times [1+\coth(\alpha r)]^{(\eta+\omega)/2}\, P_n^{(\eta-\omega,-\eta-\omega)}(\coth(\alpha r))Y_l^m(\Omega_D), \tag{59}$$

where N_n is the normalization constant.

5. Results and discussion

The well-known potentials are obtained from the generalized inverted hyperbolic potential if we make appropriate choice for the values of the parameters in the generalized inverted potentials as stated in §4. We plotted the variation of the generalized inverted hyperbolic potential as a function of r for $a = 1$, $b = 0.01$, $V_0 = 1.0$ MeV, $V_1 = 0.5$ MeV, $C_2 = 2$, $V_2 = 0.02$ MeV, $d = 2.0$ MeV at different parameters of $\alpha = 0.04$, 0.06, 0.08 and 0.10 as displayed in figure 1.

Rosen–Morse potential: For $b = d = 0$, the Rosen–Morse potential is obtained as given in eq. (16). We plotted the variation of Rosen–Morse $V(r)$ with r for $a = -1$, $V_0 =$ 1 MeV, $c = 2$ and $V_2 = 0.02$ MeV for different α parameters of $\alpha = 0.04$, 0.06, 0.08 and 0.10 in figure 2. Substituting $b = d = 0$ in eqs (43) and (59), we obtain the energy spectrum and the wave function of the Rosen–Morse potential respectively.

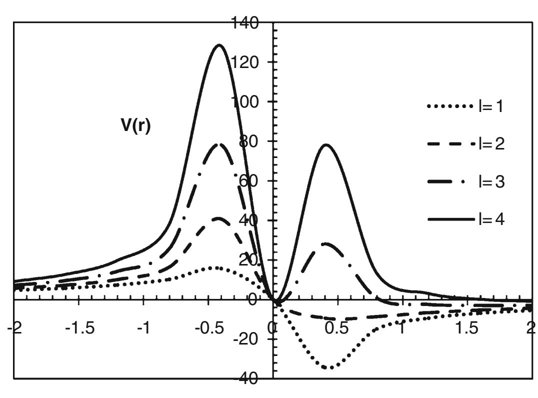

Figure 5. The variation of the effective potential as a function of r for $l = 1, 2, 3$ and 4 with $\alpha = 0.10$, $D = 3$.

Poschl–Teller potential: Poschl–Teller potential is obtained from the generalized inverted hyperbolic potential by setting $a = b = d = 0$ and $c = -c$ as given in eq. (14). The Poschl–Teller potential is plotted as a function of r for $c = -2$ and $V_2 = 0.02$ MeV in figure 3. Substituting these parameters in the energy equation of eq. (43) and the wave function eq. (59), we obtain the desired energy spectrum and the wave function of the Poschl–Teller potential.

Scarf potential: We can deduce the Scarf potential from the generalized inverted hyperbolic potential by setting $a = c = d = 0$. We display in figure 4 the plot of Scarf potential as a function of r for $b = 0.05$, $V_1 = 0.5$ MeV for different parameters of $\alpha = 0.04$, 0.06, 0.08 and 0.10. Setting the above limiting values in eqs (43) and (59) we obtain the energy eigenvalues and wave function for the Scarf potential respectively.

6. Conclusion

The NU method is used to investigate a generalized inverted hyperbolic potential in D-space. Three well-known potentials, viz., Rosen–Morse, Poschl–Teller and Scarf including the constant potential are deduced from this potential. We discussed that the bound state energy spectrum will only be valid for $\omega > 1$. This is imposed by the requirement of the NU method that the derivative of $\tau(s)$ must be negative. We also obtained the eigenfunctions of the potential for arbitrary l-state in terms of the Jacobi polynomials. In figure 5 we give a plot of the effective potential as a function of r in three dimensions ($D = 3$) for different values of $l = 1, 2, 3$ and 4.

References

[1] F Cooper, A Khare and U Sukhatme, *Phys. Rep.* **251**, 267 (1995)
[2] A N Ikot, L E Akpabio, K Essien, E E Ituen and I B Obot, *Electr. J. Chem.* **6(1)**, 113 (2009)
[3] S H Dong, *Factorization method in quantum mechanics* (Springer, Dordrecht, 2007)
[4] R Niazian, *Afri. Phys. Rev.* **4(008)**, 58 (2010)
[5] H Ciftci, R L Hall and N Saad, *J. Phys.* **A36**, 11807 (2003)
[6] O Özer, *Chin. Phys. Lett.* **25(9)**, 3111 (2008)
[7] F Taskin and G Kocak, *Chin. Phys.* **B19(9)**, 090314 (2010)
[8] X Y Gu and S H Dong, *J. Math. Chem.* **49**, 2053 (2011)
[9] A Belafhal and A Tmiri, *Phys. Chem. News* **60**, 27 (2011)
[10] B Gonul, O Ozer and M Kocak, *Comm. Theor. Phys.* **45**, 807 (2006)
[11] K J Oyewumi, F O Akinpelu and A D Agboola, *Int. J. Theor. Phys.* **47**, 1039 (2008)
[12] J L Cardoso and R Alvarez-Nodirse, *J. Phys.* **A36**, 2055 (2003)
[13] S H Dong and G H Sun, *Phys. Lett.* **A314**, 261 (2003)
[14] J L Coelho and R L P G Amaral, *J. Phys.* **A35**, 5255 (2002)
[15] S H Dong, C Y Chen and M L Cassou, *J. Phys.* **B38**, 2211 (2005)
[16] G Chen, *Chin. Phys.* **14**, 1075 (2005)
[17] S H Dong and M L Cassou, *Int. J. Mod. Phys.* **E13(5)**, 917 (2004)
[18] K J Oyewumi and A W Ogunsola, *J. Pure Appl. Sci.* **10(2)**, 343 (2004)
[19] S H Dong and M L Cassou, *Int. J. Quant. Chem.* **102(2)**, 147 (2005)
[20] Z Q Ma, S H Dong, X Y Gu, J Yu and M L Cassou, *J. Mod. Phys.* **E13(3)**, 97 (2004)
[21] H Hassanabadi, M Hamzavi, S Zarrinkanar and A A Rajabi, *Int. J. Phys. Sci.* **6(3)**, 583 (2011)

[22] G Paz, *Eur. J. Phys.* **22**, 337 (2001)
[23] S H Dong, G H Sun and Popov, *J. Math. Phys.* **44(10)**, 4467 (2003)
[24] X Y Gu, Z Q Ma and S H Dong, *Phys. Rev.* **A67(6)**, 062715 (2003)
[25] S Ikhadair and R Sever, *J. Mol. Struc: Theochem.* **855**, 13 (2008)
[26] S H Dong and Z Q Ma, *Phys. Rev.* **A65**, 042717 (2002)
[27] H Hassanabadi, S Zarrinkamar, H Hamzavi and A A Rajabi, *Arab. J. Sci. Eng.* **37**, 209 (2012)
[28] H Hassanabadi, B H Yazarloo and L L Lu, *Chin. Phys. Lett.* **29(2)**, 020303 (2012)
[29] H Hassanabadi, S Zarrinkamar and H Rahimov, *Commun. Theor. Phys.* **56(3)**, 423 (2011)
[30] Y Xu, S He and C Jia, *Phys. Scr.* **81**, 045001 (2010)
[31] H Hassanabadi, H Rahimov and S Zarrinkamar, *Adv. High Energy Phys.* **2011** (2011), DOI:10.1155/2011/4588087
[32] A N Ikot, L E Akpabio and J A Obu, *J. Vect. Relat.* **6**, 1 (2011)
[33] A N Ikot, O A Awoga, L E Akpabio and A D Antia, *Elixir Vib. Spec.* **33**, 2303 (2011)
[34] A N Ikot, A D Antia, L E Akpabio and J A Obu, *J. Vect. Relat.* **6(2)**, 65 (2011)
[35] S M Ikhadair and R Sever, *J. Math. Chem.* **42**, 461 (2007)
[36] S M Ikhadair and R Sever, *Int. J. Theor.* **46(6)**, 461 (2007)
[37] S M Ikhadair and R Sever, *Cent. Eur J. Phys.* **6**, 685 (2008)
[38] S M Ikhadair, *Chin J. Phys* **46(3)**, 291 (2008)
[39] D Agboola, *Phys. Scr.* **80(6)**, 065304 (2009)
[40] D Agboola, *Chin. Phys. Lett.* **27(4)**, 040301 (2010)
[41] H Hassanabadi, E Maghsoodi, S Zarrinkamar and H Rahimov, *J. Math. Phys.* **53**, 022104 (2012)
[42] Y P Varshni, *Rev. Mod. Phys.* **29(4)**, 664 (1957)
[43] O Yesiltas, *Phys. Scr.* **75**, 41 (2006)
[44] H Hassanabadi, S Zarrinkamar and A A Rajabi, *Commun. Theor. Phys.* **55**, 541 (2011)
[45] A F Nikiforov and V B Uvarov, *Special functions of mathematical physics* (Birkhauser, Basel, 1988)
[46] A N Ikot, *Afr. Rev. Phys.* **6(0026)**, 221 (2011)
[47] R L Greene and C Aldrich, *Phys. Rev.* **A14**, 2363 (1976)
[48] M Abramowitz and I A Stegun, *Handbook of mathematical functions with formulas, graphs and mathematical tables*, 10th edn (Washington, 1972)
[49] A D Polyanin and A V Manzhinov, *Handbook of integral equations* (CRC Press LLC, New York, 1998)
[50] I S Gradshteyn and I M Rhyzhik, *Table of integrals, series and product* (Academic Press, New York, 2007)