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A procedure to construct exact solutions of nonlinear evolution equations

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Abstract. In this paper, we implemented the functional variable method for the exact solutions of the Zakharov–Kuznetsov-modified equal-width (ZK-MEW), the modified Benjamin–Bona–Mahony (mBBM) and the modified KdV–Kadomtsev–Petviashvili (KdV–KP) equations. By using this scheme, we found some exact solutions of the above-mentioned equations. The obtained solutions include solitary wave solutions, periodic wave solutions and combined formal solutions. The functional variable method presents a wider applicability for handling nonlinear wave equations.

Keywords. Exact solutions; the functional variable method; nonlinear wave equations.

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1. Introduction

The theory of nonlinear dispersive and dissipative wave motion has recently undergone much research. Phenomena in physics and other fields are often described by nonlinear evolution equations which play a crucial role in applied mathematics and physics. Furthermore, when an original nonlinear equation is directly solved, the solution will preserve the actual physical characters of the equations. Explicit solutions of nonlinear equations are of fundamental importance. Various methods for obtaining explicit solutions of nonlinear evolution equations are proposed. Many explicit exact methods are introduced in the literature. Among these methods, the inverse scattering method [1,2], the Hirota bilinear tranformation [3], the tanh–sech method [4–7], the sine–cosine method [8,9], the exp-function method [10–13] and the (G'/G)-expansion method [14–16] are some of the methods used to develop nonlinear dispersive and dissipative problems. All these methods give different structures to the solutions.

Some direct methods for building solutions of nonlinear wave equations use symbolic computation packages like, for example, *Mathematica*, *Maple*. With the development of computer science, directly searching for solutions of nonlinear differential equations has become more and more attractive. This is due to the availability of computer symbolic systems like *Maple* which allows us to perform some complicated and tedious algebraic calculation using a computer and help us to find new exact solutions of nonlinear differential equations.

Recently, Zerarka *et al* [17] proposed the so-called functional variable method to solve nonlinear evolution equations (NLEEs) arising in mathematical physics. Right after this pioneer work, this method became popular among the research community, and many studies refining the initial idea have been published [18–20]. The advantage of this method is that one treats nonlinear problems by essentially linear methods. It originated from the well-known tanh method and the homogeneous balance method.

Our objective in this study is two-fold: First, to stress the power of the functional variable method in tackling NLEEs for solitary wave solutions and periodic wave solutions. Secondly, to compare the other results in the literature. The remainder of this paper is organized as follows. In §2, we state the procedure in brief. In §3–5, we analyse our problems. A conclusion is given in §6.

2. The functional variable method

The features of this method can be presented as follows [17]. This method is suitable for ODEs. Also a given nonlinear partial differential equation (PDE), is written in several independent variables as

$$P(u, u_t, u_x, u_y, u_z, u_{xy}, u_{yz}, u_{xz}, ...) = 0, \tag{2.1}$$

where the subscript denotes partial derivative, P is some function and $u\,\{t, x, y, z, ...\}$ is called a dependent variable or unknown function to be determined.

First we introduce the new wave variable as $\xi = k(x + ct)$ or $\xi = x - ct$.

The nonlinear partial differential equation can be converted to an ordinary differential equation (ODE) as

$$Q(U, U', U'', U''', ...) = 0. \tag{2.2}$$

Let us make a transformation in which the unknown function U is considered as a functional variable in the form

$$U_\xi = F(U) \tag{2.3}$$

and some successive derivatives of U are

$$U_{\xi\xi} = \frac{1}{2}(F^2)',$$

$$U_{\xi\xi\xi} = \frac{1}{2}(F^2)''\sqrt{F^2},$$

$$U_{\xi\xi\xi\xi} = \frac{1}{2}\left[(F^2)'''F^2 + (F^2)''(F^2)'\right],$$
$$\vdots \tag{2.4}$$

where '′' stands for d/dU. The ODE (2.2) can be reduced in terms of U, F and its derivatives on using the expressions of (2.4) into (2.2) gives

$$R(U, F, F', F'', F''', F^{(4)}, ...) = 0. \tag{2.5}$$

The key idea of this particular form (2.5) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, eq. (2.5) provides the expression of F, and this, together with (2.3), give appropriate solutions to the original problem.

To illustrate how the method works, we examine some examples treated by other approaches. This matter is explained in the following section.

3. The modified Benjamin–Bona–Mahony equation

Consider the modified Benjamin–Bona–Mahony equation (mBBM) [21]

$$u_t + u_x + u^2u_x + u_{xxt} = 0. \tag{3.1}$$

This equation models long waves in a nonlinear dispersive system. The existence of the solutions of initial value problems for the mBBM equation has been considered in [22]. Yusufoglu and Bekir [23] used the tanh and the sine–cosine methods to obtain exact solutions of the mBBM equation. By the exp-function method, Yusufoglu [24] obtained new solitary solutions for the mBBM equations. Layeni and Akinola [25] used the hyperbolic auxiliary function method and reported some new exact solutions of the mBBM equation. Abbasbandy and Shirzadi used the first integral method in this equation [26].

Now, letting $u(x, t) = U(\xi)$, $\xi = x - ct$ in (3.1), we obtain,

$$-cU' + U' + U^2U' - cU''' = 0, \tag{3.2}$$

where $U = U(\xi)$ and prime denotes derivative with respect to ξ. Integrating (3.2) with respect to ξ and considering the zero constants for intergation, we obtain

$$(1 - c)U + \frac{1}{3}U^3 = cU''. \tag{3.3}$$

Then we use the transformation

$$U_\xi = F(U), \tag{3.4}$$

that will convert eq. (3.3) to

$$\frac{1-c}{c}U + \frac{1}{3c}U^3 = \frac{(F(U)^2)'}{2}, \tag{3.5}$$

where the prime denotes differentiation with respect to ξ. According to eq. (2.4), we get from (3.5) the expression of the function $F(U)$ which reads as

$$F(U) = \sqrt{U^2(\tfrac{1-c}{c} + \tfrac{U^2}{6c})}. \tag{3.6}$$

Using transformation (3.4),

$$U_\xi = F(U) \tag{3.7}$$

and then setting the constants of integration to zero, we can obtain the following result:

$$U(\xi) = -\sqrt{6(1-c)}\mathrm{csch}\left(\sqrt{\frac{1-c}{c}}\xi\right). \tag{3.8}$$

We can easily obtain the following hyperbolic solutions:

$$u_1(x,t) = -\sqrt{6(1-c)}\mathrm{csch}\left(\sqrt{\frac{1-c}{c}}(x-ct)\right) \tag{3.9}$$

$$u_2(x,t) = \sqrt{6(c-1)}\mathrm{sech}\left(\sqrt{\frac{1-c}{c}}(x-ct)\right). \tag{3.10}$$

For $(1-c)/c < 0$, it is easy to see that solutions (3.9) and (3.10) can reduce to periodic solutions as follows:

$$u_3(x,t) = \sqrt{6(c-1)}\,\csc\left(\sqrt{\frac{c-1}{c}}(x-ct)\right), \tag{3.11}$$

$$u_4(x,t) = \sqrt{6(c-1)}\,\sec\left(\sqrt{\frac{c-1}{c}}(x-ct)\right). \tag{3.12}$$

On comparison, we observe that our solutions (3.9)–(3.12) include the solutions of Yusufoglu [23] and Abbasbandy [26].

Remark 1. Stability of this equation was studied by Pava, Banquet and Scialom [27].

4. The modified KdV–KP equation

Using the idea of Kadomtsev and Petviashvili, who relaxed the restriction that the waves be strictly one-dimensional in the KdV equation, leads to the (2+1)-dimensional modified KdV–KP equation [28]:

$$\left(u_t - \frac{3}{2}u_x + 6u^2 u_x + u_{xxx}\right)_x + u_{yy} = 0. \tag{4.1}$$

This equation was investigated in the literature because it is used to model a variety of non-linear phenomena. First integral method was used to construct travelling wave solutions of this equation in [29].

Using the wave variable $\xi = x + y - ct$ and proceeding as before we find

$$\left(-cU' - \frac{3}{2}U' + 6U^2U' + U'''\right)' + U'' = 0. \tag{4.2}$$

Integrating eq. (4.2) and neglecting constants of integration, we find

$$U\left(c + \frac{1}{2}\right) - 2U^3 = U''. \tag{4.3}$$

Following eq. (2.4), it is easy to deduce from (4.3) the expression of the function $F(U)$ as

$$F(U) = \sqrt{c + \frac{1}{2}}U\sqrt{1 - \frac{2}{2c+1}U^2}, \tag{4.4}$$

or

$$F(U) = \sqrt{c + \frac{1}{2}}U\sqrt{1 - c_1U^2}, \tag{4.5}$$

where $c_1 = 2/(2c+1)$.

The solution of eq. (4.3) is obtained as

$$U(\xi) = \sqrt{-\frac{2c+1}{2}}\operatorname{csch}\left(-\sqrt{\frac{2c+1}{2}}\xi\right). \tag{4.6}$$

We can easily obtain the following hyperbolic solutions:

$$u_1(x, y, t) = -\sqrt{-\frac{2c+1}{2}}\operatorname{csch}\left[\sqrt{\frac{2c+1}{2}}(x + y - ct)\right], \tag{4.7}$$

$$u_2(x, y, t) = \sqrt{\frac{2c+1}{2}}\operatorname{sech}\left[\sqrt{\frac{2c+1}{2}}(x + y - ct)\right]. \tag{4.8}$$

For $(2c+1)/2 < 0$, it is easy to see that solutions (4.7) and (4.8) can reduce to periodic solutions as follows:

$$u_3(x, y, t) = \sqrt{\frac{2c+1}{2}}\csc\left[\sqrt{-\frac{2c+1}{2}}(x + y - ct)\right], \tag{4.9}$$

$$u_4(x, y, t) = \sqrt{\frac{2c+1}{2}}\sec\left[\sqrt{-\frac{2c+1}{2}}(x + y - ct)\right]. \tag{4.10}$$

By comparing our results and Wazwaz's results [28] with Taghizadeh *et al*'s results [29], it can be seen that the results are the same.

5. ZK-MEW equation

We next consider the Zakharov–Kuznetsov-modified equal-width (ZK–MEW) equations [30]

$$u_t + a(u^3)_x + (bu_{xt} + ru_{yy})_x = 0. \tag{5.1}$$

In (5.1) a, b and c are real valued constants. The first term represents the evolution term while the second term is the nonlinear term, and finally the third and fourth terms together, in parentheses, are the dispersion terms. Wazwaz [31] used the tanh and the sine–cosine methods to obtain exact solutions of the ZK-MEW equation. The first integral method was used to construct travelling wave solutions of this equation in [32].

Using the wave variable $\xi = x + y - ct$, the system (5.1) is carried to a system of ODEs

$$-cU' + a(U^3)' - bcU''' + rU''' = 0. \tag{5.2}$$

Integrating eq. (5.2) and neglecting constants of integration, we find

$$aU^3 - cU + U''(r - bc) = 0. \tag{5.3}$$

Following eq. (2.4), it is easy to deduce from (5.3) the expression of the function $F(U)$ as

$$F(U) = \sqrt{U^2\left(\frac{c}{r-bc} - \frac{aU^2}{2(r-bc)}\right)} \tag{5.4}$$

or

$$F(U) = U\sqrt{\frac{c}{r-bc}}\sqrt{1 - c_1U^2}, \tag{5.5}$$

where $c_1 = a/2c$. If we get

$$U_\xi = F(U) \tag{5.6}$$

then setting the constants of integration to zero we can obtain the following result:

$$U(\xi) = \sqrt{-\frac{2c}{a}}\operatorname{csch}\left(-\sqrt{\frac{c}{r-bc}}\xi\right). \tag{5.7}$$

We can easily obtain the following hyperbolic solutions:

$$u_1(x, y, t) = -\sqrt{-\frac{2c}{a}}\operatorname{csch}\left(\sqrt{\frac{c}{r-bc}}(x + y - ct)\right), \tag{5.8}$$

$$u_2(x, y, t) = \sqrt{\frac{2c}{a}}\operatorname{sech}\left(\sqrt{\frac{c}{r-bc}}(x + y - ct)\right). \tag{5.9}$$

For $c/(r - bc) < 0$, it is easy to see that solutions (5.8) and (5.9) can reduce to periodic solutions as follows:

$$u_3(x, y, t) = \sqrt{\frac{2c}{a}}\csc\left(\sqrt{\frac{c}{bc-r}}(x + y - ct)\right), \tag{5.10}$$

$$u_4(x, y, t) = \sqrt{\frac{2c}{a}}\,\sec\left[\sqrt{\frac{c}{bc-r}}(x+y-ct)\right]. \tag{5.11}$$

It can be shown that our solutions (5.8)–(5.11) include the solutions of Wazwaz's results

[31]. By a similar discussion, we can observe that Tascan's results [32] are special cases of our results. All the solutions reported in this paper have been verified with *Maple* by putting them back into the original eq. (5.1).

Remark 2. Stability of the following type of equations was studied by Kuznetsov and Dias [33].

$$U'' \pm \alpha U \pm \lambda U^3 = 0.$$

Therefore, eqs (3.3), (4.3) and (5.3) are stable with respect to Vakhitov–Kolokolov criterion.

Remark 3. By comparing our results and Kumar's results [34], it can be seen that the results are similar.

6. Conclusion

This paper presents a wider applicability for handling nonlinear evolution equations using the new applications of the functional variable method. An implementation of the new applications of this method is given by applying it to the Zakharov–Kuznetsov-modified equal-width, the modified Benjamin–Bona–Mahony and the modified KdV–Kadomtsev–Petviashvili equations. In addition, this method is also computerizable, which allows us to perform complicated and tedious algebraic calculation using a computer by the help of symbolic programs such as *Maple*, *Mathematica*, *Matlab*, and so on.

As a result, many exact solutions are obtained with the help of the symbolic system *Maple* including soliton solutions presented by hyperbolic functions sech and cosech and periodic solutions presented by sec and cosec. It is shown that the algorithm can also be applied to other NLEEs in mathematical physics. We shall extend this method to seek soliton-like solutions for some PDEs in the forthcoming works.

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