Collapse of a Bose gas: Kinetic approach

SHYAMAL BISWAS

Department of Physics, University of Calcutta, 92 A.P.C. Road, Kolkata 700 009, India
E-mail: sbiswas.phys.cu@gmail.com

Abstract.　We have analytically explored the temperature dependence of critical number of particles for the collapse of a harmonically trapped attractively interacting Bose gas below the condensation point by introducing a kinetic approach within the Hartree–Fock approximation. The temperature dependence obtained by this easy approach is consistent with that obtained from the scaling theory.

1. Introduction

For an ultracold Bose gas, interparticle interaction is characterized by s-wave scattering length (a_s) which can be tuned arbitrarily by the Feshbach resonance method [1]. For attractive ($a_s < 0$) interaction, a harmonically trapped Bose gas tends to increase its density in the central region of a trap. This tendency, of course, is opposed by the quantum and thermal fluctuations. If the number of atoms is greater than the critical number (N_c), then the central density increases strongly, and the zero-point and thermal fluctuations are no longer able to avoid the collapse of the gas. Consequently, the gas becomes unstable even for two-body interaction.

The stability and collapse of the Bose–Einstein condensates with negative scattering lengths have already been observed in the clouds of ultracold ^{7}Li [2] and ^{85}Rb [3] for temperatures (T) close to zero or well below the condensation point (T_c). Soon after the observation, many theories for the collapse have been proposed for $T \rightarrow 0$ [4–10], as well as for $T > 0$ [11–14]. The remarkable one among these (theories) was given by Baym and Pethick [4]. They proposed a scaling theory for $T = 0$, and we generalized their theory for $0 \leq T \leq T_c$ within the Hartree–Fock (H–F) approximation [14]. In the generalized theory, different parts of the free energy (grand potential) of our system were scaled by a parameter which reduces the length scale of the system as a result of attractive

interaction; and a critical number for the collapse was eventually calculated from a critical condition of existence of a metastable minimum of the grand potential [14]. In this brief report we shall calculate the same, but in a kinetic approach which is supposed to be the easiest way.

This time, for calculating the critical number, we shall not start from the free energy, but shall adopt a mere kinetic theory like approach based on the energy and pressure of the system. We shall start from the H–F energy of the system, and shall pick up the kinetic energy and interaction energy parts of the H–F energy. While the kinetic energy of the particles causes an outward pressure, the attractive interaction causes an inward pressure. For critical number of particles, magnitudes of the two pressures would be the same. Beyond the critical number of particles, the inward pressure would be larger than the outward one, and as a consequence, the whole system would collapse. Thus, we shall calculate the critical number, and shall show its temperature dependence. Our present technique is easier than that of the already existing theory [4,14] because outward and inward pressures appear as the first-order derivative of the two parts of energy with respect to the effective volume of the system, and the already existing technique involves a second-order derivative of the grand potential with respect to the scaling parameter for obtaining the critical condition of existence of its metastable minimum.

2. Qualitative result

Before going into the details of the kinetic approach, let us estimate the critical number by qualitative manner. Our system consists of a large number (N) of Bose particles, each of which is a three-dimensional isotropic harmonic oscillator with angular frequency ω and mass m. The system, of course, is in thermodynamic equilibrium with its surroundings at temperature T. For $T \to 0$, all the particles occupy the ground state, and the system can be well described by the ground state wave function $\Psi_0(\mathbf{r}) = \sqrt{N/(l^3\pi^{3/2})}e^{-r^2/2l^2}$ in the position ($\mathbf{r}$) space, where $l = \sqrt{\hbar/m\omega}$ is the confining length scale of the oscillators [15]. Thus, for $T = 0$, the density of the condensed particles is given by [15]

$$n_0(\mathbf{r}) = |\Psi_0(\mathbf{r})|^2 = \frac{N}{l^3\pi^{3/2}}e^{-r^2/l^2}. \tag{1}$$

On the other hand, the number density of the excited particles (in the absence of interaction) is given by [14,16]

$$n_T(\mathbf{r}) = \frac{1}{\lambda_T^3}g_{3/2}(e^{-m\omega^2r^2/2k_BT}), \tag{2}$$

where $\lambda_T = \sqrt{2\pi\hbar^2/mk_BT}$ is the thermal de Broglie wavelength and $g_{3/2}(x) = x + x^2/2^{3/2} + x^3/3^{3/2} + \cdots$ is a Bose–Einstein function of the real variable x.

Let us consider the attractive interaction potential as $V_{int}(\mathbf{r}) = g\delta^3(\mathbf{r})$, where $g = -4\pi\hbar^2a/m$ is the coupling constant and $a = -a_s$ is the absolute value of the s-wave scattering length [16–18]. Typical two-body interaction energy for N number of particles is $\sim N^2g/2l^3$. Due to this, the gas tends to increase the density at the central region of the trap. Well below T_c (i.e. for $T \to 0$), this tendency is resisted by the zero-point motion of the atoms. At the critical situation, the energy ($3N\hbar\omega/2$) for the zero-point

motion must be comparable to the typical interaction energy. So, for $T \to 0$, we must have $3N_c\hbar\omega/2 \sim N_c^2 g/2l^3$ at the critical situation. From this relation we may have $(N_c a/l) \sim 1$.

On the other hand, for $0 < T < T_c$, the typical energy of the system is $3\hbar\omega[kT/\hbar\omega]^4\zeta(4) \sim N^{4/3}\hbar\omega$ [15]. So, for this range of temperatures, we must have $N_c^{4/3}\hbar\omega \sim N_c^2 g/2l^3$ at the critical situation. From this relation, we may have $(N_c a/l) \sim [l/a]^{1/2} > 1$.

However, near T_c, the length scale of the system is $L_{T_c} \sim l\sqrt{k_B T_c/\hbar\omega} \sim lN^{1/6}$ [15] so that we can write $N_c^{4/3}\hbar\omega \sim N_c^2 g/2L_{T_c}^3$ at the critical situation, and consequently, we may have $(N_c a/l) \sim [l/a]^5 \gg 1$.

3. Quantitative result from scaling theory

From the above qualitative arguments we have got $(N_c a/l) \propto 1$ for $T \to 0$, $(N_c a/l) \propto [l/a]^{1/2}$ for $0 < T < T_c$, and $(N_c a/l) \propto [l/a]^5$ for $T = T_c$. The proportionality constants were determined by a scaling theory within H–F approximation [14]. The scaling theory gives the proportionality constants as 0.671 for $T = 0$ [4,14], $1.210(T/T_c)^6/(1 - (T/T_c)^3)^3$ for $0 < T < T_c$ [14], and 2.253 for $T = T_c$ [14].

In the following, we shall also calculate the same but in a different (kinetic) approach. This approach is supposed to be the easiest one, and, we want to know whether the easiest approach reproduces similar results.

4. Kinetic approach

Within the H–F approximation we can express the energy functional as [14,16]

$$E = \int d^3\mathbf{r}\left[\frac{\hbar^2}{2m}n_0|\nabla\phi_0|^2 + \sum_{i\neq 0}\frac{\hbar^2}{2m}n_i|\nabla\phi_i|^2\right.$$
$$+ V(r)n_0(\mathbf{r}) + V(\mathbf{r})n_T(\mathbf{r}) + \frac{g}{2}n_0^2(\mathbf{r})$$
$$\left.+ 2gn_0(\mathbf{r})n_T(\mathbf{r}) + gn_T^2(\mathbf{r})\right], \tag{3}$$

where $\{\phi_i\}$ represents the set of normalized H–F wave functions and $\{n_i\}$ represent their occupations. To evaluate the above energy functional of eq. (3) we have to know $n_0(\mathbf{r})$ and $n_T(\mathbf{r})$ for the interacting case. But, for the simplicity of calculation, we keep the forms of $n_0(\mathbf{r})$ and $n_T(\mathbf{r})$ as in eqs (1) and (2) [14,16]. Thus, we evaluate the energy functional in eq. (3) as [19]

$$E(t) = [(1 + 1)c_1(t) - c_2(t)]N\frac{\hbar^2}{ml^2}, \tag{4}$$

where

$$t = T/T_c,$$
$$c_1(t) = \frac{3}{4}(1 - t^3) + \frac{3}{2}\frac{N^{1/3}\zeta(4)t^4}{[\zeta(3)]^{4/3}}, \tag{5}$$

$$c_2(t) = \frac{1}{\sqrt{2\pi}} \frac{Na}{l} [1 - t^3]^2 + \sqrt{\frac{8\zeta(3/2)}{\pi}} \frac{N^{1/2}a}{l} t^{3/2} [1 - t^3]$$
$$+ S'\sqrt{\frac{2}{\pi[\zeta(3)]^3}} \frac{N^{1/2}a}{l} t^{9/2}, \tag{6}$$

and

$$S' = \sum_{i,j=1}^{\infty} \frac{1}{(ij)^{3/2}(i+j)^{3/2}} \approx 0.6534.$$

From eq. (4) we get the kinetic energy of the system as

$$E_k(t) = c_1(t)N\frac{\hbar^2}{ml^2}, \tag{7}$$

and the interaction energy of the system as

$$E_{\text{int}}(t) = -c_2(t)N\frac{\hbar^2}{ml^2} = -c_3(t)N^2a\frac{\hbar^2}{ml^3}, \tag{8}$$

where

$$c_3(t) = c_2(t)\frac{l}{Na}. \tag{9}$$

For $0 \leq T < T_c$, the effective volume of the system is given by $V = 4\pi l^3/3$. The outward pressure is $P_{\text{out}} = \partial E_k/\partial V$, and the inward pressure is $P_{\text{in}} = \partial E_{\text{int}}/\partial V$. For the critical number of particles, P_{in} would be equal and opposite to P_{out}. From this equality relation, we can write the expression of the critical number as

$$N_c = \frac{2c_1(t)}{3ac_3(t)}\left(\frac{3V}{4\pi}\right)^{1/3}. \tag{10}$$

4.1 *Result for* $T \to 0$

Equation (10) gives the expression of the critical number as

$$\frac{N_c a}{l} = \frac{2c_1(0)}{3c_3(0)} = \sqrt{\frac{\pi}{2}} = 1.253 \qquad \text{for} \quad T \to 0. \tag{11}$$

Now we see that our new result for $T \to 0$ is approximately two times larger than the analytic result (0.671) obtained from the scaling theory [4,14]. The numerical simulation and experimental results, for $T \to 0$, however, are 0.575 [20] and 0.459 [3] respectively.

4.2 *Results for* $0 < T < T_c$

To achieve the Bose–Einstein condensation, the necessary condition is such that $a/l \ll 1$ [16]. Thus, for $0 < T \leq T_c$, we can write

$$c_1(t) \approx \frac{3}{2} \frac{N^{1/3}\zeta(4)t^4}{[\zeta(3)]^{4/3}}$$
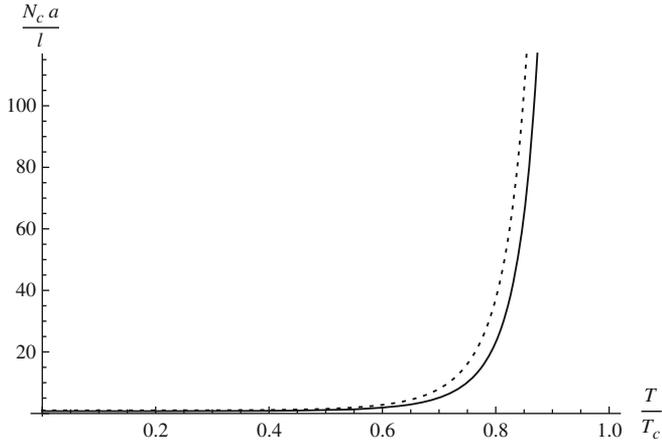
Figure 1. Solid line follows from eq. (12) and represents the plot for critical number of particles (in units of l/a) with respect to temperature (in units of T_c). Here $(l/a) = 0.0066$ [3]. Dotted line represents the results of the scaling theory [14].

from eq. (5). Similarly, for $0 < T \leq T_c$, we can write $c_3(t) \approx (1/\sqrt{2\pi})(1 - t^3)^2$ from eqs (9) and (6). Consequently, for $0 < T < T_c$, we get the expression of N_c from eq. (10) as

$$\frac{N_c a}{l} \approx \frac{2 \times \frac{3}{2} \frac{N_c^{1/3} \zeta(4) t^4}{[\zeta(3)]^{4/3}}}{3 \frac{1}{\sqrt{2\pi}}[1 - t^3]^2}.$$

From this relation we can write

$$\frac{N_c a}{l} = 0.779 \left[\frac{l}{a}\right]^{1/2} \frac{t^6}{(1 - t^3)^3} + \frac{0.750}{1 - t^3} + O\left(\frac{a}{l}\right)^{1/4} \tag{12}$$

for $0 < T < T_c$ up to a few leading orders in a/l. We plot the right-hand side of eq. (12) with respect to T in figure 1 for $a/l = 0.0066$ [3]. Now, we see that the nature of the temperature dependence of the critical number is the same as that obtained from the scaling theory [14].

4.3 *Result for $T = T_c$*

For $T = T_c$, the length scale of the system is $L_{T_c} = l N^{1/6}$ [14,16], and the effective volume of the system is given by $V = 4\pi l^3 N^{1/2}/3$. For this reason, the critical condition would be the same as that in eq. (10) except the factor V which is to be replaced by $V/N_c^{1/2}$. So, for $T = T_c$, the critical number would be obtained from

$$N_c = \frac{2 c_1(1)}{3 a c_3(1)} \left(\frac{3V}{4\pi N_c^{1/2}}\right)^{1/3}. \tag{13}$$

Putting the values of $c_1(1)$ and $c_3(1)$ into eq. (13) we get

$$\frac{N_c a}{l} = 96.062 \left[\frac{l}{a}\right]^5 \quad \text{for} \quad T = T_c. \tag{14}$$

Now we see that the critical number increases dramatically as the condensation point is approached.

Difference between our result and the scaling result [4,14] is appreciable as because the scaling theory deals with all the terms of H–F energy, and the scaled kinetic energy, although is greater than the scaled potential energy, yet comparable in particular for $T \to 0$. On the other hand, potential energy of the harmonic trap has not been considered in the analysis of our kinetic approach.

5. Conclusion

It is to be mentioned that we have considered the collapse of the condensate as well as the thermal cloud of the Bose gas only for short-ranged attractive interaction. Collapse of condensates for short-ranged repulsive and long-ranged attractive interactions have also been investigated experimentally [21] and theoretically [22,23].

Initially we explained the physics for the collapse of the attractive atomic Bose gas. Then we qualitatively estimated the critical number of particles for the collapse for $0 \leq T \leq T_c$. Finally, we calculate the same by a kinetic approach within the Hartree–Fock approximation. Our calculations support the qualitative estimations and the quantitative results obtained from the previous scaling theory [14].

Although the scaling theory was a more rigorous one, the nature of the temperature dependence of the critical number obtained by our kinetic theory (in figure 1) is similar to that obtained from the scaling theory. These two theories are equal *a priory* and self-consistent with respect to the basic physics of the collapse.

The scaling result, for $T \to 0$, is closer to the experimental data [3] which, however, is not available for the entire regime $0 < T \leq T_c$. The power of our approach is its simplicity. Future experiment for $0 < T \leq T_c$ may justify our approach.

Acknowledgements

References

[1] S Inouye, M R Andrews, J Stenger, H-J Miesner, D M S Kurn and W Ketterle, *Nature* **392** 151 (1998)
[2] C C Bradley, C A Sackett and R G Hulet, *Phys. Rev. Lett.* **78**, 985 (1997)
[3] J L Roberts, N R Claussen, S L Cornish, E A Donley, E A Cornell and C E Wieman, *Phys. Rev. Lett.* **86**, 4211 (2001)
[4] G Baym and C J Pethick, *Phys. Rev. Lett.* **76**, 6 (1996)
[5] L P Pitaevskii, *Phys. Lett.* **A221**, 14 (1996)
[6] Y Kagan, E L Surkov and G V Shlyapnikov, *Phys. Rev. Lett.* **54**, R1753 (1996)
[7] M Ueda and A J Leggett, *Phys. Rev. Lett.* **80**, 1576 (1998)

[8] S K Adhikari, *Phys. Rev.* **A66**, 013611 (2002)

[9] C M Savage, N P Robins and J J Hope, *Phys. Rev.* **A67**, 014304 (2003)

[10] V I Yukalov and E P Yukalova, *Phys. Rev.* **A72**, 063611 (2005)

[11] M Houbiers and H T C Stoof, *Phys. Rev.* **A54**, 5055 (1996)

[12] M J Davis, D A W Hutchinson and E Zaremba, *J. Phys. B: At. Mol. Opt. Phys.* **32**, 3993 (1999)

[13] E J Mueller and G Baym, *Phys. Rev.* **A62**, 053605 (2000)

[14] S Biswas, *Eur. Phys. J.* **D55**, 653 (2009)

[15] F Dalfovo, S Giorgini, L P Pitaevskii and S Stringari, *Rev. Mod. Phys.* **71**, 463 (1999)

[16] L P Pitaevskii and S Stringari, *Bose–Einstein condensation* (Oxford Sc. Pub., Oxford, UK, 2003)

[17] S Biswas, *Phys. Lett.* **A372**, 1574 (2008)

[18] S Biswas, J K Bhattacharjee, D Majumder, K Saha and N Chakravarty, *J. Phys. B: At. Mol. Opt. Phys.* **43**, 085305 (2010)

[19] It can be obtained from [14] by putting $\nu = 1$, where ν is the scaling parameter.

[20] P A Ruprecht, M J Holland, K Burnett and M Edwards, *Phys. Rev.* **A51**, 4704 (1995)

[21] T Koch, T Lahaye, J Metz, B Frohlich, A Griesmaier and T Pfau, *Nat. Phys.* **4**, 218 (2008)

[22] C Ticknor, N G Parker, A Melatos, S L Cornish, D H J O'Dell and A M Martin, *Phys. Rev.* **A78**, 061607(R) (2008)

[23] S K Haldar, B Chakrabarti and T K Das, *Phys. Rev.* **A82**, 043616 (2010)