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Lekhnitskii's formalism of one-dimensional quasicrystals and its application

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Abstract. By generalizing the complex potential approach developed by Lekhnitskii, plane problems of one-dimensional quasicrystals are solved first by using an octet formalism for which there are four pairs of complex roots. The approach uses a representation of stresses and proceeds by integration of the expressions for deformations and application of the anisotropic constitutive law and the compatibility of displacements. To illustrate its utility, the generalized Lekhnitskii's formalism is used to analyse the coupled phonon and phason fields in an infinite quasicrystal medium containing an elliptic rigid inclusion.

Keywords. Generalized Lekhnitskii's formalism; one-dimensional quasicrystals; plane problems; elliptic inclusion.

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1. Introduction

Since the discovery of icosahedral quasicrystals with five-fold symmetry in Al–Mn alloys [1], the physical properties including elasticity and defects of quasicrystals (QCs), have been intensively investigated in experimental and theoretical analyses. In particular, the linear elastic theory of QCs has been studied for many years [2–5]. Great progress has been made in the field of mechanics involving the theory of elasticity and defects (see [6,7] for details).

Among the many formalisms for two-dimensional (2D) deformations, Lekhnitskii's formalism is a very popular formalism among the engineering community. About 70 years ago, using a compliance-based formalism, Lekhnitskii developed a complex potential approach for the plate bending problems [8]. Lekhnitskii also developed a very similar but more popular formalism for 2D linear anisotropic elasticity [9]. Ting [10] gave a clear and comprehensive discussion of the method.

Nevertheless, there has been no systematic report in the Lekhnitskii's formalism for one-dimensional (1D) QCs. Due to its importance, a parallel development of the complex

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potential approach for 1D QCs which are formulated in the present paper should be allowed for applications to a broader class of problems. To illustrate the applications of the generalized Lekhnitskii's formalism developed in the present paper, an elliptic hole and a crack subjected to uniform strains at infinity are investigated.

2. Generalized elastic theory of 1D QCs

He *et al* [11] first found 1D QCs derived from the 2D decagonal QCs in rapidly solidified Al–Ni–Si, Al–Cu–Mn and Al–Cu–Co alloys. A 1D QC refers to a 3D solid structure with periodic arrangement in a plane and quasiperiodic arrangement in the orthogonal direction [12], where phonon and phason fields exist simultaneously. Wang *et al* [5] derived all 31 possible 1D QCs point groups, which can be further categorized into ten Laue classes and six 1D QCs systems: triclinic, monoclinic, orthorhombic, tetragonal, trigonal and hexagonal systems, and obtained a generalized Hooke law for 1D QCs. Recently, the general solutions for 3D problems of 1D QCs were presented by introducing several displacement functions [13,14].

According to the generalized elastic theory [3], the general equations governing the 3D theory of 3D QCs in the absence of body forces can be written as

$$\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \quad w_{3j} = \partial_j w_3, \tag{1}$$

$$\partial_j \sigma_{ij} = 0, \quad \partial_j H_{3j} = 0, \tag{2}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + R_{ij3l}w_{3l}, \quad H_{3j} = R_{kl3j}\varepsilon_{kl} + K_{3j3l}w_{3l}, \tag{3}$$

where $\partial_j = \partial/\partial x_j$, $i, j, k, l = 1, 2, 3$. Let u_i, σ_{ij} and ε_{ij} be, respectively, displacements, stresses and strains in the phonon filed, and w_3, H_{3j} and w_{3j} be, respectively, those in the phason field. C_{ijkl} and K_{3j3l} denote elastic constants in the phonon and phason fields, respectively, R_{ij3l} denote phonon–phason coupling elastic constants. An alternative expression using σ_{ij} and H_{3j} as controlling quantities is

$$\varepsilon_{ij} = \bar{s}_{ijkl}\sigma_{kl} + \bar{r}_{ij3l}H_{3l}, \quad w_{3j} = \bar{r}_{kl3j}\sigma_{kl} + \bar{k}_{3j3l}H_{3l}, \tag{4}$$

where $\bar{s}_{ijkl}$ and $\bar{k}_{3j3l}$ are elastic compliances in the phonon and phason fields, respectively, $\bar{r}_{ij3l}$ are phonon–phason coupling elastic compliances. The moduli in eqs (3) and (4) are obviously related. The material constants have the following symmetries:

$$\bar{s}_{ijkl} = \bar{s}_{jikl} = \bar{s}_{klij}, \quad \bar{r}_{ij3l} = \bar{r}_{ji3l}, \quad \bar{k}_{3j3l} = \bar{k}_{3l3j}.$$

Moreover, $\bar{s}_{ijkl}$ and $\bar{k}_{3j3l}$ are positive definite in the sense that

$$\bar{s}_{ijkl}\sigma_{ij}\sigma_{kl} > 0, \quad \bar{k}_{3j3l}H_{3j}H_{3l} > 0. \tag{5}$$

Introducing the contracted notation $\sigma_{11} = \sigma_1$, $\sigma_{22} = \sigma_2$, $\sigma_{33} = \sigma_3$, $\sigma_{23} = \sigma_4$, $\sigma_{31} = \sigma_5$, $\sigma_{12} = \sigma_6$, $\varepsilon_{11} = \varepsilon_1$, $\varepsilon_{22} = \varepsilon_2$, $\varepsilon_{33} = \varepsilon_3$, $2\varepsilon_{23} = \varepsilon_4$, $2\varepsilon_{31} = \varepsilon_5$, $2\varepsilon_{12} = \varepsilon_6$, the generalized Hooke's laws (4) can be rewritten as

$$\varepsilon_m = \bar{s}_{mn}\sigma_n + \bar{r}_{mj}H_{3j}, \quad w_{3i} = \bar{r}_{ni}\sigma_n + \bar{k}_{ij}H_{3j}, \tag{6}$$

where $m, n = 1, 2, ..., 6$.

For 2D deformations for which the displacements depend on x_1 and x_2 only, eq. (1) is written as

$$\varepsilon_1 = \partial_1 u_1, \quad \varepsilon_2 = \partial_2 u_2, \quad \varepsilon_3 = 0, \quad \varepsilon_4 = \partial_2 u_3, \quad \varepsilon_5 = \partial_1 u_3,$$
$$\varepsilon_6 = \partial_1 u_2 + \partial_2 u_1, \quad w_{31} = \partial_1 w_3, \quad w_{32} = \partial_2 w_3, \quad w_{33} = 0. \tag{7}$$

Furthermore, the strain components satisfy the compatibility relations

$$\partial_2^2 \varepsilon_1 + \partial_1^2 \varepsilon_2 = \partial_1 \partial_2 \varepsilon_6, \quad \partial_2 \varepsilon_5 = \partial_1 \varepsilon_4, \quad \partial_2 w_{31} = \partial_1 w_{32}. \tag{8}$$

The stresses σ_3 and H_{33} are determined by the formula

$$\sigma_3 = \frac{1}{\bar{r}_{33}^2 - \bar{s}_{33}\bar{k}_{33}}[(\bar{s}_{3p}\bar{k}_{33} - \bar{r}_{p3}\bar{r}_{33})\sigma_p + (\bar{r}_{3\alpha}\bar{k}_{33} - \bar{k}_{3\alpha}\bar{r}_{33})H_{3\alpha}],$$
$$H_{33} = \frac{1}{\bar{r}_{33}^2 - \bar{s}_{33}\bar{k}_{33}}[(\bar{r}_{p3}\bar{s}_{33} - \bar{s}_{3p}\bar{r}_{33})\sigma_p + (\bar{k}_{3\alpha}\bar{s}_{33} - \bar{r}_{3\alpha}\bar{r}_{33})H_{3\alpha}]. \tag{9}$$

Elimination of σ_3 and H_{33} in eq. (6) leads to

$$\varepsilon_p = s_{pq}\sigma_q + r_{p\beta}H_{3\beta}, \quad w_{3\alpha} = r_{p\alpha}\sigma_p + k_{\alpha\beta}H_{3\beta}, \tag{10}$$

where $p, q = 1, 2, 4, 5, 6$ and $\alpha, \beta = 1, 2$, and s_{pq}, $r_{p\beta}$ and $k_{\alpha\beta}$ are the reduced elastic compliances in eq. (10)

$$s_{pq} = s_{qp} = \bar{s}_{pq} + \frac{1}{\bar{r}_{33}^2 - \bar{s}_{33}\bar{k}_{33}}[\bar{s}_{p3}(\bar{s}_{3q}\bar{k}_{33} - \bar{r}_{q3}\bar{r}_{33}) + \bar{r}_{p3}(\bar{r}_{q3}\bar{s}_{33} - \bar{s}_{3q}\bar{r}_{33})],$$
$$r_{p\beta} = \bar{r}_{p\beta} + \frac{1}{\bar{r}_{33}^2 - \bar{s}_{33}\bar{k}_{33}}[\bar{s}_{p3}(\bar{r}_{3\beta}\bar{k}_{33} - \bar{k}_{3\beta}\bar{r}_{33}) + \bar{r}_{p3}(\bar{k}_{3\beta}\bar{s}_{33} - \bar{r}_{3\beta}\bar{r}_{33})],$$
$$k_{\alpha\beta} = k_{\beta\alpha} = \bar{k}_{\alpha\beta} + \frac{1}{\bar{r}_{33}^2 - \bar{s}_{33}\bar{k}_{33}}[\bar{r}_{3\alpha}(\bar{r}_{3\beta}\bar{k}_{33} - \bar{k}_{3\beta}\bar{r}_{33}) + \bar{k}_{\alpha 3}(\bar{k}_{3\beta}\bar{s}_{33} - \bar{r}_{3\beta}\bar{r}_{33})].$$

3. The Lekhnitskii's formalism of 1D QCs

Now extend the Lekhnitskii's formalism [8,9] for elastic solids to 1D QC solids by introducing the following potential function representation:

$$\sigma_1 = \partial_2^2 U, \quad \sigma_2 = \partial_1^2 U, \quad \sigma_6 = -\partial_1 \partial_2 U,$$
$$\sigma_5 = \partial_2 \varphi, \quad \sigma_4 = -\partial_1 \varphi,$$
$$H_{31} = \partial_2 \psi, \quad H_{32} = -\partial_1 \psi, \tag{11}$$

where $U(x_1, x_2)$ is the Airy function while $\psi(x_1, x_2)$ and $\varphi(x_1, x_2)$ are the stress functions. It can be shown that the equilibrium equations are automatically satisfied by eq. (11). Substitution of eq. (11) into eq. (10) and then into eq. (8) yields the differential equations in matrix notation,

$$\mathbf{LU} = \mathbf{0}, \tag{12}$$

where the vector $\mathbf{U} = [U, \psi, \varphi]^{\mathrm{T}}$ (the superscript T denotes the transpose) and $\mathbf{L}$ is a 3×3 differential operator matrix. The components of $\mathbf{L}$ are:

$$\begin{aligned}
L_{11} &= s_{22}\partial_1^4 - 2s_{26}\partial_1^3\partial_2 + (2s_{12} + s_{66})\,\partial_1^2\partial_2^2 - 2s_{16}\partial_1\partial_2^3 + s_{11}\partial_2^4,\\
L_{12} &= L_{21} = -r_{22}\partial_1^3 + (r_{21} + r_{62})\,\partial_1^2\partial_2 - (r_{12} + r_{61})\,\partial_1\partial_2^2 + r_{11}\partial_2^3,\\
L_{13} &= L_{31} = -s_{24}\partial_1^3 + (s_{25} + s_{64})\,\partial_1^2\partial_2 - (s_{14} + s_{65})\,\partial_1\partial_2^2 + s_{15}\partial_2^3,\\
L_{22} &= k_{22}\partial_1^2 - 2k_{12}\partial_1\partial_2 + k_{11}\partial_2^2,\\
L_{23} &= L_{32} = r_{42}\partial_1^2 - (r_{41} + r_{52})\,\partial_1\partial_2 + r_{51}\partial_2^2,\\
L_{33} &= s_{44}\partial_1^2 - 2s_{45}\partial_1\partial_2 + s_{55}\partial_2^2.
\end{aligned} \tag{13}$$

If we eliminate ψ and φ, eq. (11) is rewritten with respect to U

$$\begin{aligned}
&(L_{22}L_{33} - L_{23}^2)\psi = (L_{13}L_{23} - L_{12}L_{33})U,\\
&(L_{22}L_{33} - L_{23}^2)\varphi = (L_{12}L_{23} - L_{13}L_{22})U,\\
&L_0 U = 0,
\end{aligned} \tag{14}$$

where L_0 is the 'determinant' of the differential operator matrix $\mathbf{L}$.

The third equation of eq. (14) is a homogeneous differential equation of eighth order. A general solution depends on a composite variable which is a linear combination of x_1 and x_2. Without loss in generality, we set the coefficient of x_1 in the linear combination to be unity. Thus the solution of eq. (14) will be constituted by means of a function $U(z)$

$$U = U(z)\,, \quad \psi = \psi(z)\,, \quad \varphi = \varphi(z)\,, \quad z = x_1 + \mu x_2, \tag{15}$$

where z is a generalized complex variable and μ is a complex parameter to be determined. Insertion of eq. (15) into eq. (14) leads to

$$\begin{aligned}
&(l_{22}l_{33} - l_{23}^2)\psi^{(4)}(z) = (l_{13}l_{23} - l_{12}l_{33})U^{(5)}(z)\,,\\
&(l_{22}l_{33} - l_{23}^2)\varphi^{(4)}(z) = (l_{12}l_{23} - l_{13}l_{22})U^{(5)}(z)\,,\\
&l_0(\mu) = 0,
\end{aligned} \tag{16}$$

where the symbolic $(\ )^{(n)}$ denotes the n-differential operator, l_0 is the 'determinant' of $\mathbf{l}$ and l_{ij} are the components of $\mathbf{l}$,

$$\begin{aligned}
l_{11} &= s_{22} - 2s_{26}\mu + (2s_{12} + s_{66})\mu^2 - 2s_{16}\mu^3 + s_{11}\mu^4,\\
l_{12} &= l_{21} = -r_{22} + (r_{21} + r_{62})\mu - (r_{12} + r_{61})\mu^2 + r_{11}\mu^3,\\
l_{13} &= l_{31} = -s_{24} + (s_{25} + s_{64})\mu - (s_{14} + s_{65})\mu^2 + s_{15}\mu^3,\\
l_{22} &= k_{22} - 2k_{12}\mu + k_{11}\mu^2,\\
l_{23} &= l_{32} = r_{42} - (r_{41} + r_{52})\mu + r_{51}\mu^2,\\
l_{33} &= s_{44} - 2s_{45}\mu + s_{55}\mu^2.
\end{aligned} \tag{17}$$

Integration of the first two equations of eq. (16) yields

$$\psi(z) = \lambda(\mu)U^{(1)}(z)\,, \quad \varphi(z) = \rho(\mu)U^{(1)}(z)\,, \tag{18}$$

where the complex variables λ and ρ are defined as

$$\lambda(\mu) = \frac{l_{13}l_{23} - l_{12}l_{33}}{l_{22}l_{33} - l_{23}^2}, \quad \rho(\mu) = \frac{l_{12}l_{23} - l_{13}l_{22}}{l_{22}l_{33} - l_{23}^2}. \tag{19}$$

The third equation of eq. (16) is the eighth-degree characteristic equation of μ. As in the case of anisotropic elastic body [9], the generalized Lekhnitskii's formalism leads to an eighth-degree polynomial with real coefficients in a parameter μ whose eight roots μ_t ($t = 1, 2, ..., 8$) must be determined in order to effect an elastostatic solution; elastic stability requires that the roots occur in four pairs of complex conjugates. We assume the roots are distinct and let

$$\mathrm{Im}\,\mu_r > 0, \quad \mu_{r+4} = \bar{\mu}_r,\ r = 1, 2, 3, 4, \tag{20}$$

where Re and Im stand for the real and imaginary parts, respectively, and the overbar denotes the complex conjugate.

Once the roots μ_r are known, the solution is written as

$$U = 2\delta_{rR}\,\mathrm{Re}(U_r), \quad \psi = 2\mathrm{Re}(\lambda_r\phi_r), \quad \varphi = 2\mathrm{Re}(\rho_r\phi_r), \tag{21}$$

where δ_{ij} is the Kronecker delta symbol. The Einstein summation over repeated lower case indices is applied, while capital indices take on the same numbers as the corresponding lower indices but are not summed. Insertion of eq. (21) into eq. (11) yields the stress components

$$\begin{aligned} &\sigma_1 = 2\mathrm{Re}(\mu_r^2\phi_r'), \quad \sigma_2 = 2\delta_{rR}\,\mathrm{Re}(\phi_r'), \quad \sigma_6 = -2\mathrm{Re}(\mu_r\phi_r'),\\ &\sigma_5 = 2\mathrm{Re}(\rho_R\mu_r\phi_r'), \quad \sigma_4 = -2\mathrm{Re}(\mu_r\phi_r'),\\ &H_{31} = 2\mathrm{Re}(\lambda_R\mu_r\phi_r'), \quad H_{32} = -2\mathrm{Re}(\lambda_r\phi_r'), \end{aligned} \tag{22}$$

where complex function $\phi_r(z_r) = U_r'(z_r)$ and the superscript prime denotes differentiation with respect to z_r. It should be noted that the stress components σ_3 and H_{33} can be obtained from eq. (9). Substitution of eq. (22) into eq. (10) and then into eq. (7) leads to the displacement components without rigid body displacements,

$$\begin{aligned} &u_1 = 2\mathrm{Re}(p_r\phi_r), \quad u_2 = 2\mathrm{Re}(q_r\phi_r),\\ &u_3 = 2\mathrm{Re}(s_r\phi_r), \quad w_3 = 2\mathrm{Re}(t_r\phi_r), \end{aligned} \tag{23}$$

where the complex variables p_r, q_r, t_r, s_r are given as

$$\begin{aligned} p_r &= s_{11}\mu_r^2 + s_{12} - s_{14}\rho_r + s_{15}\rho_R\mu_r - s_{16}\mu_r + r_{11}\lambda_R\mu_r - r_{12}\lambda_r,\\ q_r &= \frac{1}{\mu_R}(s_{21}\mu_r^2 + s_{22} - s_{24}\rho_r + s_{25}\rho_R\mu_r - s_{26}\mu_r + r_{21}\lambda_R\mu_r - r_{22}\lambda_r t),\\ s_r &= s_{51}\mu_r^2 + s_{52} - s_{54}\rho_r + s_{55}\rho_R\mu_r - s_{56}\mu_r + r_{51}\lambda_R\mu_r - r_{52}\lambda_r,\\ t_r &= r_{11}\mu_r^2 + r_{21} - r_{41}\rho_r + r_{51}\rho_R\mu_r - r_{61}\mu_r + k_{11}\lambda_R\mu_r - k_{12}\lambda_r. \end{aligned} \tag{24}$$

Thus, the plane problem of 1D QCs now reduces to determining the four unknown complex functions ϕ_r, where $r = 1, 2, 3, 4$, which satisfy a given boundary condition.

4. Elliptic rigid cylinder inclusion

A QC material is a complex system composed of crystallites, defects (cracks or pores) and inclusions (fibres or particles). The existence of these defects and inclusions greatly affects the optic, magnetic, thermal and mechanical properties of such materials. It is essential to study the 2D problem of a single inclusion in an infinite matrix. In general, this is a classic elastic equilibrium problem of a two-body. Since the pioneering works of Eshelby [15,16] for an ellipsoidal inclusion in which a uniform eigenstrain is prescribed, the inclusion problems continue to attract researchers' attention. Most of the work related to the inclusion problems can be found in the book by Mura and his review articles [17–19].

Consider an infinite anisotropic 1D QC medium containing an elliptic cylinder inclusion which penetrates through the solid along the quasiperiodic direction (x_3-direction). Let the boundary L of the ellipse be given by

$$x_1 = a\cos\theta, \quad x_2 = b\sin\theta, \tag{25}$$

where a and b are the major and minor axes of the ellipse and θ is a real parameter. For an elliptic region, one can give a simple function that maps the elliptic region within a unit circle, and thus the solutions of elliptic inclusion can be obtained in closed form. Consider the mapping [9]

$$z_r = \frac{1}{2}\left[(a - \mathrm{i}\mu_R b)\zeta_r + (a + \mathrm{i}\mu_R b)\zeta_r^{-1}\right]. \tag{26}$$

Notice that when $a = b$ we still need the mapping (26) unless $\mu = \mathrm{i}$. If (x_1, x_2) is on the elliptic boundary L, eq. (26) gives $\zeta_r = \exp(\mathrm{i}\theta)$. For the region outside the ellipse we obtain from eq. (25)

$$\zeta_r = \frac{z_r + \sqrt{z_r^2 - a^2 - \mu_r^2 b^2}}{a - \mathrm{i}\mu_r b}. \tag{27}$$

Thus the mapping is one-to-one for points outside the ellipse in the z_r-plane to points outside the unit circle in the ζ_r-plane.

We consider a generalized 2D problem of 1D QC medium containing an elliptic inclusion. The QC medium undergoes uniform phonon strains ε_p^∞ and uniform phason strains $w_{3\alpha}^\infty$ at infinity. The associated stresses are denoted by σ_p^∞ and $H_{3\alpha}^\infty$ which are determined from eq. (3). Therefore, we assume that both the stresses and strains applied at infinity are known.

The gradient of the phonon displacement field figures out the local rearrangement of atoms in a cell in QCs. External forces are needed to drive the atoms through barriers when they make local rearrangement in a cell, such that, besides the conventional body forces and surface forces in the physical space $E_\parallel^3$, the generalized body forces and surface forces in the perpendicular space $E_\perp^3$ should be introduced in the elasticity of QCs. Similarly, in addition to the conventional stress components σ_{ij}, H_{ij} describe the stress components along the x_i direction in $E_\perp^3$ acting on the surface orthogonal to the x_j direction in $E_\parallel^3$.

The displacements u_i^∞ and w_3^∞ which produce the given constant strains can be written in the simplified form

$$\left\langle u_1^\infty, u_2^\infty, u_3^\infty, w_3^\infty\right\rangle = x_1\left\langle \varepsilon_1^\infty, 0, \varepsilon_5^\infty, w_{31}^\infty\right\rangle + x_2\left\langle \varepsilon_6^\infty, \varepsilon_2^\infty, \varepsilon_4^\infty, w_{32}^\infty\right\rangle. \tag{28}$$

The displacements are unique up to a rigid-body translation and rotation. In the above, we have assumed no translation and no rotation of the x_1-axis which explains the zero element in the right side of eq. (28).

The generalized displacement components in eq. (23) have the following form:

$$\langle u_1, u_2, u_3, w_3\rangle = \langle u_1^\infty, u_2^\infty, u_3^\infty, w_3^\infty\rangle + 2\mathrm{Re}\langle p_r, q_r, s_r, t_r\rangle\, \phi_r. \tag{29}$$

If the ellipse is a rigid inclusion with vanishing elastic displacement, we have $u_1 = u_2 = u_3 = w_3 = 0$ on L.

$$2\mathrm{Re}\,\langle p_r, q_r, s_r, t_r\rangle\, \phi_r(t) = -\frac{1}{2}\left[a\left(t+\frac{1}{t}\right)\langle \varepsilon_1^\infty, 0, \varepsilon_5^\infty, w_{31}^\infty\rangle - \mathrm{i}b\left(t-\frac{1}{t}\right)\langle \varepsilon_6^\infty, \varepsilon_2^\infty, \varepsilon_4^\infty, w_{32}^\infty\rangle\right]. \tag{30}$$

Taking the operator $\mathrm{d}t/(t-\zeta_r)$ on both sides of eq. (30), and then integrating around the boundary of the unit circle, one obtains

$$\phi_r(\zeta_R) = -\frac{1}{2}[a\langle \varepsilon_1^\infty, 0, \varepsilon_5^\infty, w_{31}^\infty\rangle + \mathrm{i}b\langle \varepsilon_6^\infty, \varepsilon_2^\infty, \varepsilon_4^\infty, w_{32}^\infty\rangle]\zeta_r^{-1}\langle p_r, q_r, s_r, t_r\rangle^{-1}. \tag{31}$$

Thus the complex functions ϕ_r can be solved from eq. (31).

It is well known that the elliptic rigid inclusion is the basis of the rigid line inclusion in elastic analysis. When the elliptic rigid inclusion degenerates into a rigid line inclusion by taking certain limit, i.e., $b = 0$, then eq. (31) becomes

$$\phi_r(z_R) = -\frac{1}{2}\left\langle \varepsilon_1^\infty, 0, \varepsilon_5^\infty, w_{31}^\infty\right\rangle\left(z_R - \sqrt{z_R^2 - a^2}\right)\langle p_r, q_r, s_r, t_r\rangle^{-1}. \tag{32}$$

The explicit expressions for the displacement field can be obtained by substituting eq. (32) into eq. (29)

$$\begin{aligned}\langle u_1, u_2, u_3, w_3\rangle &= x_1\left\langle \varepsilon_1^\infty, 0, \varepsilon_5^\infty, w_{31}^\infty\right\rangle + x_2\left\langle \varepsilon_6^\infty, \varepsilon_2^\infty, \varepsilon_4^\infty, w_{32}^\infty\right\rangle \\ &\quad - \left\langle \varepsilon_1^\infty, 0, \varepsilon_5^\infty, w_{31}^\infty\right\rangle \\ &\quad \times \mathrm{Re}\left\{\langle p_r, q_r, s_r, t_r\rangle\left(z_R - \sqrt{z_R^2 - a^2}\right)\langle p_r, q_r, s_r, t_r\rangle^{-1}\right\}.\end{aligned} \tag{33}$$

The analytical solutions of the strain field can be given through eqs (7) and (33) as follows:

$$\begin{aligned}\left\langle \varepsilon_1, \varepsilon_6', \varepsilon_5, w_{31}\right\rangle &= \left\langle \varepsilon_1^\infty, 0, \varepsilon_5^\infty, w_{31}^\infty\right\rangle \\ &\quad \times \mathrm{Re}\left\{\langle p_r, q_r, s_r, t_r\rangle \frac{z_R}{\sqrt{z_R^2 - a^2}}\langle p_r, q_r, s_r, t_r\rangle^{-1}\right\}, \\ \left\langle \varepsilon_6'', \varepsilon_2, \varepsilon_4, w_{32}\right\rangle &= \left\langle \varepsilon_6^\infty, \varepsilon_2^\infty, \varepsilon_4^\infty, w_{32}^\infty\right\rangle + \left\langle \varepsilon_1^\infty, 0, \varepsilon_5^\infty, w_{31}^\infty\right\rangle \\ &\quad \times \mathrm{Re}\left\{\langle p_r, q_r, s_r, t_r\rangle\left(\frac{\mu_R z_R}{\sqrt{z_R^2 - a^2}} - \mu_R\right)\langle p_r, q_r, s_r, t_r\rangle^{-1}\right\},\end{aligned} \tag{34}$$

where

$$\varepsilon_6' = \partial_1 u_2, \quad \varepsilon_6'' = \partial_2 u_1, \quad \varepsilon_6 = \varepsilon_6' + \varepsilon_6''.$$

From eqs (33) and (34) the interdependence between phonon and phason quantities for the entire fields can be recognized. This indicates that the existence of the phason field and the coupling between phonon and phason fields strongly affect the configuration and the mechanical properties of the materials.

5. Conclusions

On the basis of the complex potential approach, the generalized Lekhnitskii's formalism is obtained. The formalism automatically accounts for the satisfaction of the equilibrium equations and seeks to enforce the equations of compatibility. In virtue of the generalized Lekhnitskii's formalism, an elliptic rigid inclusion in 1D QC bodies subjected to uniform strains at infinity is investigated. This work reveals the physical sense of the results relative to the phason field and the difference between mechanical behaviours of inclusion problems in conventional crystals and QCs. These provide important information for studying the deformation and defects of the new solid phase.

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