PRAMANA — journal of physics

Vol. 79, No. 2
August 2012
pp. 233–241

Quantum Jarzynski equality with multiple measurement and feedback for isolated system

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MS received 14 December 2011; revised 16 February 2012; accepted 1 March 2012

Abstract. In this paper, we derive the Jarzynski equality (JE) for an isolated quantum system in three different cases: (i) the full evolution is unitary with no intermediate measurements, (ii) with intermediate measurements of arbitrary observables being performed, and (iii) with intermediate measurements whose outcomes are used to modify the external protocol (feedback). We assume that the measurements will involve errors that are purely classical in nature. Our treatment is based on path probability in state space for each realization. This is in contrast with the formal approach based on projection operator and density matrices. We find that the JE remains unaffected in the second case, but gets modified in the third case where the mutual information between the measured values with the actual eigenvalues must be incorporated into the relation.

Keywords. Fluctuation theorems; Jarzynski equality; measurement and feedback; mutual information.

PACS Nos 05.40.–a; 05.70.Ln; 03.65.Yz

1. Introduction

In the last couple of decades, a lot of work has been directed towards nonequilibrium statistical mechanics, which has given birth to several equalities that are valid even when the system is far from equilibrium. They are collectively known as the fluctuation theorems [1–5]. These theorems also shed new light on some fundamental problems such as how irreversibility arises from underlying time-reversible dynamics. Moreover, these theorems have important application in nanotechnology and nanophysics. One of the pioneering works was due to Jarzynski [3], who had derived a relation between the nonquilibrium work performed on a system and the change in its equilibrium free energy. Let us consider a system that is initially at canonical equilibrium with a heat bath at inverse temperature $\beta = 1/k_{\mathrm{B}}T$. Subsequently, an external perturbation $\lambda(t)$, called protocol, is applied to the system that takes it out of equilibrium. At time $t = \tau$, the process is terminated when the parameter value reaches $\lambda(\tau)$. The work W done on the system will in general vary for different phase-space trajectories, owing to the randomness of the initial state and thermal fluctuations due to coupling with the environment during the evolution. The Jarzynski equality (JE) states that,

$$\langle \mathrm{e}^{-\beta W} \rangle = \mathrm{e}^{-\beta \Delta F}. \tag{1}$$

Here, the angular brackets denote ensemble averaging over a large number of repetitions of the experiment. $\Delta F \equiv F(\lambda(\tau)) - F(\lambda(0))$ is the difference in the equilibrium free energy of the system between the final and the initial states. The JE has been extended to quantum domain [6] in the presence of measurement [7] and feedback [8,9]. JE in the presence of feedback has also been verified experimentally [10]. Quantum feedbacks are important in nanosystems or mesoscopic systems and can be applied to produce the cooling of nanomechanical resonators and atoms [11,12].

In our present study we derive quantum extended JE with multiple measurements and feedback for an isolated system. Our treatment is based on path probability in state space for each realization as opposed to formal approach dealing with projection operator and density matrices [8,9]. All the results are simple extensions of the theorems for fixed protocol, and the latter in turn depends on the principle of microscopic reversibility. It may be noted that only with the choice of von Neumann-type measurements, corresponding to the measurement operator $\Pi_j \equiv |j\rangle\langle j|$, the earlier approaches reduce to the present approach. Here, $|j\rangle$ is an eigenstate of the measured observable.

For the quantum case to obtain the work values, we perform measurement (von Neumann-type) of system energies (or Hamiltonian $H(t)$) at the beginning and end of protocol. The measured energy eigenvalues are denoted by $E_{i_0}(\lambda(0))$ and $E_{i_\tau}(\lambda(\tau))$ and the corresponding instantaneous eigenstates by $|i_0\rangle$ and $|i_\tau\rangle$ respectively. The work done on the system by changing external protocol $\lambda(t)$ is given by

$$W = E_{i_\tau}(\lambda(\tau)) - E_{i_0}(\lambda(0)). \tag{2}$$

W is a realization-dependent random variable. Initially the system is brought into contact with large reservoir at temperature T, thereby allowing the system to equilibrate. Subsequently, the system is decoupled from the bath and the system evolves unitarily with a given Hamiltonian $H(t)$. Our treatment closely follows [13] wherein Hamiltonian derivation of JE under feedback control is derived for classical case.

Probability of the system being in state $|i_0\rangle$ is given by

$$p(i_0) = \frac{\mathrm{e}^{-\beta E_{i_0}(\lambda(0))}}{Z_0}. \tag{3}$$

The partition function is defined as

$$Z_0 = \sum_{i_0} \mathrm{e}^{-\beta E_{i_0}(\lambda(0))}. \tag{4}$$

Between measurements, the system undergoes unitary evolution with an operator U given by

$$U_\lambda(t_2, t_1) = T \exp\left(-\frac{i}{\hbar}\int_{t_1}^{t_2} H(t, \lambda(t))\mathrm{d}t\right), \tag{5}$$

where T denotes time ordering and $H(t)$ is the system Hamiltonian. The probability of the system initially in the state $|i_0\rangle$ to be found in state $|i_\tau\rangle$ at time τ is given by

$$P(i_\tau|i_0) = |\langle i_\tau|U_\lambda(\tau, 0)|i_0\rangle|^2. \tag{6}$$

Thus the joint probability of state being in $|i_0\rangle$ and $|i_\tau\rangle$ is

$$P(i_\tau, i_0) = P(i_\tau|i_0)p(i_0) \qquad \text{(Bayes' theorem)}. \tag{7}$$

In §2, we rederive the JE for a quantum particle to make the paper self-consistent. In §3, we derive the same with measurements of arbitrary observables being performed in-between. In §4, we derive the extended JE for a system with the protocol being monitored by a feedback control that changes the protocol according to the outcomes of the measurements performed. In §5 generalized JE involving efficacy parameter is derived.

2. Jarzynski equality

For deriving JE we need to calculate $\langle \mathrm{e}^{-\beta W}\rangle$ which is given by

$$\langle \mathrm{e}^{-\beta W}\rangle = \sum_{i_\tau, i_0} \mathrm{e}^{-\beta W} P(i_\tau, i_0). \tag{8}$$

Substituting the expression for realization-dependent work (eq. (2)) and joint probability $P(i_\tau, i_0)$ (eq. (7)) and using eqs (3) and (6) we get

$$\begin{aligned}\langle \mathrm{e}^{-\beta W}\rangle &= \sum_{i_0, i_\tau} \mathrm{e}^{-\beta(E_{i_\tau}(\lambda(\tau)) - E_{i_0}(\lambda(0)))} |\langle i_\tau|U_\lambda(\tau, 0)|i_0\rangle|^2 \frac{\mathrm{e}^{-\beta E_{i_0}(\lambda(0))}}{Z_0} \\ &= \sum_{i_0, i_\tau} \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda(\tau))}}{Z_0} \langle i_\tau|U_\lambda(\tau, 0)|i_0\rangle\langle i_0|U_\lambda^\dagger(\tau, 0)|i_\tau\rangle. \end{aligned} \tag{9}$$

Making use of completeness relation $\sum_{i_0} |i_0\rangle\langle i_0| = 1$ and normalization condition $\langle i_\tau|i_\tau\rangle = 1$ and unitarity of evolution, $U_\lambda^\dagger U_\lambda = 1$, we have

$$\langle \mathrm{e}^{-\beta W}\rangle = \sum_{i_\tau} \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda(\tau))}}{Z_0} = \frac{Z_\tau}{Z_0} = \mathrm{e}^{-\beta \Delta F}, \tag{10}$$

where $Z_\tau = \sum_{i_\tau} \mathrm{e}^{-\beta E_{i_\tau}(\lambda(\tau))}$ is the partition function of the system with the control parameter held fixed at $\lambda(\tau)$ and $\Delta F = \ln(Z_0/Z_\tau)$ is the equilibrium free energy difference between final and initial states. This is the quantum version of the JE [6]. Using Jensen's inequality, we retrieve the second law from the above relation:

$$\langle W\rangle \geq \Delta F, \tag{11}$$

implying second law is valid for average W although for some individual realizations, W can be less than ΔF.

3. JE in the presence of measurement

This time, one intermediate measurement (of arbitrary observables, not necessarily the Hamiltonian) at time t_1 has been carried out but the entire protocol $\lambda(t)$ is predetermined. At time t_1 the state collapses to one of the eigenstates of the measured observable, say

$|i_1\rangle$, after which it evolves according to the unitary operator $U_\lambda(\tau, t_1)$ up to the final time τ. It is to be noted that the projective measurements result in the collapse of the system state to one of the eigenstates and leads to decoherence and dephasing. If along two paths, intermediate measurements are performed, then the interference between alternative paths disappears and quantum effects are suppressed. Hence in the presence of measurement, path probabilities in state space obey simple classical probability rules. For example, the path probability is simply the product of the transition probabilities between subsequent measured states. However, it may be noted that quantum mechanics enters through the explicit calculation of transition probabilities between states. The joint probability of the state trajectory is

$$P(i_\tau, i_1, i_0) = p(i_\tau|i_1)p(i_1|i_0)p(i_0) \tag{12}$$

$$= |\langle i_\tau|U_{\lambda_{y_1}}(\tau, t_1)|i_1\rangle|^2 |\langle i_1|U_\lambda(t_1, 0)|i_0\rangle|^2 p(i_0). \tag{13}$$

Then,

$$\langle \mathrm{e}^{-\beta W}\rangle = \sum_{i_\tau, i_1, i_0} \mathrm{e}^{-\beta W} P(i_\tau, i_1, i_0).$$

Using eqs (2), (13) and (3)

$$\begin{aligned}
\langle \mathrm{e}^{-\beta W}\rangle &= \sum_{i_0, i_1, i_\tau} \mathrm{e}^{-\beta(E_{i_\tau}(\lambda(\tau)) - E_{i_0}(\lambda(0)))} |\langle i_1|U_\lambda(t_1, 0)|i_0\rangle|^2 |\langle i_\tau|U_\lambda(\tau, t_1)|i_1\rangle|^2 \\
&\quad \times \frac{\mathrm{e}^{-\beta E_{i_0}(\lambda(0))}}{Z_0} \\
&= \sum_{i_0, i_1, i_\tau} \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda(\tau))}}{Z_0} \langle i_1|U_\lambda(t_1, 0)|i_0\rangle\langle i_0|U_\lambda^\dagger(t_1, 0)|i_1\rangle |\langle i_\tau|U_\lambda(\tau, t_1)|i_1\rangle|^2 \\
&= \sum_{i_1, i_\tau} \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda(\tau))}}{Z_0} \langle i_\tau|U_\lambda(\tau, t_1)|i_1\rangle\langle i_1|U_\lambda^\dagger(\tau, t_1)|i_\tau\rangle \\
&= \sum_{i_\tau} \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda(\tau))}}{Z_0} = \frac{Z_\tau}{Z_0} = \mathrm{e}^{-\beta \Delta F}.
\end{aligned} \tag{14}$$

In the above simplification we have used completeness relation, normalization condition and unitarity of U_λ as in §2. Thus, we find that the JE remains unaffected even if measurements are performed on the system in between $(0, \tau)$. The above treatment can be readily generalized to the case of multiple measurements (see Appendix A). Even though the form of JE is not altered in the presence of measurements, the statistics of the work performed on the system changes (strongly influenced by measurements). This is due to the fact that path probabilities for a given value of work are modified in the presence of measurements. This is clearly illustrated in [14], wherein work distribution has been calculated for the Landau–Zener model in the presence of measurement.

4. Extended JE in the presence of feedback

The extended JE in the presence of feedback has been given by Sagawa and Ueda for both the classical [13,15] and the quantum [9] cases. Feedback means that the system will be controlled by the measurement output. After each measurement, the protocol is changed accordingly. Suppose the initial protocol was $\lambda(t)$; at time t_1 a measurement of some observable A is performed on the system and outcome y_1 is obtained. We then modify our protocol from $\lambda_0(t)$ to $\lambda_{y_1}(t)$ and evolve the system up to time τ. We assume that the intermediate measurements can involve errors that are purely classical in nature. The error probability is given by $p(y_1|i_1)$, where $|i_1\rangle$ is the actual collapsed eigenstate of A. The final value of the protocol $\lambda_{y_1}(\tau)$ depends on y_1 and hence equilibrium free energy at the end of the protocol depends on y_1. The mutual information between the actual state $|i_1\rangle$ and the measured value y_1 is

$$I = \ln \frac{p(y_1|i_1)}{p(y_1)}. \tag{15}$$

Here, $p(y_1)$ is the probability density of the outcome y_1. The mutual information I quantifies a change in uncertainty about the state of the system upon making measurement [16]. Note that I can be positive or negative for a given realization; however, $\langle I\rangle$ is always positive. The probability of the state trajectory $|i_0\rangle \to |i_1\rangle \to |i_\tau\rangle$ with single measurement is

$$\begin{aligned} P(i_\tau, i_1, i_0, y_1) &= p(i_\tau|i_1)p(y_1|i_1)p(i_1|i_0)p(i_0) \\ &= |\langle i_\tau|U_{\lambda_{y_1}}(\tau, t_1)|i_1\rangle|^2 p(y_1|i_1)|\langle i_1|U_\lambda(t_1, 0)|i_0\rangle|^2 p(i_0). \end{aligned} \tag{16}$$

Now we have,

$$\langle \mathrm{e}^{-\beta(W-\Delta F)-I}\rangle = \int \mathrm{d}y_1 \sum_{i_\tau, i_1, i_0} P(i_\tau, i_1, i_0, y_1)\mathrm{e}^{-\beta(W-\Delta F(y_1))-I}. \tag{17}$$

Substituting the expressions of joint probability $P(i_\tau, i_1, i_0, y_1)$ (eq. (16)), work W (eq. (2)), free energy difference $\Delta F = Z_0/(Z_\tau(y_1))$, and mutual information I (eq. (15)) and simplifying we get

$$\begin{aligned} \langle \mathrm{e}^{-\beta(W-\Delta F)-I}\rangle &= \int \mathrm{d}y_1 \sum_{i_\tau, i_1, i_0} |\langle i_\tau|U_{\lambda_{y_1}}(\tau, t_1)|i_1\rangle|^2 |\langle i_1|U_\lambda(t_1, 0)|i_0\rangle|^2 p(y_1) \\ &\quad \times \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y)} \\ &= \int \mathrm{d}y_1 \sum_{i_\tau, i_1, i_0} |\langle i_\tau|U_{\lambda_{y_1}}(\tau, t_1)|i_1\rangle|^2 \\ &\quad \times \langle i_1|U_\lambda(t_1, 0)|i_0\rangle\langle i_0|U_\lambda^\dagger(t_1, 0)|i_1\rangle p(y_1) \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y)} \\ &= \int \mathrm{d}y_1 \sum_{i_\tau, i_1} |\langle i_\tau|U_{\lambda_{y_1}}(\tau, t_1)|i_1\rangle|^2 p(y_1) \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y)} \end{aligned}$$

$$= \int \mathrm{d}y_1 \sum_{i_\tau, i_1} \langle i_\tau | U_{\lambda_{y_1}}(\tau, t_1) | i_1 \rangle \langle i_1 | U^\dagger_{\lambda_{y_1}}(\tau, t_1) | i_\tau \rangle p(y_1)$$
$$\times \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y)}$$
$$= \int \mathrm{d}y_1 p(y_1) \sum_{i_\tau} \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y)}$$
$$= \int \mathrm{d}y_1 p(y_1) = 1. \tag{18}$$

In the second and fourth steps, the modulus squared terms have been rewritten in expanded form and completeness relation is used. The relation (18) constitutes the extended JE in the presence of information. Using Jensen's inequality, one arrives at the generalized version of the second law in the presence of feedback:

$$\langle W \rangle \geq \langle \Delta F \rangle - k_{\mathrm{B}} T \langle I \rangle, \tag{19}$$

where the average mutual entropy $\langle I \rangle$ is always non-negative on account of being a relative entropy [17]. Thus, the lower bound of the mean work done on the system can be lowered by a term that is proportional to the average of the mutual information. In other words, with the help of an efficiently designed feedback, we can extract more work from the system. The above treatment can be readily extended to the case of multiple measurements between $(0, \tau)$, not necessarily at equal intervals of time. This is given in Appendix B.

5. Generalized JE and efficacy parameter in the presence of feedback

The efficacy parameter γ [8,13,15] provides a measure of how efficiently our feedback is able to extract work from the system. It is defined as

$$\gamma \equiv \langle \mathrm{e}^{-\beta(W-\Delta F)} \rangle = \int \mathrm{d}y_1 \sum_{i_\tau, i_1, i_0} P(i_\tau, i_1, i_0, y_1) \mathrm{e}^{-\beta(W-\Delta F)}. \tag{20}$$

Here we have assumed single intermediate measurement. Substituting the expressions of joint probability $P(i_\tau, i_1, i_0, y_1)$ (eq. (16)), work W (eq. (2)), free energy difference $\Delta F = Z_0/(Z_\tau(y_1))$, and information I (eq. (15)), we get

$$\langle \mathrm{e}^{-\beta(W-\Delta F)} \rangle = \int \mathrm{d}y_1 \sum_{i_\tau, i_1, i_0} |\langle i_\tau | U_{\lambda_{y_1}}(\tau, t_1) | i_1 \rangle|^2 |\langle i_1 | U_\lambda(t_1, 0) | i_0 \rangle|^2 p(y_1 | i_1)$$
$$\times \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y)}$$
$$= \int \mathrm{d}y_1 \sum_{i_\tau, i_1, i_0} |\langle i_\tau | U_{\lambda_{y_1}}(\tau, t_1) | i_1 \rangle|^2$$
$$\times \langle i_1 | U_\lambda(t_1, 0) | i_0 \rangle \langle i_0 | U^\dagger_\lambda(t_1, 0) | i_1 \rangle p(y_1 | i_1) \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y)}$$

$$= \int \mathrm{d}y_1 \sum_{i_\tau, i_1} |\langle i_\tau | U_{\lambda_{y_1}}(\tau, t_1) | i_1 \rangle|^2 p(y_1|i_1) \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y)}. \tag{21}$$

For further calculations we need to take into account time-reversed path. For this we introduce time reversal operator Θ with the properties $\Theta^\dagger = \Theta$ and $\Theta^\dagger \Theta = 1$. Let $|i_0^*\rangle$ denote the time-reversed state of $|i_0\rangle$, i.e. $|i_0^*\rangle = \Theta |i_0\rangle$. It follows [18]

$$\Theta U_{\lambda_{y_1}}(\tau, t_1) \Theta^\dagger = U_{\lambda_{y_1}^\dagger}(\tilde{\tau}, \tilde{t}_1), \tag{22}$$

where $\tilde{t} = \tau - t$, i.e. the time calculated along reverse process. We assume time-reversibility of measurements, $p(y_1^*|i_1^*) = p(y_1|i_1)$ [8], y_1^* being the time-reversed value of y_1. As i^* and i have one-to-one correspondence, the summation over i_1, i_τ is equivalent to that over i_1^*, i_τ^*. We get

$$\begin{aligned}
\langle \mathrm{e}^{-\beta(W-\Delta F)} \rangle &= \int \mathrm{d}y_1 \sum_{i_\tau^*, i_1^*} |\langle i_\tau | \Theta^\dagger \Theta U_{\lambda_{y_1}}(\tau, t_1) \Theta^\dagger \Theta | i_1 \rangle|^2 p(y_1|i_1) \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y)} \\
&= \int \mathrm{d}y_1 \sum_{i_\tau^*, i_1^*} |\langle i_\tau^* | U_{\lambda_{y_1}^\dagger}(\tilde{\tau}, \tilde{t}_1) | i_1^* \rangle|^2 p(y_1^*|i_1^*) \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y)} \\
&= \int \mathrm{d}y_1 \sum_{i_\tau^*, i_1^*} |\langle i_1^* | U_{\lambda_{y_1}^\dagger}^\dagger(\tilde{\tau}, \tilde{t}_1) | i_\tau^* \rangle|^2 p(y_1^*|i_1^*) \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y)} \\
&= \int \mathrm{d}y_1 \sum_{i_\tau^*, i_1^*} |\langle i_1^* | U_{\lambda_{y_1}^\dagger}(\tilde{t}_1, \tilde{\tau}) | i_\tau^* \rangle|^2 p(y_1^*|i_1^*) \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y)} \\
&= \int \mathrm{d}y_1 \sum_{i_\tau^*, i_1^*} P_{\lambda_{y_1}^\dagger}(i_1^*|i_\tau^*) p(y_1^*|i_1^*) P(i_\tau),
\end{aligned} \tag{23}$$

where

$$P_{\lambda_{y_1}^\dagger}(i_1^*|i_\tau^*) = |\langle i_1^* | U_{\lambda_{y_1}^\dagger}(\tilde{t}_1, \tilde{\tau}) | i_\tau^* \rangle|^2, \tag{24}$$

is the conditional probability of time-reversed trajectory from state $|i_\tau^*\rangle$ to $|i_1^*\rangle$. We also have

$$P(i_\tau^*) = P(i_\tau) = \frac{\mathrm{e}^{-\beta E_{i_\tau}(\lambda_{y_1}(\tau))}}{Z_\tau(y_1)}, \tag{25}$$

which is the initial probability distribution of the time-reversed process with fixed protocol $\lambda_{y_1}^\dagger(\tau)$. Substituting eq. (25) in eq. (23) and using Bayes' theorem we get

$$\gamma = \langle \mathrm{e}^{-\beta(W-\Delta F)} \rangle = \int \mathrm{d}y_1 \sum_{i_1^*} p(y_1^*|i_1^*) P_{\lambda_{y_1}^\dagger}(i_1^*) = \int \mathrm{d}y_1 P_{\lambda_{y_1}^\dagger}(y_1^*). \tag{26}$$

The physical meaning of the efficacy parameter is apparent now: it is the total probability of observing time-reversed outcomes along time-reversed protocols. Thus expression for the efficacy parameter remains the same as in the classical case. For multiple measurements, efficacy parameter is given by $\gamma = \int \mathrm{d}y_1 \cdots \mathrm{d}y_n P_{\lambda^\dagger}(y_1^* \cdots y_n^*)$. The derivation is simple and we are not reproducing it here.

In conclusion, we have shown that the quantum extension of JE with multiple measurements and measurement accompanied feedback and quantum efficacy parameter retain the same expressions as in the classical case. This is mainly due to the measurements performed being of von Neumann projective type accompanied by classical errors, and system being isolated. We have also shown that in quantum case, entropy production fluctuation theorems retain the same form as in the classical case with measurement and feedback. The results will be published elsewhere.

Acknowledgement

AMJ thanks DST, India for financial support.

Appendix A. JE in the presence of multiple measurements

We consider n number of intermediate measurements of any observable being performed at time $t_1, t_2, ..., t_n$ and the system collapses to its corresponding eigenstate at $|i_1\rangle, |i_2\rangle, ..., |i_n\rangle$ respectively. Here we have considered that the system evolves with the predetermined protocol $\lambda(t)$. The probability of the corresponding state trajectory

$$\begin{aligned} P(i_\tau, ..., i_2, i_1, i_0) &= p(i_\tau|i_n)\cdots p(i_2|i_1)p(i_1|i_0)p(i_0) \\ &= |\langle i_\tau|U_\lambda(\tau, t_n)|i_n\rangle|^2 \cdots |\langle i_2|U_\lambda(t_2, t_1)|i_1\rangle|^2 \\ &\quad \times |\langle i_1|U_\lambda(t_1, 0)|i_0\rangle|^2 p(i_0). \end{aligned} \tag{27}$$

$$\begin{aligned} \langle e^{-\beta W}\rangle &= \sum_{i_0, i_1, ..., i_\tau} e^{-\beta(E_{i_\tau}(\lambda(\tau)) - E_{i_0}(\lambda(0)))} P(i_\tau, ..., i_2, i_1, i_0) \\ &= \sum_{i_0, i_1, ..., i_\tau} e^{-\beta(E_{i_\tau}(\lambda(\tau)) - E_{i_0}(\lambda(0))} |\langle i_\tau|U_\lambda(\tau, t_n)|i_n\rangle|^2 \cdots |\langle i_2|U_\lambda(t_2, t_1)|i_1\rangle|^2 \\ &\quad \times |\langle i_1|U_\lambda(t_1, 0)|i_0\rangle|^2 p(i_0). \end{aligned} \tag{28}$$

Using completeness and normalization of eigenstates $|i_0\rangle, |i_1\rangle, ..., |i_n\rangle$ and unitarity of evolution, we get after simplification

$$\langle e^{-\beta W}\rangle = \sum_{i_\tau} \frac{e^{-\beta E_{i_\tau}(\lambda(\tau))}}{Z_0} = \frac{Z_\tau}{Z_0} = e^{-\beta \Delta F}. \tag{29}$$

Thus JE retains the same classical form even in the presence of multiple measurements.

Appendix B. Multiple measurement and feedback

Let the outcome of measurement values at time $t_1, t_2, ..., t_n$ is $y_1, y_2, ..., y_n$ with a classical measurement error $p(y_1|i_1)$, $p(y_2|i_2)$, ..., $p(y_n|i_n)$ respectively when actual intermediate states are $|i_1\rangle, |i_2\rangle \cdots |i_n\rangle$. The state $|i_0\rangle$ and $|i_\tau\rangle$ are the observed projected eigenstates of energy observable in the beginning and end of the protocol. The total path probability can be expressed as

$$\begin{aligned} &P(i_\tau, ..., i_1, i_0, y_n, ..., y_1) \\ &= |\langle i_\tau|U_{\lambda_{y_n}}(\tau, t_1)|i_n\rangle|^2 \cdots p(y_2|i_2)|\langle i_2|U_{\lambda_1}(t_2, t_1)|i_1\rangle|^2 p(y_1|i_1) \\ &\quad \times |\langle i_1|U_\lambda(t_1, 0)|i_0\rangle|^2 p(i_0). \end{aligned} \tag{30}$$

Now,

$$\langle e^{-\beta(W-\Delta F)-I}\rangle = \int dy_n, ..., dy_1 \sum_{i_\tau,...,i_1,i_0} P(i_\tau, ..., i_1, i_0, y_n, ..., y_1) e^{-\beta(W-\Delta F)-I}. \tag{31}$$

Substituting the value of work W (eq. (2)), mutual information $I = \ln(p(y_n|i_n)\cdots p(y_2|i_2)p(y_1|i_1))/(p(y_n, ..., y_2, y_1))$, free energy difference $\Delta F = Z_0/(Z_\tau(y_n))$, and simplifying we get

$$\begin{aligned}
&\langle e^{-\beta(W-\Delta F)-I}\rangle \\
&= \int dy_n \cdots dy_1 \sum_{i_\tau,..,i_1,i_0} |\langle i_\tau|U_{\lambda_{y_n}}(\tau, t_1)|i_n\rangle|^2 \cdots |\langle i_2|U_{\lambda_1}(t_2, t_1)|i_1\rangle|^2 \\
&\quad \times |\langle i_1|U_\lambda(t_1, 0)|i_0\rangle|^2 \\
&\quad \times p(y_n, ..., y_2, y_1)\frac{e^{-\beta E_{i_\tau}(\lambda_{y_n}(\tau))}}{Z_\tau(y_n)} \\
&= \int dy_n \cdots dy_1 p(y_n, ..., y_2, y_1) \sum_{i_\tau} \frac{e^{-\beta E_{i_\tau}(\lambda_{y_n}(\tau))}}{Z_\tau(y_n)} \\
&= \int dy_n \cdots dy_1 p(y_n, ..., y_2, y_1) = 1.
\end{aligned} \tag{32}$$

This is the extended quantum JE in the presence of multiple measurements accompanied by feedback.

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