PRAMANA — journal of physics Vol. 79, No. 2
August 2012
pp. 185–198

Solitons and cnoidal waves of the Klein–Gordon–Zakharov equation in plasmas

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MS received 24 August 2011; revised 16 March 2012; accepted 28 March 2012

Abstract. This paper studies the Klein–Gordon–Zakharov equation with power-law nonlinearity. This is a coupled nonlinear evolution equation. The solutions for this equation are obtained by the travelling wave hypothesis method, (G'/G) method and the mapping method.

Keywords. Solitons; cnoidal waves; integrability.

PACS Nos **02.30.Jr; 02.30.Ik**

1. Introduction

The theory and integrability issues of nonlinear evolution equations (NLEEs) are very important in the areas of Applied Mathematics and Mathematical Physics. This theory has advanced considerably in the past couple of decades particularly with regard to the integrability aspect [1–15]. Numerous methods have been developed in the past two decades that allow the integrability of these NLEEs. Previously, there was only one method, which was the most powerful, to integrate these NLEEs. This was called the inverse scattering transform (IST) that is the nonlinear version of Fourier transform.

Some of the modern methods of integrability are variational iteration method, homotopy analysis method, semi-inverse variational principle, exponential function method, (G'/G) method, Riccati equation method, Fan's F-expansion method and so on. In this paper, the travelling wave hypothesis, the (G'/G) method and the mapping methods are going to be employed to carry out the integration of the Klein–Gordon–Zakharov (KGZ) equation that is a coupled NLEE.

The dimensionless form of the KGZ equation that is going to be studied in this paper is given by [6]

$$q_{tt} - k^2 q_{xx} + aq + brq + c|q|^{2n}q = 0 \tag{1}$$

DOI: 10.1007/s12043-012-0307-4; ***e*Publication: 11 July 2012**

$$r_{tt} - k^2 r_{xx} = d(|q|^{2n})_{xx}. \tag{2}$$

In (1) and (2) the dependent variables are q and r while the independent variables are x and t which are respectively referred to as the spatial and temporal variables. The dependent variable $q(x, t)$ is a complex valued function while the independent variable $r(x, t)$ is a real valued function. The parameter n is the power-law nonlinearity parameter. Equations (1) and (2) represent the KGZ equation with power-law nonlinearity. The coefficients a, b, c, d and k^2 are constants which are otherwise arbitrary.

Incidentally, this equation was studied before by the ansatz method and 1-soliton solution of this equation was retrieved in 2010 [6]. The numerical simulations were also given in 2010 [6].

2. Travelling waves

In this section the travelling wave hypothesis will be used to carry out the integration of (1) and (2). The search will be for soliton solutions only. The starting hypothesis for the dependent functions [6] are given by

$$q(x, t) = g(x - vt)\mathrm{e}^{i(-\kappa x + \omega t + \theta)} \tag{3}$$

and

$$r(x, t) = h(x - vt), \tag{4}$$

where g and h represent the soliton profiles for the two wave functions and v is the velocity of the soliton. In (3), κ represents the wave number of the soliton while ω represents the frequency of the soliton and θ is the phase constant. The following notation is used for convenience:

$$s = x - vt. \tag{5}$$

Substituting (3) and (4) into (2) implies

$$(v^2 - k^2)h'' = d(g^{2n})'', \tag{6}$$

where g'' represents $\mathrm{d}^2 g/\mathrm{d}s^2$ and $h'' = \mathrm{d}^2 h/\mathrm{d}s^2$. Integrating (6) twice and taking the integration constant to be zero, since the search is for soliton solutions only, gives

$$h(x, t) = \frac{d}{v^2 - k^2} g^{2n}. \tag{7}$$

Again substituting (3) and (4) into (1) and (2) and then decomposing into real and imaginary parts give

$$v = \frac{\kappa k^2}{\omega} \tag{8}$$

and

$$(v^2 - k^2)g'' + (a - \omega^2 + k^2\kappa^2)g + bgh + cg^{2n+1} = 0. \tag{9}$$

Now, eq. (9), by virtue of (7) reduces to

$$(v^2 - k^2)g'' + (a - \omega^2 + k^2\kappa^2)g + \left(c + \frac{bd}{v^2 - k^2}\right) g^{2n+1} = 0. \tag{10}$$

Multiplying both sides of (10) by $g' = \mathrm{d}g/\mathrm{d}s$ and integrating gives

$$(v^2 - k^2)(g')^2 = (\omega^2 - a - k^2\kappa^2)g^2 - \lambda^2 g^{2n+2}, \tag{11}$$

where

$$\lambda = \left[\frac{c(v^2 - k^2) + bd}{(n+1)\left(v^2 - k^2\right)}\right]^{1/2}. \tag{12}$$

Separating variables in (11) and integrating by taking the integration constant to be zero, gives

$$\frac{x - vt}{\sqrt{v^2 - k^2}} = \int \frac{dg}{g\sqrt{\omega^2 - a - k^2\kappa^2 - \lambda^2 g^{2n}}} \tag{13}$$

so that

$$\frac{\lambda g^n}{\sqrt{\omega^2 - a - \kappa^2 k^2}} = \operatorname{sech}\left[v\sqrt{\frac{\omega^2 - a - k^2\kappa^2}{v^2 - k^2}}(x - vt)\right]. \tag{14}$$

Equation (14) finally reduces to

$$g(x - vt) = A_1 \operatorname{sech}^{1/n}[B(x - vt)], \tag{15}$$

where the amplitude A_1 is given by

$$A_1 = \left[\frac{(n+1)\left(v^2 - k^2\right)\left(\omega^2 - a - k^2\kappa^2\right)}{c\left(v^2 - k^2\right) + bd}\right]^{1/2n} \tag{16}$$

and the width is given by

$$B = n\sqrt{\frac{\omega^2 - a - k^2\kappa^2}{v^2 - k^2}}. \tag{17}$$

Thus the soliton solution of the q wave function is given by

$$q(x, t) = A_1 \operatorname{sech}^{1/n}[B(x - vt)]\mathrm{e}^{i(-\kappa x + \omega t + \theta)}. \tag{18}$$

The relations (12), (16) and (17) pose the restrictions given by

$$(v^2 - k^2)(\omega^2 - a - k^2\kappa^2)\{c(v^2 - k^2) + bd\} > 0 \tag{19}$$

$$(\omega^2 - a - k^2\kappa^2)(v^2 - k^2) > 0 \tag{20}$$

and

$$\{c(v^2 - k^2) + bd\}(v^2 - k^2) > 0. \tag{21}$$

From (7), the soliton solution of the wave function $r(x, t)$ is given by

$$r(x,t) = A_2 \operatorname{sech}^2[B(x - vt)], \tag{22}$$

where the amplitude A_2 is given by

$$A_2 = \frac{(n+1)d(\omega^2 - a - k^2\kappa^2)}{c(v^2 - k^2) + bd}. \tag{23}$$

Hence, finally the travelling wave 1-soliton solutions to (1) and (2) are given by (18) and (22) respectively, where the two amplitudes A_1 and A_2 are in (16) and (23) and the width B of the soliton is given by (17) and the velocity v of the soliton is in (8). These relations impose the constraints that are given by (19), (20) and (21).

3. Mapping methods

3.1 *Description of the method*

Consider a nonlinear evolution equation in two variables

$$F(u,\ u_t,\ u_x, ...) = 0. \tag{24}$$

We search for its travelling wave solution in the form

$$u(x,t) \equiv u(\xi),\ \ \xi = k(x - ct), \tag{25}$$

where k and c are constants to be determined.

Substitution of eq. (25) into eq. (24) gives rise to an ordinary differential equation, for which the solution is searched in the form

$$u(\xi) = \sum_{i=0}^{n} A_i f^i, \tag{26}$$

where n is a positive integer which can be determined by balancing the linear term of the highest order with the nonlinear term in eq. (24), A_i are constants to be determined and f satisfies an elliptic equation of the first kind, for example, of the form

$$f'' = pf + qf^3,\ \ f'^2 = pf^2 + \frac{1}{2}qf^4 + r. \tag{27}$$

Here the prime denotes the derivative with respect to ξ, and $p,\ q$ and r are parameters to be determined.

After substituting eq. (26) into the ordinary differential equation and using eq. (27), the constants will be determined. A mapping relation is thus established in eq. (26) between the solution to eq. (27) and eq. (24).

This mapping method can be generalized and the solution can be searched in the form

$$u(\xi) = \sum_{i=0}^{n} A_i f^i + \sum_{i=1}^{n} B_i f^{-i} \tag{28}$$

for the modified mapping method and

$$u(\xi) = \sum_{i=0}^{n} A_i f^i + \sum_{i=1}^{n} B_i g^i \tag{29}$$

for the extended mapping method where f satisfies eq. (27) and g satisfies

$$g'' = g\,(c_1 + c_2 f^2), \quad g^2 = c_3 + c_4 f^2, \tag{30}$$

where c_i are also constants to be determined.

3.2 *Mapping method*

Now, we solve eq. (10) for $n = 1$ by mapping methods [7,8] which generate a variety of periodic wave solutions (PWSs) in terms of Jacobi elliptic functions and we subsequently derive their infinite period counterparts in terms of hyperbolic functions which are either solitary wave solutions (SWSs) or explode decay mode solutions. When $n = 1$, eq. (10) reduces to

$$Ag'' + Bg + Cg^3 = 0, \tag{31}$$

where

$$A = v^2 - k^2, \quad B = a - \omega^2 + k^2\kappa^2, \quad C = \frac{c(v^2 - k^2) + bd}{v^2 - k^2}. \tag{32}$$

We assume that eq. (31) has a solution in the form

$$g = A_0 + A_1 f, \tag{33}$$

where

$$f'' = pf + qf^3, \quad f'^2 = pf^2 + \frac{1}{2}qf^4 + r. \tag{34}$$

Equation (33) is the mapping relation between the solution to eq. (34) and that of eq. (31). Substituting eq. (33) into eq. (31), using eq. (34) and equating the coefficients of like powers of f to zero, we arrive at the set of relations

$$qAA_1 + CA_1^3 = 0, \tag{35}$$

$$3CA_0 A_1^2 = 0, \tag{36}$$

$$(pA + B)A_1 + 3\,CA_0^2 A_1 = 0, \tag{37}$$

$$BA_0 + CA_0^3 = 0, \tag{38}$$

from which we obtain

$$A_0 = 0, \quad A_1 = \pm\sqrt{\frac{qB}{pC}}, \quad pA + B = 0. \tag{39}$$

Using eq. (32), we obtain the exact solution of eq. (31) as

$$g(s) = \pm\sqrt{\frac{q}{p}\,\frac{(a-\omega^2+k^2\kappa^2)\,(v^2-k^2)}{c(v^2-k^2)+bd}}\;f(s). \tag{40}$$

Now, we consider the specific expressions of f according to eq. (32).

Case 1. $p = -2,\ q = 2,\ r = 1.$

In this case, the solution of eq. (34) is $f(s) = \tanh(s)$. So, we have the shock wave solution of eq. (31) as

$$g(s) = \pm\sqrt{-\frac{(a-\omega^2+k^2\kappa^2)\,(v^2-k^2)}{c(v^2-k^2)+bd}}\;\tanh(s). \tag{41}$$

Case 2. $p = 1,\ q = -2,\ r = 0.$

Here, the solution of eq. (34) is $f(s) = \mathrm{sech}(s)$. Now, we have the SWS of eq. (31) as

$$g(s) = \pm\sqrt{-2\,\frac{(a-\omega^2+k^2\kappa^2)\,(v^2-k^2)}{c(v^2-k^2)+bd}}\;\mathrm{sech}(s). \tag{42}$$

Case 3. $p = -(1+m^2),\ q = 2m^2,\ r = 1.$

The solution of eq. (34) is $f(s) = \mathrm{sn}(s)$ or $f(s) = \mathrm{cd}(s)$. Thus, we have the PWSs of eq. (31) as

$$g(s) = \pm\sqrt{-\frac{2m^2}{1+m^2}\,\frac{(a-\omega^2+k^2\kappa^2)\,(v^2-k^2)}{c(v^2-k^2)+bd}}\;\mathrm{sn}(s) \tag{43}$$

and

$$g(s) = \pm\sqrt{-\frac{2m^2}{1+m^2}\,\frac{(a-\omega^2+k^2\kappa^2)\,(v^2-k^2)}{c(v^2-k^2)+bd}}\;\mathrm{cd}(s). \tag{44}$$

As $m \to 1$, the shock wave solution (41) is recovered from eq. (43).

Case 4. $p = 2-m^2,\ q = -2,\ r = m^2-1.$

Now, the solution of eq. (34) is $f(s) = \mathrm{dn}(s)$ So, the PWSs of eq. (31) is

$$g(s) = \pm\sqrt{-\frac{2}{2-m^2}\,\frac{(a-\omega^2+k^2\kappa^2)\,(v^2-k^2)}{c(v^2-k^2)+bd}}\;\mathrm{dn}(s). \tag{45}$$

As $m \to 1$, the SWS of (42) is recovered from eq. (45).

Case 5. $p = -(1+m^2),\ q = 2,\ r = m^2.$

Here, the solution of eq. (34) is $f(s) = \mathrm{ns}(s)$ or $f(s) = \mathrm{dc}(s)$. So, the PWSs of eq. (31) are

$$g(s) = \pm\sqrt{-\frac{2}{1+m^2}\,\frac{(a-\omega^2+k^2\kappa^2)\,(v^2-k^2)}{c(v^2-k^2)+bd}}\;\mathrm{ns}(s) \tag{46}$$

and

$$g(s) = \pm\sqrt{-\frac{2}{1+m^2}\,\frac{(a-\omega^2+k^2\kappa^2)\,(v^2-k^2)}{c(v^2-k^2)+bd}}\,\mathrm{dc}(s). \tag{47}$$

As $m \to 0$, eqs (46) and (47) degenerate as

$$g(s) = \pm\sqrt{-2\,\frac{(a-\omega^2+k^2\kappa^2)\,(v^2-k^2)}{c(v^2-k^2)+bd}}\;\mathrm{cosec}(s) \tag{48}$$

and

$$g(s) = \pm\sqrt{-2\,\frac{(a-\omega^2+k^2\kappa^2)\,(v^2-k^2)}{c(v^2-k^2)+bd}}\;\sec(s). \tag{49}$$

When $m \to 1$, we obtain from eq. (46) the solution in the form

$$g(s) = \pm\sqrt{-\frac{(a-\omega^2+k^2\kappa^2)\,(v^2-k^2)}{c(v^2-k^2)+bd}}\;\coth(s). \tag{50}$$

3.3 *Modified mapping method*

Now, we use the modified mapping method in which we assume a solution of eq. (31) in the form

$$g = A_0 + A_1 f + B_1 f^{-1}, \tag{51}$$

where f satisfies eq. (34). We substitute eq. (51) into eq. (31), use eq. (34) and equate the coefficients of like powers of f to zero to arrive at a set of equations from which it can be found that

$$A_0 = 0, \quad A_1 = \pm\sqrt{-\frac{qA}{C}} \tag{52}$$

$$B_1 = \pm\sqrt{-\frac{2rA}{C}}, \quad pA + B + 3CA_1B_1 = 0. \tag{53}$$

Thus for real solutions of eq. (31) to exist, when q and r are both positive, A and C should be of opposite signs and when q and r are both negative, A and C should be of the same signs. So, we have another set of new exact solution of eq. (31) which is given by

$$g(s) = \pm\sqrt{-\frac{q(v^2-k^2)^2}{c(v^2-k^2)+bd}}\,f(s) \pm \sqrt{-\frac{2r(v^2-k^2)^2}{c(v^2-k^2)+bd}}\,f^{-1}(s). \tag{54}$$

Case 1. $p = -2,\ q = 2,\ r = 1.$

Here, the solution of eq. (34) is $f(s) = \tanh(s)$. So, the solution of eq. (31) is

$$g(s) = \sqrt{-\frac{2(v^2-k^2)^2}{c(v^2-k^2)+bd}}\,\{\pm\tanh(s) \pm \coth(s)\}. \tag{55}$$

Case 2. $p = -(1 + m^2),\ q = 2m^2,\ r = 1.$

So, the solution of eq. (34) is $f(s) = \text{sn}(s)$ or $f(s) = \text{cd}(s)$. Thus, the PWSs of eq. (31) are

$$g(s) = \sqrt{-\frac{2(v^2-k^2)^2}{c(v^2-k^2)+bd}}\{\pm m^2\,\text{sn}(s) \pm \text{ns}(s)\} \tag{56}$$

and

$$g(s) = \sqrt{-\frac{2(v^2-k^2)^2}{c(v^2-k^2)+bd}}\{\pm m^2\,\text{cd}(s) \pm \text{dc}(s)\}. \tag{57}$$

When $m \to 0$, eqs (56) and (57) will give rise respectively to the solutions

$$g(s) = \pm\sqrt{-\frac{2(v^2-k^2)^2}{c(v^2-k^2)+bd}}\,\text{cosec}(s) \tag{58}$$

and

$$g(s) = \pm\sqrt{-\frac{2(v^2-k^2)^2}{c(v^2-k^2)+bd}}\,\sec(s) \tag{59}$$

and, when $m \to 1$, eq. (56) degenerates to eq. (55).

Case 3. $p = 2 - m^2,\ q = 2,\ r = 1 - m^2.$

So, the solution of eq. (34) is $f(s) = \text{cs}(s)$. Hence, the PWS of eq. (31) is

$$g(s) = \sqrt{-\frac{2(v^2-k^2)^2}{c(v^2-k^2)+bd}}\{\pm \text{cs}(s) \pm (1-m^2)\,\text{sc}(s)\}. \tag{60}$$

When $m \to 0$, eq. (60) will reduce to

$$g(s) = \sqrt{-\frac{2(v^2-k^2)^2}{c(v^2-k^2)+bd}}\ \{\pm \tan(s) \pm \cot(s)\}, \tag{61}$$

and when $m \to 1$, eq. (60) will give rise to the explode decay mode solution

$$g(s) = \sqrt{-\frac{2(v^2-k^2)^2}{c(v^2-k^2)+bd}}\ \text{cosech}(s). \tag{62}$$

Case 4. $p = 2 - m^2,\ q = -2,\ r = -(1 - m^2).$

Here, the solution of eq. (34) is $f(s) = \text{dn}(s)$. So, the PWS of eq. (31) is

$$g(s) = \sqrt{\frac{2(v^2-k^2)^2}{c(v^2-k^2)+bd}}\{\pm \text{dn}(s) \pm (1-m^2)\,\text{nd}(s)\}. \tag{63}$$

When $m \to 1$, eq. (63) will give rise to the SWS

$$\sqrt{\frac{2(v^2-k^2)^2}{c(v^2-k^2)+bd}}\ \mathrm{sech}(s). \tag{64}$$

3.4 *Extended mapping method*

Now, we use the extended mapping method in which we assume a solution of eq. (31) in the form

$$g = A_0 + A_1 f + B_1 f^*, \tag{65}$$

where f satisfies eq. (34) and f^* satisfies

$$f^{*\prime\prime} = f^*(c_1 + c_2 f^2), \quad f^{*2} = c_3 + c_4 f^2. \tag{66}$$

We substitute eq. (65) into eq. (31), use eqs (34) and (66) and then equate the coefficients of like powers of f to zero to arrive at a set of equations from which we obtain

$$A_0 = 0, \tag{67}$$

$$A_1 = \pm\sqrt{\frac{(c_4 p - c_3 q)(v^2-k^2)^2 + c_4(a-\omega+k^2\kappa^2)(v^2-k^2)}{c_3\{c(v^2-k^2)+bd\}}}, \tag{68}$$

$$B_1 = \pm\sqrt{-\frac{p(v^2-k^2)^2 + (a-\omega+k^2\kappa^2)(v^2-k^2)}{3c_3\{c(v^2-k^2)+bd\}}}. \tag{69}$$

By this method, the new exact solution of eq. (31) is given by

$$\begin{aligned} g(s) = {}& \pm\sqrt{\frac{(c_4 p - c_3 q)(v^2-k^2)^2 + c_4(a-\omega+k^2\kappa^2)(v^2-k^2)}{c_3\{c(v^2-k^2)+bd\}}}\ f(s) \\ & \pm\sqrt{-\frac{p(v^2-k^2)^2 + (a-\omega+k^2\kappa^2)(v^2-k^2)}{3c_3\{c(v^2-k^2)+bd\}}}\ f^*(s). \end{aligned} \tag{70}$$

Case 1. $p = 2m^2 - 1$, $q = -2m^2$, $r = 1 - m^2$, $c_1 = m^2$, $c_2 = -2m^2$, $c_3 = 1 - m^2$, $c_4 = m^2$.

In this case, $f(s) = \mathrm{cn}(s)$ and $f^*(s) = \mathrm{dn}(s)$. Thus the PWS of eq. (31) is

$$\begin{aligned} g(s) = {}& \pm\sqrt{\frac{(v^2-k^2)^2 + (a-\omega+k^2\kappa^2)(v^2-k^2)}{(1-m^2)\{c(v^2-k^2)+bd\}}}\ m\,\mathrm{cn}(s) \\ & \pm\sqrt{-\frac{(2m^2-1)(v^2-k^2)^2 + (a-\omega+k^2\kappa^2)(v^2-k^2)}{3(1-m^2)\{c(v^2-k^2)+bd\}}}\ \mathrm{dn}(s). \end{aligned} \tag{71}$$

Case 2. $p = 2 - m^2$, $q = -2(1 - m^2)$, $r = -1$, $c_1 = 1$, $c_2 = -2(1 - m^2)$, $c_3 = -1/m^2$, $c_4 = 1/m^2$.

In this case, $f(s) = \mathrm{nd}(s)$ and $f^*(s) = \mathrm{sd}(s)$. So, the PWS of eq. (31) is

$$g(s) = \pm\sqrt{-\frac{m^2(v^2-k^2)^2+(a-\omega+k^2\kappa^2)(v^2-k^2)}{\{c(v^2-k^2)+bd\}}}\,\mathrm{nd}(s)$$
$$\pm\sqrt{\frac{(v^2-k^2)^2+(a-\omega+k^2\kappa^2)(v^2-k^2)}{3\{c(v^2-k^2)+bd\}}}\,m\,\mathrm{sd}(s). \quad (72)$$

When $m \to 1$, eq. (72) gives rise to the solution

$$g(s) = \pm\sqrt{-\frac{(v^2-k^2)^2+(a-\omega+k^2\kappa^2)(v^2-k^2)}{\{c(v^2-k^2)+bd\}}}\,\cosh(s)$$
$$\pm\sqrt{\frac{(v^2-k^2)^2+(a-\omega+k^2\kappa^2)(v^2-k^2)}{3\{c(v^2-k^2)+bd\}}}\,\mathrm{sinhs}(s). \quad (73)$$

Case 3. $p = 2-m^2$ $q = 2,\ r = 1-m^2, c_1 = 1,\ c_2 = 2,\ c_3 = 1-m^2,\ c_4 = 1.$

In this case, $f(s) = \mathrm{cs}(s)$ and $f^*(s) = \mathrm{ds}(s)$. Thus the PWS of eq. (31) is

$$g(s) = \pm\sqrt{\frac{m^2(v^2-k^2)^2+(a-\omega+k^2\kappa^2)(v^2-k^2)}{(1-m^2)\{c(v^2-k^2)+bd\}}}\,\mathrm{cs}(s)$$
$$\pm\sqrt{-\frac{(2-m^2)(v^2-k^2)^2+(a-\omega+k^2\kappa^2)(v^2-k^2)}{3(1-m^2)\{c(v^2-k^2)+bd\}}}\,\mathrm{ds}(s). \quad (74)$$

When $m \to 0$, eq. (74) will reduce to the solution

$$g(s) = \pm\sqrt{\frac{(a-\omega+k^2\kappa^2)(v^2-k^2)}{\{c(v^2-k^2)+bd\}}}\,\cot(s)$$
$$\pm\sqrt{-\frac{2(v^2-k^2)^2+(a-\omega+k^2\kappa^2)(v^2-k^2)}{3\{c(v^2-k^2)+bd\}}}\,\mathrm{cosec}(s). \quad (75)$$

4. (G'/G) Method

In this section the (G'/G) method will be applied to solve the KGZ equation. First the method will be described and subsequently it will be applied to solve the equation.

4.1 *Description of the method*

Suppose that a nonlinear partial differential equation is given by

$$P(q, q_x, q_t, q_{xx}, q_{xt}, q_{tt}, \ldots) = 0, \quad (76)$$

where $q = q(x,t)$ is an unknown function, P is a polynomial in $q = q(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the (G'/G)-expansion method [1–3].

Step 1. We consider the travelling wave variable $q(x,t) = q(\xi), \xi = x - vt$ that reduce (70) to the ODE for $q = q(\xi)$

$$P(q, vq', q', v^2q'', -vq'', q'', \ldots) = 0. \tag{77}$$

Step 2. Suppose that the solution of ODE (77) can be expressed by the following polynomial in (G'/G):

$$q(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \quad a_m \neq 0, \tag{78}$$

where $G = G(\xi)$ satisfies the second-order ODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{79}$$

where λ, μ and a_i's are constants to be determined later. The general solution of (79) is well known, and hence based on the sign of $\Delta = \lambda^2 - 4\mu$ we get the following cases:

$$\frac{G'(\xi)}{G(\xi)} = \beta_1\left(\frac{c_1 \cosh(\beta_1\xi) + c_2 \sinh(\beta_1\xi)}{c_1 \sinh(\beta_1\xi) + c_2 \cosh(\beta_1\xi)}\right) - \frac{\lambda}{2}, \quad \lambda^2 - 4\mu > 0, \tag{80}$$

where $\beta_1 = \sqrt{(\lambda^2 - 4\mu)}/2$ and

$$\frac{G'(\xi)}{G(\xi)} = \beta_2\left(\frac{c_1 \cosh(\beta_2\xi) + c_2 \sinh(\beta_2\xi)}{c_1 \sinh(\beta_2\xi) + c_2 \cosh(\beta_2\xi)}\right) - \frac{\lambda}{2}, \quad \lambda^2 - 4\mu < 0, \tag{81}$$

where $\beta_2 = \sqrt{(4\mu - \lambda^2)}/2$ and finally

$$\frac{G'(\xi)}{G(\xi)} = \frac{c_1}{c_1\xi + c_2} - \frac{\lambda}{2}, \quad \lambda^2 - 4\mu = 0. \tag{82}$$

The positive integer m in (78) can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (77).

Step 3. By substituting (78) into (77) and using second-order ODE (77), collecting all terms with the same order of (G'/G) together, the left-hand side of (77) is converted into another polynomial in (G'/G). Equating each coefficient of this polynomial to zero yields a set of algebraic equations for v, μ, λ and a_i's.

Step 4. Assuming that the unknown constants v, μ, λ and a_i can be obtained by solving the algebraic equations in Step 3, then substituting v, a_i's and the general solutions of (79) into (78), more travelling wave solutions of the nonlinear evolution eq. (76) can be obtained.

4.2 *Application to KGZ equation*

We shall apply the (G'/G)-expansion method to the KGZ systems (1) and (2). For this we use variables (3), (4) and $\xi = x - vt$. Substituting (3) and (4) into (2) implies (6). Integrating (6) twice gives

$$h(x,t) = \frac{d}{v^2 - k^2} g^{2n}(\xi) + \eta_1\xi + \eta_2. \tag{83}$$

Again substituting (3) and (4) into (1) and (2) and then decomposing into real and imaginary parts give

$$v = \frac{\kappa k^2}{\omega} \tag{84}$$

and

$$(v^2 - k^2)g'' + (a - \omega^2 + k^2\kappa^2)g + bgh + cg^{2n+1} + b(\eta_1\xi + \eta_2)g = 0, \tag{85}$$

where η_1 and η_2 are the integration constants to be determined later. Now, eq. (85), by virtue of (83) reduces to

$$(v^2 - k^2)g'' + (a - \omega^2 + k^2\kappa^2)g + \left(c + \frac{bd}{v^2 - k^2}\right) g^{2n+1} + b(\eta_1\xi + \eta_2)g = 0. \tag{86}$$

Suppose that the solution of ODE (86) can be expressed by a polynomial in (G'/G) as follows:

$$g(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{87}$$

where $G = G(\xi)$ satisfies the second-order ODE in the form

$$G'' + \lambda G' + \mu G = 0. \tag{88}$$

By using (87) and (88) one can easily derive

$$g^{2n+1} = a_m^{2n+1} \left(\frac{G'}{G}\right)^{(2n+1)m}, \ldots, \tag{89}$$

$$g'' = m(m+1)a_m \left(\frac{G'}{G}\right)^{m+2}, \ldots. \tag{90}$$

Considering the homogeneous balance between g'' and g^{2n+1} in (86), based on (89) and (90) we required that $m + 2 = (2n + 1)m \Rightarrow m = 1/n$. Thus, using transformation

$$g = W^{1/n}, \tag{91}$$

convert (86) into the following ordinary differential equation:

$$(k^2\kappa^2 - w^2)k^2(k^2(1-n)W'^2 + k^2 nWW'' + (a + w^2 - k^2\kappa^2 + b(\eta_1\xi + \eta_2))n^2w^2W^2 + cn^2w^2W^4) + bdn^2w^4W^4 = 0. \tag{92}$$

Suppose that the solutions of (92) can be expressed by a polynomial in (G'/G) as follows:

$$W(\xi) = \sum_{i=0}^{M} A_i \left(\frac{G'}{G}\right)^i, \tag{93}$$

where $G = G(\xi)$ satisfies the second-order ODE (88). Balancing W^4 with $W''W$ in (92) gives

$$\left(\frac{G'}{G}\right)^{4M} = \left(\frac{G'}{G}\right)^{2M+2}, \tag{94}$$

so that

$$M = 1. \tag{95}$$

Thus we can write (93) as

$$W(\xi) = A_0 + A_1\left(\frac{G'(\xi)}{G(\xi)}\right), \tag{96}$$

where A_0 and A_1 are constants to be determined.

Substituting the general solution of (88) together with W, W^2, W^4, W'^2, W'' and WW'' as polynomials of (G'/G) into eq. (92) yield algebraic equations involving powers of (G'/G) and ξ. Equating the coefficients of $(G'/G)^i$'s and $\xi(G'/G)^i$'s to zero gives a system of algebraic equations for A_0, A_1, λ, μ, η_1, η_2, κ and w. Then, solving this system by Maple gives

$$A_1 = \frac{\sqrt{(-(-w^2ck^2 + ck^4\kappa^2 + w^2bd)(1+m))}(w^2 - k^2\kappa^2)k^2}{(-w^2ck^2 + ck^4\kappa^2 + w^2bd)wm}, \tag{97}$$

$$A_0 = \pm\frac{l}{2}A_1, \tag{98}$$

$$\eta_1 = 0, \quad \eta_2 = \frac{w^2 - k^2\kappa^2 - a}{b} \tag{99}$$

and

$$\mu = \frac{\lambda^2}{4}. \tag{100}$$

Since $\Delta = \lambda^2 - 4\mu = 0$,

$$\frac{G'}{G} = \frac{-\lambda}{2} + \frac{c_1}{c_1 s + c_2} \tag{101}$$

and also using relations (3)–(5) gives the following rational travelling wave solutions for (1) and (2):

$$r(x,t) = \frac{d}{(\kappa k^2/w)^2 - k^2} g^{2m}\left(x - \frac{\kappa k}{w}t\right) + \frac{w^2 - k^2\kappa^2 - a}{b} \tag{102}$$

and

$$q(x,t) = g\left(x - \frac{\kappa k}{w}t\right)\mathrm{e}^{i(-\kappa x + wt + \theta)}, \tag{103}$$

where substituting (98) into (96) gives

$$g\left(x-\frac{\kappa k}{w}t\right) = W^{1/n} = \left(A_1\left(\pm\frac{\lambda}{2}+\frac{G'}{G}\right)\right)^{1/n}$$
$$= \left(A_1\left(\pm\frac{\lambda}{2}+\frac{-\lambda}{2}+\frac{c_1}{c_1(x-(\kappa k^2/w)t)+c_2}\right)\right)^{1/n} \quad (104)$$

and $c_1, c_2, k, \lambda, w, \kappa$ and θ are arbitrary constants.

The relations (32), (47) and (52) pose the restrictions given by

$$|k|< v, \quad b\neq 0, \quad w\neq 0 \quad (105)$$

and

$$ck^2(w^2-\kappa^2) > w^2bd. \quad (106)$$

5. Conclusions

This paper studies and solves the KGZ equation with power-law nonlinearity. First the travelling wave hypothesis is applied to obtain a 1-soliton solution to this equation. Subsequently, the mapping methods and their variations are applied to solve the KGZ equation with cubic nonlinearity. By using this technique, cnoidal and snoidal waves are extracted for the KGZ equation. Finally, the (G'/G) method is employed to extract more solutions to the KGZ equation with power-law nonlinearity. These solutions are going to be very useful for the further analysis of this equation in various areas of Applied Mathematics and Mathematical Physics.

In future, the plan is to study the numerical simulations for this equation along with other techniques such as variational iteration method, homotopy perturbation method and so on. Those results will be reported in future publications.

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