PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 79, No. 2
August 2012
pp. 173–183

Complex dynamical invariants for two-dimensional complex potentials

J S VIRDI[1,*], F CHAND[2], C N KUMAR[1] and S C MISHRA[2]
[1]Department of Physics, Punjab University, Chandigarh 160 014, India
[2]Department of Physics, Kurukshetra University, Kurukshetra 136 119, India
[*]Corresponding author. E-mail: jpsvirdi@gmail.com

MS received 15 February 2011; revised 23 December 2011; accepted 1 March 2012

Abstract. Complex dynamical invariants are searched out for two-dimensional complex potentials using rationalization method within the framework of an extended complex phase space characterized by $x = x_1 + ip_3,\ y = x_2 + ip_4,\ p_x = p_1 + ix_3,\ p_y = p_2 + ix_4$. It is found that the cubic oscillator and shifted harmonic oscillator admit quadratic complex invariants. The obtained invariants may be useful for studying non-Hermitian Hamiltonian systems.

Keywords. Complex Hamiltonian; exact complex invariant.

PACS Nos 02.30.IK; 03.65.Fd; 03.20+i

1. Introduction

The theory of invariants is very important in the study dynamical systems. In the past, invariants have been successfully utilized in a variety of problems in different branches of science, viz. for testing the stability of solutions of differential equations (onset of chaos) [1], to reduce the order of differential equations [2], to diagonalize the Hamiltonian of a system or to derive the eigenfunctions in Schrödinger picture [3], to find solutions of some time-dependent quantum mechanical problems via eigenvalues and eigenfunctions of invariants [4], to find the solution of time-dependent Cauchy's problem [5], to check the accuracy of numerical simulation [6] etc. Thus, many methods have been devised for the construction of invariants and many studies on the construction of invariants and their possible applications have been reported [7–9].

It is well known that a real Hamiltonian representation of a system can provide a good amount of information about it, but in some cases, particularly in dissipative systems, a complex Hamiltonian form can be more appropriate for providing a better insight of the underlying dynamics. Moreover, with the advent of PT-symmetric quantum mechanics, the study of dynamical systems in complex space becomes more significant for explaining many physical problems [10]. Therefore, presently a great deal of research activities is

DOI: 10.1007/s12043-012-0300-y; ***e*Publication: 11 July 2012**

going on to study different aspects of complex/non-Hermitian Hamiltonian systems [10]. In this endeavor, search of invariants for non-Hermitian Hamiltonian systems may also be interesting.

In the past, some authors found complex invariants [5,9,11–15] and their possible applications particularly in particle physics [16,17]. Recently, with a view to explore the role of invariants for complex systems, Kaushal *et al* [18] found invariants for some one-dimensional systems within the framework of an extended complex phase space (ECPS). Some quantum mechanical studies within the ECPS are also reported [19]. But most of such studies are restricted in one dimension only. Thus, the generalization of such studies in higher dimensions is desirable from the intrinsic mathematical interest, to check the validity of various methods/theories and to find solutions of some realistic physical problems. With this motivation, recently we generalized the ECPS in two dimensions and studied some classical and quantum systems [20,21]. With the same spirit, in the present work, we construct dynamical invariants for two complex systems within the ECPS using rationalization method.

The presentation of the paper as follows. In §2, we briefly develop the basic formalism which enables us to construct invariants. In §3, construction of complex invariants for two physical systems is carried out and finally, concluding remarks are presented in §4.

2. The formalism

Consider a two-dimensional real phase space (x, y, p_x, p_y), which may be transformed into the corresponding ECPS $(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4)$ by defining

$$x = x_1 + ip_3; \quad y = x_2 + ip_4; \quad p_x = p_1 + ix_3; \quad p_y = p_2 + ix_4. \tag{1}$$

The above transformations add four additional degrees of freedom, (x_3, x_4, p_3, p_4), which can make mathematical analysis of a problem a bit more involved. But nevertheless, these types of transformations are used in many studies [18–22]. Note that (x_1, p_1), (x_2, p_2), (x_3, p_3) and (x_4, p_4) form canonical pairs [18]. From eq. (1) one can easily obtain

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_3}; \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4}; \qquad \frac{\partial}{\partial p_x} = \frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3};$$
$$\frac{\partial}{\partial p_y} = \frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}. \tag{2}$$

Therefore, the Hamiltonian $H(x, y, p_x, p_y)$ of a two-dimensional autonomous system in ECPS is expressed as

$$\begin{aligned} H = {} & H_1(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4) \\ & + iH_2(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4). \end{aligned} \tag{3}$$

Now consider a function $I(x, y, p_x, p_y, t)$ in complex phase space as

$$I = I_1(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4) + iI_2(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4). \tag{4}$$

The function I will be a dynamical invariant of the system, provided it satisfies the invariance condition

$$\frac{\mathrm{d}I}{\mathrm{d}t} = [I, H] = 0, \tag{5}$$

where [., .] is the Poisson bracket which in the ECPS becomes

$$\begin{aligned}[I, H] = {} & [I, H]_{(x_1,p_1)} - i[I, H]_{(x_1,x_3)} - i[I, H]_{(p_3,p_1)} - [I, H]_{(p_3,x_3)} \\ & + [I, H]_{(x_2,p_2)} - i[I, H]_{(x_2,x_4)} - i[I, H]_{(p_4,p_2)} - [I, H]_{(p_4,x_4)} = 0.\end{aligned} \tag{6}$$

Now, using eqs (3) and (4) in eq. (5) and after equating real and imaginary parts separately to zero, one obtains the following pair of equations:

$$\begin{aligned}&\left(\frac{\partial I_1}{\partial x_1}+\frac{\partial I_2}{\partial p_3}\right)\left(\frac{\partial H_1}{\partial p_1}+\frac{\partial H_2}{\partial x_3}\right)-\left(\frac{\partial I_2}{\partial x_1}-\frac{\partial I_1}{\partial p_3}\right)\left(\frac{\partial H_2}{\partial p_1}-\frac{\partial H_1}{\partial x_3}\right)\\ &-\left(\frac{\partial I_1}{\partial p_1}+\frac{\partial I_2}{\partial x_3}\right)\left(\frac{\partial H_1}{\partial x_1}+\frac{\partial H_2}{\partial p_3}\right)+\left(\frac{\partial I_2}{\partial p_1}-\frac{\partial I_1}{\partial x_3}\right)\left(\frac{\partial H_2}{\partial x_1}-\frac{\partial H_1}{\partial p_3}\right)\\ &+\left(\frac{\partial I_1}{\partial x_2}+\frac{\partial I_2}{\partial p_4}\right)\left(\frac{\partial H_1}{\partial p_2}+\frac{\partial H_2}{\partial x_4}\right)-\left(\frac{\partial I_2}{\partial x_2}-\frac{\partial I_1}{\partial p_4}\right)\left(\frac{\partial H_2}{\partial p_2}-\frac{\partial H_1}{\partial x_4}\right)\\ &-\left(\frac{\partial I_1}{\partial p_2}+\frac{\partial I_2}{\partial x_4}\right)\left(\frac{\partial H_1}{\partial x_2}+\frac{\partial H_2}{\partial p_4}\right)+\left(\frac{\partial I_2}{\partial p_2}-\frac{\partial I_1}{\partial x_4}\right)\left(\frac{\partial H_2}{\partial x_2}-\frac{\partial H_1}{\partial p_4}\right)=0,\end{aligned} \tag{7}$$

$$\begin{aligned}&\left(\frac{\partial I_2}{\partial x_1}-\frac{\partial I_1}{\partial p_3}\right)\left(\frac{\partial H_1}{\partial p_1}+\frac{\partial H_2}{\partial x_3}\right)+\left(\frac{\partial I_1}{\partial x_1}+\frac{\partial I_2}{\partial p_3}\right)\left(\frac{\partial H_2}{\partial p_1}-\frac{\partial H_1}{\partial x_3}\right)\\ &-\left(\frac{\partial I_2}{\partial p_1}-\frac{\partial I_1}{\partial x_3}\right)\left(\frac{\partial H_1}{\partial x_1}+\frac{\partial H_2}{\partial p_3}\right)-\left(\frac{\partial I_1}{\partial p_1}+\frac{\partial I_2}{\partial x_3}\right)\left(\frac{\partial H_2}{\partial x_1}-\frac{\partial H_1}{\partial p_3}\right)\\ &+\left(\frac{\partial I_2}{\partial x_2}-\frac{\partial I_1}{\partial p_4}\right)\left(\frac{\partial H_1}{\partial p_2}+\frac{\partial H_2}{\partial x_4}\right)+\left(\frac{\partial I_1}{\partial x_2}+\frac{\partial I_2}{\partial p_4}\right)\left(\frac{\partial H_2}{\partial p_2}-\frac{\partial H_1}{\partial x_4}\right)\\ &-\left(\frac{\partial I_2}{\partial p_2}-\frac{\partial I_1}{\partial x_4}\right)\left(\frac{\partial H_1}{\partial x_2}+\frac{\partial H_2}{\partial p_4}\right)-\left(\frac{\partial I_1}{\partial p_2}+\frac{\partial I_2}{\partial x_4}\right)\left(\frac{\partial H_2}{\partial x_2}-\frac{\partial H_1}{\partial p_4}\right)=0.\end{aligned} \tag{8}$$

Equations (7) and (8) essentially form a basis for obtaining invariants of a system. In brief, the procedure for the construction of invariants is as follows. For a given $H(x, p)$ make an ansatz for I, preferably a polynomial in momenta, and reduce both H and I to the forms $I = I_1 + iI_2$ and $H = H_1 + iH_2$ and then substitute I_1, I_2, H_1 and H_2 in eqs (7) and (8). The rationalization of the resultant expressions with respect to powers of p_1, x_3, p_2 and x_4 and their different combinations will yield a set of coupled partial differential equations (PDEs) for the arbitrary complex coupling coefficient functions appearing in the ansatz for I. Finally, solve these equations successively and substitution of solutions of these PDEs in the ansatz for I yields the invariant.

In the present work, we are mainly concerned with the construction of quadratic complex invariants. Thus, the ansatz for an invariant is made as

$$I = a_0(x, y) + a_1(x, y)(p_x^2 + p_y^2) + a_2(x, y)p_x p_y, \tag{9}$$

where a_0, a_1 and a_2 are unknown complex functions, which are to be determined later, of the form $a_k(x, y) = a_{krx} + a_{kry} + i(a_{kix} + a_{kiy})$ with $k = 0, 1, 2$ and the third subscripts x and y represent the arguments of the coefficients. From the above equation, the real and imaginary parts of I are respectively written as

$$\begin{aligned} I_1 &= a_{0rx} + a_{0ry} + (a_{1rx} + a_{1ry})\left(p_1^2 + p_2^2 - x_3^2 - x_4^2\right) \\ &\quad - 2(p_1x_3 + p_2x_4)(a_{1ix} + a_{1iy}) + (a_{2rx} + a_{2ry})(p_1p_2 - x_1x_2) \\ &\quad - (a_{2ix} + a_{2iy})(p_1x_4 - p_2x_3), \end{aligned} \tag{10}$$

$$\begin{aligned} I_2 &= a_{0ix} + a_{0iy} + (a_{1ix} + a_{1iy})\left(p_1^2 + p_2^2 - x_3^2 - x_4^2\right) \\ &\quad + 2(p_1x_3 + p_2x_4)(a_{1rx} + a_{1ry}) + (a_{2rx} + a_{2ry})(p_1x_4 - p_2x_3) \\ &\quad + (a_{2xi} + a_{2iy})(p_1p_2 - x_4x_3). \end{aligned} \tag{11}$$

After developing the formalism, in what follows we find invariants of two specific complex potentials.

3. Complex cubic potential

With a view of constructing complex invariants for some cases, in this section we use the method discussed in the previous section. We first consider the case of a complex cubic potential described by

$$H = p_x^2 + p_y^2 + \delta_1(ix + iy) + \delta_2\left\{(ix)^2 + (iy)^2\right\} + \delta_3\left\{(ix)^3 + (iy)^3\right\}. \tag{12}$$

This system can be transformed into ECPS by invoking eq. (1) and hence the corresponding real and imaginary parts of the Hamiltonian respectively become

$$\begin{aligned} H_1 &= \left(p_1^2 + p_2^2 - x_3^2 - x_4^2\right) - \delta_1(p_3 + p_4) + \delta_2\left(p_3^2 + p_4^2 - x_1^2 - x_2^2\right) \\ &\quad - \delta_3\left(p_3^3 + p_4^3 + 3x_1^2p_3 + 3x_2^2p_4\right), \\ H_2 &= 2p_1x_3 + 2p_2x_4 + \delta_1(x_1 + x_2) - 2\delta_2(x_1p_3 + x_2p_4) \\ &\quad - \delta_3\left(x_1^3 + x_2^3 + 3x_1p_3^2 + 3x_2p_4^2\right). \end{aligned} \tag{13}$$

Now assume that the system (12) admits an invariant, a quadratic polynomial in momenta, of the form given in eq. (9). Thus, in order to determine the unknown complex coupling functions a_0, a_1 and a_2 for the given system (12), insert eqs (10), (11) and (13) in eq. (7) and then rationalization of the resultant expression with respect to the powers of p_1, x_3, p_2, x_4 and their different combinations, give the following set of 12 coupled PDEs:

$$\frac{\partial a_{1rx}}{\partial x_1} + \frac{\partial a_{1ix}}{\partial p_3} = 0, \tag{14}$$

$$\frac{\partial a_{1ix}}{\partial x_1} - \frac{\partial a_{1rx}}{\partial p_3} = 0, \tag{15}$$

$$\frac{\partial a_{2rx}}{\partial x_1} + \frac{\partial a_{2ix}}{\partial p_3} = 0, \tag{16}$$

$$\frac{\partial a_{2ix}}{\partial x_1} - \frac{\partial a_{2rx}}{\partial p_3} = 0, \tag{17}$$

$$\frac{\partial a_{0rx}}{\partial x_1} + \frac{\partial a_{0ix}}{\partial p_3} - 4(2\delta_2 x_1 + 6\delta_3 x_1 p_3)(a_{1rx} + a_{1ry}) + 4(\delta_1 - 2\delta_2 p_3 - 3\delta_3 x_1)(a_{1ix} + a_{1iy}) = 0, \tag{18}$$

$$\frac{\partial a_{0rx}}{\partial p_3} - \frac{\partial a_{0ix}}{\partial x_1} + 4(2\delta_2 x_1 + 6\delta_3 x_1 p_3)(a_{1ix} + a_{1iy}) + 4(\delta_1 - 2\delta_2 p_3 - 3\delta_3 x_1)(a_{1rx} + a_{1ry}) = 0, \tag{19}$$

$$\frac{\partial a_{1ry}}{\partial x_2} + \frac{\partial a_{1iy}}{\partial p_4} = 0, \tag{20}$$

$$\frac{\partial a_{1iy}}{\partial x_2} - \frac{\partial a_{1ry}}{\partial p_4} = 0, \tag{21}$$

$$\frac{\partial a_{2ry}}{\partial x_2} + \frac{\partial a_{2iy}}{\partial p_4} = 0, \tag{22}$$

$$\frac{\partial a_{2iy}}{\partial x_2} - \frac{\partial a_{2ry}}{\partial p_4} = 0, \tag{23}$$

$$\frac{\partial a_{0ry}}{\partial x_2} + \frac{\partial a_{0iy}}{\partial p_4} - 4(2\delta_2 x_2 + 6\delta_3 x_2 p_4)(a_{1rx} + a_{0ry}) + 4(\delta_1 - 2\delta_2 p_4 - 3\delta_3 x_2)(a_{1ix} + a_{1iy}) = 0, \tag{24}$$

$$\frac{\partial a_{0ry}}{\partial p_4} - \frac{\partial a_{0iy}}{\partial x_2} + 4(2\delta_2 x_2 + 6\delta_3 x_2 p_4)(a_{1ix} + a_{1iy}) + 4(\delta_1 - 2\delta_2 p_4 - 3\delta_3 x_2)(a_{1rx} + a_{1ry}) = 0. \tag{25}$$

So, for obtaining the complex invariant for the present two-dimensional complex system, now we solve the above set of PDEs.

(i) Solutions for a_{1rx} and a_{1ix}: For the solutions for a_{1rx} and a_{1ix} , eqs (14) and (15) can be reduced to similar second-order PDEs respectively as

$$\frac{\partial^2 a_{1rx}}{\partial x_1^2} + \frac{\partial^2 a_{1rx}}{\partial p_3^2} = 0, \qquad \frac{\partial^2 a_{1ix}}{\partial x_1^2} + \frac{\partial^2 a_{1ix}}{\partial p_3^2} = 0. \tag{26}$$

Assuming separability of a_{1rx} and a_{1ix} by adding $a_{1rx} = X_{1rx}(x_1) + P_{1rx}(p_3)$ and $a_{1ix} = X_{1ix}(x_1) + P_{1ix}(p_3)$, the solutions of eq. (26) are obtained as

$$a_{1rx} = \frac{\alpha}{2}\left(x_1^2 - p_3^2\right) + \alpha_1 x_1 + \alpha_2 p_3 + \delta_3,$$
$$a_{1ix} = \frac{\beta}{2}\left(x_1^2 - p_3^2\right) + \beta_1 x_1 + \beta_2 p_3 + \delta_4, \tag{27}$$

where $\alpha, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \delta_3$ and δ_4 are arbitrary constants of integration.

(ii) Solutions for a_{2rx} and a_{2ix}: Similar to the previous case, eqs (16) and (17) are reduced to similar second-order forms for the functions a_{2rx} and a_{2ix}, respectively as

$$\frac{\partial^2 a_{2rx}}{\partial x_1^2} + \frac{\partial^2 a_{2rx}}{\partial p_3^2} = 0, \qquad \frac{\partial^2 a_{2ix}}{\partial x_1^2} + \frac{\partial^2 a_{2ix}}{\partial p_3^2} = 0, \tag{28}$$

and the solutions of the above equations are written as

$$\begin{aligned} a_{2rx} &= \frac{\nu}{2}\left(x_1^2 - p_3^2\right) + \nu_1 x_1 + \nu_2 p_3 + \delta_1, \\ a_{2ix} &= \frac{\rho}{2}\left(x_1^2 - p_3^2\right) + \rho_1 x_1 + \rho_2 p_3 + \delta_2, \end{aligned} \tag{29}$$

where ν, ρ, ν_1, ν_2, ρ_1, ρ_2, δ_1 and δ_2 are arbitrary constants of integration.

(iii) Solutions for a_{0rx} and a_{0ix}: In order to solve a_{0rx} and a_{0ix}, differentiate eq. (18) with respect to x_1 and eq. (19) with respect to p_3 and on adding, we get

$$\begin{aligned} \frac{\partial^2 a_{0rx}}{\partial x_1^2} + \frac{\partial^2 a_{0rx}}{\partial p_3^2} &= 4(-2\delta_2 x_1 - 6\delta_3 x_1 p_3)\left(-2\frac{\partial a_{1ix}}{\partial p_3}\right) \\ &\quad - 4\left(2\frac{\partial a_{1ix}}{\partial x_1}\right)(\delta_1 - 2\delta_2 p_3 - 3\delta_3 x_1) \\ &= -8\{(-2\delta_2 x_1 - 6\delta_3 x_1 p_3)\beta_2 \\ &\quad - (\delta_1 - 2\delta_2 p_3 - 3\delta_3 x_1)\beta_1\}, \end{aligned} \tag{30}$$

where we have used eqs (14) and (15) and then expression (29) to simplify the right-hand side. For the solution of eq. (30), we again assume a separable form for $a_{0rx} = X_{0rx}(x_1) + P_{0rx}(p_3)$, and substitution of this leads to a pair of ordinary PDEs whose solution immediately will yield

$$\begin{aligned} a_{0rx} &= \beta_1\delta_3\left(x_1^4 - p_3^4\right) - 2\beta_2\delta_3\left(x_1 p_3^3 + x_1^3 p_3\right) + 2\delta_2\left(\beta_2 x_1 p_3^2 + \beta_1 x_1^2 p_3\right) \\ &\quad - \beta_1\delta_1\left(x_1^2 + p_3^2\right) + \frac{2}{3}\delta_2\left(\beta_1 p_3^3 + \beta_2 x_1^3\right). \end{aligned} \tag{31}$$

Similarly, a_{0ix} can be determined by following the same procedure as for a_{0rx} and the coefficient function a_{0ix} is given in the form

$$\begin{aligned} a_{0ix} &= \beta_2\delta_3\left(x_1^4 - p_3^4\right) + 2\beta_1\delta_3\left(x_1 p_3^3 + x_1^3 p_3\right) + 2\delta_2\left(\beta_2 x_1^2 p_3 - \beta_1 x_1 p_3^2\right) \\ &\quad - \beta_2\delta_1\left(x_1^2 + p_3^2\right) - \frac{2}{3}\delta_2\left(-\beta_2 p_3^3 + \beta_1 x_1^3\right). \end{aligned} \tag{32}$$

(iv) Solutions for a_{1ry} and a_{1iy}: To obtain the solutions for these coefficients, eqs (20) and (21) are reduced to similar second-order forms respectively as

$$\frac{\partial^2 a_{1ry}}{\partial x_2^2} + \frac{\partial^2 a_{1ry}}{\partial p_4^2} = 0, \qquad \frac{\partial^2 a_{2iy}}{\partial x_2^2} + \frac{\partial^2 a_{2iy}}{\partial p_4^2} = 0. \tag{33}$$

The solutions of eq. (33) are obtained as

$$
\begin{aligned}
a_{1ry} &= \frac{\alpha}{2}\left(x_2^2 - p_4^2\right) + \alpha_1 x_2 + \alpha_2 p_4 + \delta_7, \\
a_{1iy} &= \frac{\beta}{2}\left(x_2^2 - p_4^2\right) + \beta_1 x_2 + \beta_2 p_4 + \delta_8,
\end{aligned} \tag{34}
$$

where α, β, α_1, α_2, β_1, β_2, δ_7 and δ_8 are arbitrary constants of integration.

(v) Solution for a_{2ry} and a_{2iy}: For the solutions for a_{2ry} and a_{2iy}, eqs (22) and (23) are reduced to

$$
\frac{\partial^2 a_{2ry}}{\partial x_2^2} + \frac{\partial^2 a_{2ry}}{\partial p_4^2} = 0, \qquad \frac{\partial^2 a_{2iy}}{\partial x_2^2} + \frac{\partial^2 a_{2iy}}{\partial p_4^2} = 0, \tag{35}
$$

and the solutions of eq. (35) become

$$
\begin{aligned}
a_{2ry} &= \frac{\nu}{2}\left(x_2^2 - p_4^2\right) + \nu_1 x_2 + \nu_2 p_4 + \delta_5, \\
a_{2iy} &= \frac{\rho}{2}\left(x_2^2 - p_4^2\right) + \rho_1 x_2 + \rho_2 p_4 + \delta_6,
\end{aligned} \tag{36}
$$

where ν, ρ, ν_1, ν_2, ρ_1, ρ_2, δ_5 and δ_6 are arbitrary constants of integration.

(vi) Solutions for a_{0ry} and a_{0iy}: Similarly, to solve a_{0ry} and a_{0iy}, differentiate eq. (24) with respect to x_2 and eq. (25) with respect to p_4 and on adding, we obtain

$$
\begin{aligned}
\frac{\partial^2 a_{0ry}}{\partial x_2^2} + \frac{\partial^2 a_{0ry}}{\partial p_4^2} &= 4(-2\delta_2 x_1 - 6\delta_3 x_1 p_3)\left(-2\frac{\partial a_{1iy}}{\partial p_4}\right) \\
&\quad - 4(\delta_1 - 2\delta_2 p_4 - 3\delta_3 x_2)\left(2\frac{\partial a_{1iy}}{\partial x_2}\right) \\
&= 8\{(-2\delta_2 x_1 - 6\delta_3 x_1 p_3)\beta_2 - (\delta_1 - 2\delta_2 p_4 - 3\delta_3 x_2)\beta_1\}.
\end{aligned} \tag{37}
$$

The solution of this equation immediately will yield

$$
\begin{aligned}
a_{0ry} &= \beta_1\delta_3\left(x_2^4 - p_4^4\right) - 2\beta_2\delta_3\left(x_2 p_4^3 + x_2^3 p_4\right) + 2\delta_2\left(\beta_2 x_2 p_4^2 + \beta_1 x_2^2 p_4\right) \\
&\quad - \beta_2\delta_1\left(x_2^2 + p_4^2\right) + \frac{2}{3}\delta_2\left(\beta_1 p_4^3 + \beta_2 x_2^3\right).
\end{aligned} \tag{38}
$$

Also following same procedure as above, the coefficient function a_{0iy} is derived as

$$
\begin{aligned}
a_{0iy} &= \beta_2\delta_3\left(x_2^4 - p_4^4\right) + 2\beta_1\delta_3\left(x_2 p_4^3 + x_2^3 p_4\right) + 2\delta_2\left(\beta_2 x_2^2 p_4 - \beta_1 x_2 p_4^2\right) \\
&\quad - \beta_2\delta_1\left(x_2^2 + p_4^2\right) - \frac{2}{3}\delta_2\left(-\beta_2 p_4^3 + \beta_1 x_2^3\right).
\end{aligned} \tag{39}
$$

Note that the solutions ((27), (29), (31), (32), (34), (36), (38) and (39)) of various $a_{kr}(x, y)$ and $a_{ki}(x, y)$ are determined only using eq. (7). With these expressions for the coefficient functions, when eq. (8) is rationalized, we obtained several constraint relations among the arbitrary integration constants, thereby reducing the number of arbitrary integration constants in the final solutions. The constrained relations so obtained are: all δ's are zero, and $\nu = \rho = \alpha = \beta = 0$, $\rho_1 = -\nu_2$, $\rho_2 = \nu_1$, $\beta_2 = -\alpha_1$, $\beta_1 = \alpha_2$. Thus, under the above restrictions, the solutions of various coefficient functions become

$$\begin{aligned}
a_{0rx} &= \beta_1\delta_3\left(x_1^4 - p_3^4\right) - 2\beta_2\delta_3\left(x_1p_3^3 + x_1^3p_3\right) + 2\delta_2\left(\beta_2x_1p_3^2 + \beta_1x_1^2p_3\right)\\
&\quad - \beta_1\delta_1\left(x_1^2 + p_3^2\right) + \frac{2\delta_2}{3}\left(\beta_1p_3^3 + \beta_2x_1^3\right),\\
a_{0ix} &= \beta_2\delta_3\left(x_1^4 - p_3^4\right) + 2\beta_1\delta_3\left(x_1p_3^3 + x_1^3p_3\right) + 2\delta_2\left(\beta_2x_1^2p_3 - \beta_1x_1p_3^2\right)\\
&\quad - \beta_2\delta_1\left(x_1^2 + p_3^2\right) - \frac{2\delta_2}{3}\left(-\beta_2p_3^3 + \beta_1x_1^3\right),\\
a_{1rx} &= -\beta_2x_1 + \beta_1p_3; \qquad a_{1ix} = \beta_1x_1 + \beta_2p_3,\\
a_{2rx} &= -\rho_2x_1 + \rho_1p_3; \qquad a_{2iy} = \rho_1x_1 + \rho_2p_3,\\
a_{0ry} &= \beta_1\delta_3\left(x_2^4 - p_4^4\right) - 2\beta_2\delta_3\left(x_2p_4^3 + x_2^3p_4\right) + 2\delta_2\left(\beta_2x_2p_4^2 + \beta_1x_2^2p_4\right)\\
&\quad - \beta_2\delta_1\left(x_2^2 + p_4^2\right) + \frac{2\delta_2}{3}\left(\beta_1p_4^3 + \beta_2x_2^3\right),\\
a_{0iy} &= \beta_2\delta_3\left(x_2^4 - p_4^4\right) + 2\beta_1\delta_3\left(x_2p_4^3 + x_2^3p_4\right) + 2\delta_2\left(\beta_2x_2^2p_4 - \beta_1x_2p_4^2\right)\\
&\quad - \beta_2\delta_1\left(x_2^2 + p_4^2\right) - \frac{2\delta_2}{3}\left(-\beta_2p_4^3 + \beta_1x_2^3\right),\\
a_{1ry} &= -\beta_2x_2 + \beta_1p_4; \qquad a_{1iy} = +\beta_1x_2 + \beta_2p_4,\\
a_{2ry} &= -\rho_2x_2 + \rho_1p_4; \qquad a_{2iy} = \rho_1x_2 + \rho_2p_4.
\end{aligned} \tag{40}$$

Thus, combining the above set of equations, we have

$$\begin{aligned}
a_{1r} &= -\beta_2(x_1 + x_2) + \beta_1(p_3 + p_4); \quad a_{1i} = \beta_1(x_1 + x_2) + \beta_2(p_3 + p_4),\\
a_{2r} &= -\rho_2(x_1 + x_2) + \rho_1(p_3 + p_4); \quad a_{2i} = \rho_1(x_1 + x_2) + \rho_2(p_3 + p_4),\\
a_{0r} &= \beta_1\delta_3\left(x_1^4 + x_2^4 - p_3^4 - p_3^4\right) - 2\beta_2\delta_3\left(x_1p_3^3 + x_1^3p_3 + x_2p_4^3 + x_2^3p_4\right)\\
&\quad + 2\delta_2\beta_2\left(x_1p_3^2 + x_2p_4^2\right) + 2\delta_2\beta_1\left(x_1^2p_3 + x_2^2p_4\right)\\
&\quad - \beta_1\delta_1\left(x_1^2 + x_2^2 + p_3^2 + p_3^2\right) + \frac{2\delta_2}{3}\{\beta_1\left(p_3^3 + p_4^3\right) + \beta_2\left(x_1^3 + x_2^3\right)\},\\
a_{0i} &= \beta_1\delta_3\left(x_1^4 + x_2^4 - p_3^4 - p_3^4\right) - 2\beta_2\delta_3\left(x_1p_3^3 + x_1^3p_3 + x_2p_4^3 + x_2^3p_4\right)\\
&\quad + 2\delta_2\beta_2\left(x_1p_3^2 + x_2p_4^2\right) + 2\delta_2\beta_1\left(x_1^2p_3 + x_2^2p_4\right)\\
&\quad - \beta_1\delta_1\left(x_1^2 + x_2^2 + p_3^2 + p_3^2\right)\\
&\quad + \frac{2\delta_2}{3}\{\beta_1\left(p_3^3 + p_4^3\right) + \beta_2\left(x_1^3 + x_2^3\right)\}.
\end{aligned} \tag{41}$$

Now using the results of eq. (41) for $a_{0r}, a_{0i}, a_{1r}, a_{1i}, a_{2r}$ and a_{2i} in eqs (10) and (11), one obtains the real and imaginary parts I_1 and I_2 as

$$\begin{aligned}
I_1 &= \beta_1\delta_3\left(x_1^4 + x_2^4 - p_3^4 - p_3^4\right) - 2\beta_2\delta_3\left(x_1p_3^3 + x_1^3p_3 + x_2p_4^3 + x_2^3p_4\right)\\
&\quad + 2\delta_2\beta_2\left(x_1p_3^2 + x_2p_4^2\right) + 2\delta_2\beta_1\left(x_1^2p_3 + x_2^2p_4\right)\\
&\quad - \beta_1\delta_1\left(x_1^2 + x_2^2 + p_3^2 + p_4^2\right) + \frac{2}{3}\delta_2\left\{\beta_1\left(p_3^3 + p_4^3\right) + \beta_2\left(x_1^3 + x_2^3\right)\right\}\\
&\quad - 2\left(p_1x_3 + p_4x_4\right)\{\beta_1\left(x_1 + x_2\right) + \beta_2\left(p_3 + p_4\right)\}\\
&\quad + \left(p_1^2 + p_2^2 - x_3^2 - x_4^2\right)\{-\beta_2\left(x_1 + x_2\right) + \beta_1\left(p_3 + p_4\right)\}\\
&\quad + \left(p_1p_2 - x_3x_4\right)\{-\rho_2\left(x_1 + x_2\right) + \rho_1\left(p_3 + p_4\right)\}\\
&\quad - \left(p_1x_4 + p_2x_3\right)\{\rho_1\left(x_1 + x_2\right) + \rho_2\left(p_3 + p_4\right)\},
\end{aligned} \tag{42}$$

$$
\begin{aligned}
I_2 = {} & \beta_1\delta_3\left(x_1^4 + x_2^4 - p_3^4 - p_3^4\right) - 2\beta_2\delta_3\left(x_1 p_3^3 + x_1^3 p_3 + x_2 p_4^3 + x_2^3 p_4\right) \\
& + 2\delta_2\beta_2\left(x_1 p_3^2 + x_2 p_4^2\right) + 2\delta_2\beta_1\left(x_1^2 p_3 + x_2^2 p_4\right) \\
& - \beta_1\delta_1\left(x_1^2 + x_2^2 + p_3^2 + p_4^2\right) + \frac{2}{3}\delta_2\left\{\beta_1\left(p_3^3 + p_4^3\right) + \beta_2\left(x_1^3 + x_2^3\right)\right\} \\
& + 2\left(p_1 x_3 + p_4 x_4\right)\left\{-\beta_2\left(x_1 + x_2\right) + \beta_1\left(p_3 + p_4\right)\right\} \\
& + \left(p_1^2 + p_2^2 - x_3^2 - x_4^2\right)\left\{\beta_1\left(x_1 + x_2\right) + \beta_2\left(p_3 + p_4\right)\right\} \\
& + \left(p_1 p_2 - x_3 x_4\right)\left\{\rho_1\left(x_1 + x_2\right) + \rho_2\left(p_3 + p_4\right)\right\} \\
& + \left(p_1 x_4 + p_2 x_3\right)\left\{-\rho_2\left(x_1 + x_2\right) + \rho_1\left(p_3 + p_4\right)\right\}. \qquad (43)
\end{aligned}
$$

And finally the complex invariant I is obtained by combining I_1 and iI_2 as

$$
\begin{aligned}
I = {} & \frac{ib\delta_2}{3}\left\{x^*\left(x^{*2} - 3x^2\right) + y^*\left(y^{*2} - 3y^2\right)\right\} + \frac{b\delta_1}{2}\left(xx^* + yy^*\right) \\
& - i\delta_3\left(x^* x^3 + y^* y^3\right) + b\left(x^* + y^*\right)\left(p_x^2 + p_y^2\right) + e\left(x^* + y^*\right)p_x p_y, \qquad (44)
\end{aligned}
$$

where $x^* = x_1 - ip_3,\ y^* = x_2 - ip_4,\ p_x^* = p_1 - ix_3,\ p_y^* = p_2 - ix_4, e = -\rho_2 + i\rho_1$ and $b = -\beta_2 + i\beta_1$. It is to be noted that from the general expression of the invariant for complex cubic potential, one can find invariants of simple harmonic oscillator and shifted harmonic oscillator in two-dimensional complex space by imposing some restrictions on potential coupling parameters. In what follows, we consider one such special case.

Special case

An invariant for a shifted harmonic oscillator in complex plane can be derived by substituting $\delta_3 = 0$, $\delta_2 = -\frac{1}{2}$ and $\delta_1 = \gamma$ in eq. (12). Thus after appropriate scaling of x and p (with $\omega = 1$), the Hamiltonian for a shifted harmonic oscillator is expressed as

$$
H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + i\gamma(x + y). \qquad (45)
$$

The system is transformed into ECPS by using eq. (1) and hence the real and imaginary components of the Hamiltonian become

$$
\begin{aligned}
H_1 &= \frac{1}{2}(p_1^2 + p_2^2 + x_1^2 + x_2^2 - p_3^2 - p_4^2 - x_3^2 - x_4^2) - \gamma(x_1 p_4 + x_2 p_3), \\
H_2 &= p_1 x_3 + p_2 x_4 + x_1 p_3 + x_2 p_4 + \gamma(x_1 x_2 - p_3 p_4). \qquad (46)
\end{aligned}
$$

Similar to the previous case, suppose that the above system possesses a quadratic invariant of the form given by eq. (9). So, to determine the unknown coefficients $a_{0r}, a_{0i}, a_{1r}, a_{1i}, a_{2r}$ and a_{2i}, substitute eqs (7), (8) and (46) in eq. (7) and rationalization of the resultant expressions with respect to the powers of p_1, x_3, p_2, x_4 and their combinations give a set of 12 coupled PDEs. These PDEs can again be solved by following the procedure adopted in the previous case. Finally, using solutions obtained from PDEs, the complex invariant I for the shifted harmonic oscillator is written as

$$I = \frac{b}{3}\left\{x^*\left(3x^2 + x^*\right) + y^*\left(3y^2 + y^*\right)\right\} + ib\gamma\left(xx^* + yy^*\right) + \sigma_1\left(x^* + y^*\right) + b\left(x^* + y^*\right)\left(p_x^2 + p_y^2\right) + e\left(x^* + y^*\right)p_x p_y, \quad (47)$$

where $\sigma_1 = c_1 + id_1$ and the definitions of b and e are the same as given in eq. (44).

4. Conclusion

Keeping in view the growing demand of complex Hamiltonians in different branches of science, in the present work, we have searched for quadratic invariants for two general non-Hermitian Hamiltonian systems in a two-dimensional ECPS. To this effect, the rationalization method has been employed.

In the past, complex invariants have been discussed in the context of understanding fermion masses and quark mixing, and CP-conserving two-Higgs-doublet model scalar potentials in Particle Physics [16,17]. Since invariants of real Hamiltonian systems have played a vital role in understanding the underlying dynamics of the systems, we expect that the complex invariants can also be helpful in exploring some deep insights into features of complex dynamical systems.

Acknowledgement

The authors are grateful to the referee for useful suggestions which improved the presentation of this paper.

References

[1] K J Whiteman, *Rep. Prog. Phys.* **40**, 1033 (1977)
[2] M Prelle and M Singer, *Trans. Am. Math. Soc.* **279**, 215 (1983)
V K Chandersekar, M Senthilvelan and M Lakshmanan, *Proc. R. Soc. London* **A461**, 2451 (2005)
[3] H R Lewis and W B Riesenfield, *J. Math. Phys.* **10**, 1498 (1969)
[4] R S Kaushal, *Phys. Rev.* **A46**, 2941 (1992)
F Chand and S C Mishra, *Pramana – J. Phys.* **68**, 891 (2007)
[5] M S Abdalla and P G L Leach, *J. Math. Phys.* **52**, 083504 (2011)
[6] J Strukmeier and C Riedel, *Phys. Rev. Lett.* **85**, 3830 (2000); *Phys. Rev.* **E64**, 26503 (2001)
[7] J Hietarinta, *Phys. Rep.* **147**, 87 (1987)
[8] X C Gao, J B Xu and T Z Qian, *Europhys. Lett.* **17**, 485 (1992)
[9] R S Kaushal, *Classical and quantum mechanics of noncentral potentials* (Narosa Publishing House, New Delhi, 1998)
[10] C M Bender, *Rep. Prog. Phys.* **70**, 947 (2007)
[11] R K Colegrave, P Croxson and M A Mannan, *Phys. Lett.* **A131**, 407 (1988)
R K Colegrave and P Croxson, *J. Math. Phys.* **32**, 3361 (1991)
[12] Roshan Lal and S C Mishra, *Ind. J. Phys.* **B77**, 567 (2003)
[13] S C Mishra and Fakir Chand, *Pramana – J. Phys.* **66**, 601 (2006)

[14] S P Kim and D N Page, *Phys. Rev.* **A64**, 012104 (2001)
[15] I Kovacic, *Appl. Math. Comp.* **215**, 3482 (2010)
[16] A Kusenko and R Shrock, *Phys. Lett.* **B323**, 18 (1994); *Phys. Rev.* **D50**, R30 (1994)
[17] J F Gunion and H E Haber, *Phys. Rev.* **D72**, 095002 (2005)
[18] R S Kaushal and H J Korsch, *Phys. Lett.* **A276**, 47 (2000)
R S Kaushal and S Singh, *Ann. Phys.* **288**, 253 (2001)
S Singh and R S Kaushal, *Phys. Scr.* **67**, 181 (2003)
R S Kaushal, *Pramana – J. Phys.* **73**, 287 (2009)
[19] R S Kaushal, *J. Phys. A: Math. Gen.* **34**, L709 (2001)
R S Kaushal and Parthasarathi, *J. Phys.* **A35**, 8743 (2002)
Parthasarathi and R S Kaushal, *Phys. Scr.* **68**, 115 (2003)
[20] F Chand and S C Mishra, *Pramana – J. Phys.* **67**, 999 (2006)
F Chand, R M Singh, N Kumar and S C Mishra, *J. Phys.* **A40**, 10171 (2007)
R M Singh, F Chand and S C Mishra, *Comm. Theor. Phys.* **51**, 397 (2009)
F Chand, S C Mishra and R M Singh, *Pramana – J. Phys.* **73**, 349 (2009)
[21] J S Virdi, F Chand, C N Kumar and S C Mishra, *Can. J. Phys.* **90**, 2 (2012)
[22] A L Xavier Jr and M A M de Aguiar, *Phys. Rev. Lett.* **79**, 3323 (1996); *Ann. Phys. (N.Y.)* **252**, 458 (1997)