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Dynamics of fractional-ordered Chen system with delay

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Abstract. In the present paper the effect of delay on chaos in fractional-order Chen system is investigated. It is observed that inclusion of delay changes chaotic behaviour to limit cycles or stable systems.

Keywords. Caputo derivative; fractional-order dynamical systems; attractor; delay differential equations; predictor–corrector method.

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1. Introduction

Fractional calculus is a branch of Mathematics which deals with differentiation and integration of arbitrary orders and is as old as Calculus. Its origins can be traced back to the end of the seventeenth century, the time when Newton and Leibniz developed the foundations of differential and integral calculus. In particular, Leibniz introduced the symbol $\mathrm{d}^n f(x)/\mathrm{d}x^n$ to denote the nth derivative of a function f. When he reported this in a letter to L'Hospital (apparently with the implicit assumption that $n \in N$), L'Hospital asked: "What does it mean when $n = 1/2$?" This correspondence is now considered as the first occurrence of the concept of fractional-ordered derivative. The subject has developed since then through the pioneering works of Leibniz, Bernoulli, Euler, Lagrange, Abel, Riemann, Liouville and many others.

Though this subject has a history of more than 300 years, utility and applicability of fractional calculus to various branches of science and engineering have been realized only recently. It is becoming more and more clear that derivatives and integrals of non-integer orders are not mere mathematical curiosities but many processes, such as dielectric polarization [1], diffusion [2–5], viscoelastic systems [6], signal processing [7]

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etc. can be modelled effectively by incorporating fractional-ordered derivatives and integrals. Fractional calculus is also proven useful in modelling biological, biophysical and bio-engineering phenomena [8].

There has been an explosion of activity in dynamical systems theory in the past two decades. Most activity is centred on ordinary differential equations (ODE) and difference equations. The qualitative behaviour of ordinary differential equations is now well understood. Different qualitative behaviours and transitions between those are studied using bifurcation theory. In one-dimensional dynamical systems, the only possible attractor is a fixed point while in two dimensions two attractors are possible; a fixed point and a limit cycle. In three dimensions or above, we can see a chaotic attractor as well. What happens to this classification for fractional differential equations? Fractional differential equations open new possibilities since the evolution depends on entire history.

Fractional differential equations are more stable than their ODE counterpart. For example, fractional version of the harmonic oscillator equations behaves like a damped harmonic oscillator [9]. A marginally stable case with a limit cycle can appear for fractional differential equations of order less than two [10]. Several interesting differences are spotted in ref. [10]. The two-norm is no longer conserved. In fact, the radius of the final contour is larger than the location of the initial condition and as the dimensionality tends to zero, radius diverges to infinity. Nonetheless, marginally stable oscillator shows a periodic behaviour even in dimension close to zero in commensurate order system. Thus, it is possible to have periodic behaviour in systems with dimensionality close to zero.

How about chaotic systems? One of the oldest systems where chaos has been discovered is the Lorenz system. In 1963, Lorenz [11] discovered chaotic solutions to a system of three autonomous ordinary differential equations. Later, several dynamical systems exhibiting chaos have been presented in various branches of science [12]. Grigorenko and Grigorenko [13] extended the study of this prototypical system to equations of fractional-order and observed that it is possible to observe chaos in such systems. Fractional systems with dimensionality (sum of orders of differential equations) less than three have been reported to exhibit chaos; though this is not very surprising since fractional-order systems are systems with long memory. The chaos does not disappear abruptly as the dimension is reduced from three. It smoothly changes its nature and disappears at some lower order. This work has been extended to several other systems (see [14], and references therein) in later works from the viewpoints of theoretical understanding as well as applications such as secure communication. One problem encountered in this context has been to find ‘minimum dimension for which the system displays chaos’.

However, there are other questions of interest. Though Lorenz system is one of the oldest, most studied and in some sense prototypical example of chaos, questions have been posed on all possible types of nonlinearities which could yield chaos in three dimensions. Lorenz system has two quadratic nonlinear terms, and there have been attempts to find different classes of such systems. Rössler [15] gave another model of differential equations which will yield chaos. Sprott worked extensively on the problem of finding various possible examples of three differential equations which can yield chaos and even proposed a ‘simplest’ such flow [16]. One system which is qualitatively different from Lorenz system and displays chaos, is the Chen system [17]. It is a system which shows a chaotic attractor which is not equivalent to Lorenz attractor in the sense that there is no diffeomorphism connecting the two [18]. However, Lorenz and Chen’s equations are very similar except the evolution of one of the variables. Thus, it is still considered a dual of Lorenz system [19]. Essentially they look at the linear part of evolution equations and classify the systems according to the sign of the product of off-diagonal terms. Thus there is yet another system called Lu system which describes the case when this product is zero [20]. The Chen system is a relatively new system and we have much less understanding of Chen system compared to Lorenz system. Thus it is worthwhile to consider different variants of Chen system and see the qualitative changes in attractor and routes to chaos, if any. In this connection, there have been attempts to introduce delay in Chen system and to introduce fractional Chen system. In this work, we have attempted to simulate a fractional Chen system with delay and study if it is still able to display chaos and how the chaos is achieved.

It should be remarked here that performing simulations of equation with delay or fractional differential equations is a rather cumbersome task and combining these two makes the simulations rather formidable. The fractional differential equations tend to lower the dimensionality of the differential equations in question. However, introducing delay in differential equation makes it infinite-dimensional. Thus, even a single ordinary differential equation with delay can display chaos [21]. We note that delay differential equations (DDE) have been useful in several contexts [22–26]. Because of their wide applicability, study of fractional DDEs is of theoretical and practical interest.

In the next sections, we present the time-delayed version of fractional-order Chen system [27] and investigate the existence of chaos. It is observed that chaotic systems get stabilized for suitable values of delay. Periodic motion and limit cycles are also observed in some cases.

2. Preliminaries

Basic definitions and properties of fractional derivative/integrals are given below [28,29].

2.1 *Fractional calculus*

A real function $f(t), t > 0$ is said to be in space $C_\alpha, \alpha \in \Re$ if there exists a real number $p(> \alpha)$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C[0, \infty)$.

A real function $f(t), t > 0$ is said to be in space $C_\alpha^m, m \in \mathbb{N} \bigcup \{0\}$ if $f^{(m)} \in C_\alpha$.

Let $f \in C_\alpha$ and $\alpha \geq -1$, then the (left-sided) Riemann–Liouville integral of order μ, $\mu > 0$ is given by

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau)\, \mathrm{d}\tau, \quad t > 0. \tag{1}$$

The (left-sided) Caputo fractional derivative of f, $f \in C_{-1}^m, m \in \mathbb{N} \bigcup \{0\}$, is defined as

$$\begin{aligned} D^\mu f(t) &= \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), \quad \mu = m \\ &= I^{m-\mu} \frac{\mathrm{d}^m f(t)}{\mathrm{d}t^m}, \quad m-1 < \mu < m, \quad m \in \mathbb{N}. \end{aligned} \tag{2}$$

Note that for $m-1<\mu \leq m, m \in \mathbb{N}$,

$$I^{\mu} D^{\mu} f(t)=f(t)-\sum_{k=0}^{m-1} \frac{\mathrm{d}^{k} f}{\mathrm{d} t^{k}}(0) \frac{t^{k}}{k!}, \tag{3}$$

$$I^{\mu} t^{\nu}=\frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}. \tag{4}$$

2.2 *Numerical method for solving fractional differential equations*

Standard numerical methods used for solving ordinary differential equations have to be modified for solving fractional differential equations (FDE). A modification of Adams–Bashforth–Moulton algorithm is proposed by Diethelm *et al* [30,31] to solve FDEs. Bhalekar and Daftardar-Gejji [32] have extended this algorithm for fractional differential equations involving delay, as follows: Consider the following FDDE:

$$D_{t}^{\alpha} y(t)=f(t, y(t), y(t-\tau)), \quad t \in[0, T], \quad 0<\alpha \leq 1 \tag{5}$$

$$y(t)=g(t), \quad t \in[-\tau, 0]. \tag{6}$$

Consider a uniform grid

$$\{t_{n}=n h: n=-k,-k+1, \ldots,-1,0,1, \ldots, N\}$$

where k and N are integers such that $h=T/N$ and $h=\tau/k$. Let

$$y_{h}(t_{j})=g(t_{j}), \quad j=-k,-k+1, \ldots,-1,0 \tag{7}$$

and note that

$$y_{h}(t_{j}-\tau)=y_{h}(j h-k h)=y_{h}(t_{j-k}), \quad j=0,1, \ldots, N. \tag{8}$$

Suppose we have already calculated approximations $y_{h}(t_{j}) \approx y(t_{j})$, $(j=-k,-k+1, \ldots,-1,0,1, \ldots, n)$ and we want to calculate $y_{h}(t_{n+1})$ using

$$y(t_{n+1})=g(0)+\frac{1}{\Gamma(\alpha)} \int_{0}^{t_{n+1}}(t_{n+1}-\xi)^{\alpha-1} f(\xi, y(\xi), y(\xi-\tau)) \mathrm{d} \xi. \tag{9}$$

Note that eq. (9) is obtained by applying $I_{t_{n+1}}^{\alpha}$ on both sides of (5) and using (6). We use approximations $y_{h}(t_{n})$ for $y(t_{n})$ in (9). Further, the integral in eq. (9) is evaluated using product trapezoidal quadrature formula. The corrector formula is thus

$$\begin{aligned} y_{h}(t_{n+1}) = & \ g(0)+\frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_{n+1}, y_{h}(t_{n+1}), y_{h}(t_{n+1}-\tau)) \\ & +\frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j, n+1} f(t_{j}, y_{h}(t_{j}), y_{h}(t_{j}-\tau)) \\ = & \ g(0)+\frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_{n+1}, y_{h}(t_{n+1}), y_{h}(t_{n+1-k})) \\ & +\frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j, n+1} f(t_{j}, y_{h}(t_{j}), y_{h}(t_{j-k})), \end{aligned} \tag{10}$$

where $a_{j,n+1}$ are given by

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} & \\ -2(n-j+1)^{\alpha+1}, & \text{if } 1 \le j \le n, \\ 1, & \text{if } j = n+1. \end{cases} \tag{11}$$

The unknown term $y_h(t_{n+1})$ appears on both sides of (10) and due to nonlinearity of f, eq. (10) cannot be solved explicitly for $y_h(t_{n+1})$. So we replace the term $y_h(t_{n+1})$ on the right-hand side by an approximation $y_h^{\mathrm{P}}(t_{n+1})$, called predictor. Product rectangle rule is used in (9) to evaluate the predictor term

$$\begin{aligned} y_h^{\mathrm{P}}(t_{n+1}) &= g(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j), y_h(t_j - \tau)) \\ &= g(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j), y_h(t_{j-k})), \end{aligned} \tag{12}$$

where $b_{j,n+1}$ is given by

$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha} \left((n+1-j)^{\alpha} - (n-j)^{\alpha}\right). \tag{13}$$

3. Fractional-order Chen system with time delay

Li and Peng [27] proposed the follwing fractional version of the Chen system:

$$\begin{aligned} D^{\alpha} x(t) &= a(y(t) - x(t)), \\ D^{\alpha} y(t) &= (c-a)x(t) - x(t)z(t) + cy(t), \\ D^{\alpha} z(t) &= x(t)y(t) - bz(t), \end{aligned} \tag{14}$$

where $a = 35$, $b = 3$, $c = 27$ and $\alpha \in (0, 1)$.

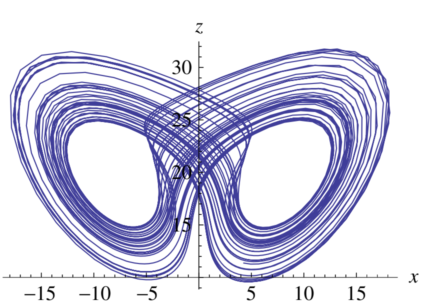

Figure 1. Phase portraits for $\alpha = 1$, $\tau = 0.009$.

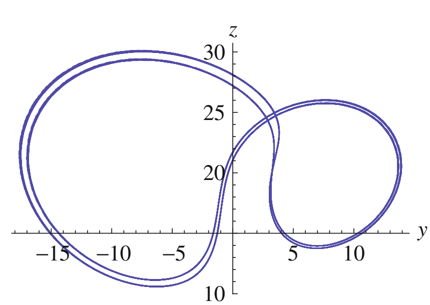

Figure 2. Phase portraits for $\alpha = 1$, $\tau = 0.010$.

In this work we consider the generalization of the system (14) which involves time-delay:

$$\begin{aligned}
D^\alpha x(t) &= a(y(t) - x(t-\tau)),\\
D^\alpha y(t) &= (c-a)x(t-\tau) - x(t)z(t) + cy(t),\\
D^\alpha z(t) &= x(t)y(t) - bz(t-\tau),\\
x(t) &= 0.2, \quad y(t) = 0, \quad z(t) = 0.5 \quad \text{for } t \in [-\tau, 0]. \qquad (15)
\end{aligned}$$

4. Numerical simulations

- For $\alpha = 1$, the system (15) shows chaotic behaviour for $0 \le \tau < 0.010$. Figure 1 shows xz-phase portrait for the system with $\tau = 0.009$. Two-cycle is observed for

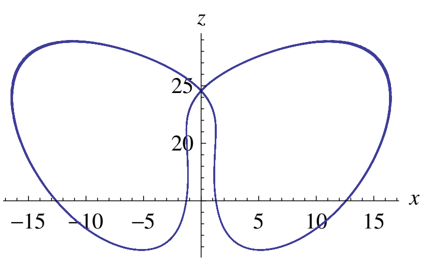

Figure 3. Phase portraits for $\alpha = 1$, $\tau = 0.013$

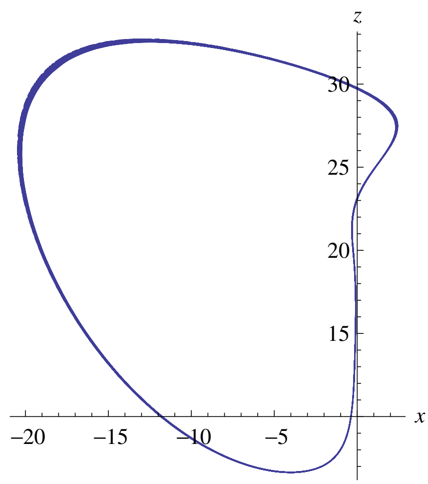

Figure 4. Phase portraits for $\alpha = 1$, $\tau = 0.016$.

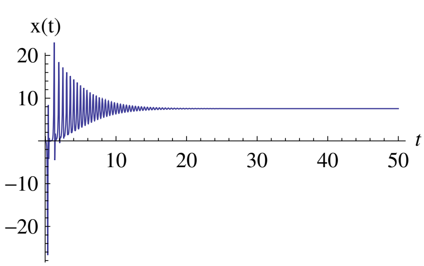

Figure 5. Time series for $\alpha = 1$, $\tau = 0.017$.

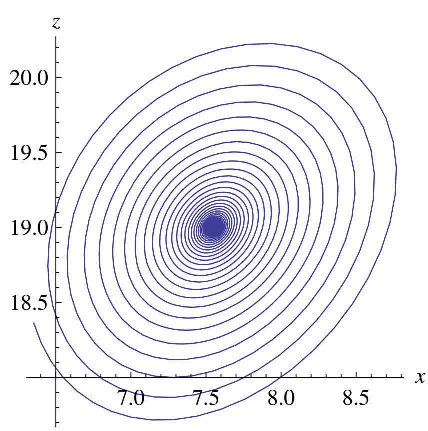

Figure 6. Phase portraits for $\alpha = 1$, $\tau = 0.017$.

$\tau = 0.010$ (figure 2). The system shows periodic behaviour for $0.010 \leq \tau < 0.017$. Cycles are shown in figures 3 and 4 for $\tau = 0.013$ and $\tau = 0.016$ respectively. Stable orbits are observed at $\tau = 0.017$ (figures 5 and 6).

- For $\alpha = 0.97$, chaos is observed for $0 \leq \tau \leq 0.005$. Figure 7 shows chaotic yz-phase portrait for $\tau = 0.005$. A limit cycle is observed for $\tau = 0.007$ (figure 8). The system then becomes chaotic once again at $\tau = 0.009$ (figure 9). If τ is increased further, system tends to periodicity (figure 10, $\tau = 0.013$) and becomes stable finally (figures 11 and 12, $\tau = 0.015$).

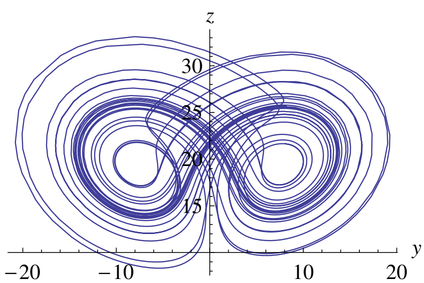

Figure 7. Phase portraits for $\alpha = 0.97, \tau = 0.005$.

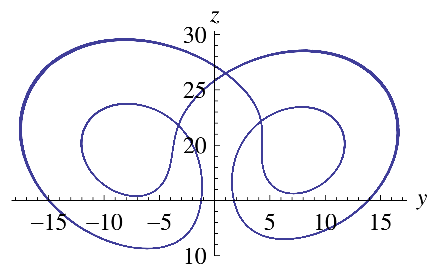

Figure 8. Phase portraits for $\alpha = 0.97, \tau = 0.007$.

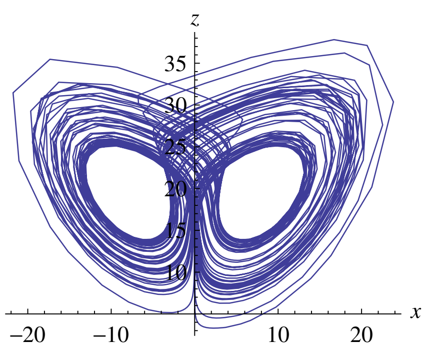

Figure 9. Phase portraits for $\alpha = 0.97, \tau = 0.009$.

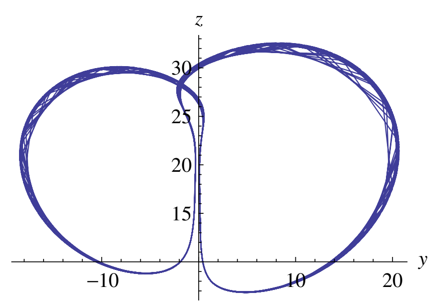

Figure 10. Phase portraits for $\alpha = 0.97, \tau = 0.013$.

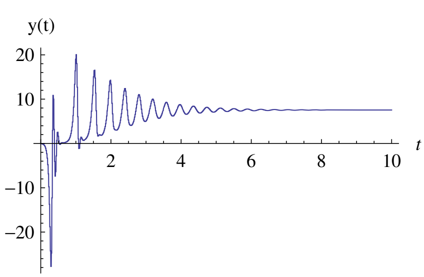

Figure 11. Time series for $\alpha = 0.97$, $\tau = 0.015$.

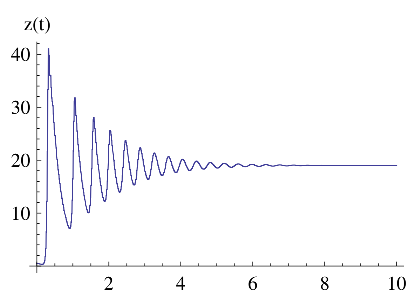

Figure 12. Time series for $\alpha = 0.97$, $\tau = 0.015$.

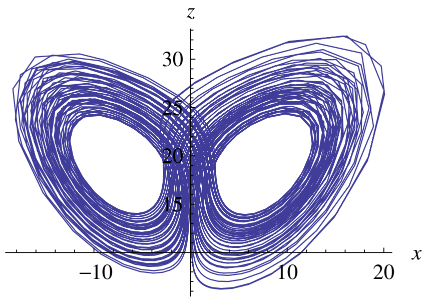

Figure 13. Phase portraits for $\alpha = 0.94$, $\tau = 0.009$.

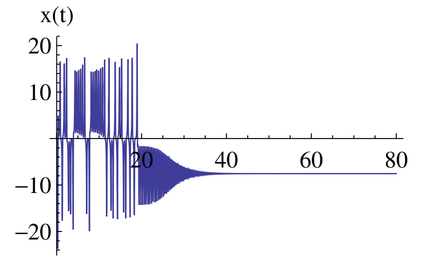

Figure 14. Time series for $\alpha = 0.94$, $\tau = 0.010$.

- For $\alpha = 0.94$, aperiodic behaviour is observed for $0 \leq \tau \leq 0.009$. Chaotic phase portrait for $\tau = 0.009$ is shown in figure 13. The system gets stabilized for larger values of τ. Figure 14 shows waveform $x(t)$ for $\tau = 0.01$.

Mathematica 7 has been used for computations in the present paper.

5. Conclusions

Fractional-order Chen system with time-delay is investigated. Numerical experiments are performed for different values of the derivative and the time-delay term. It is observed that chaotic system gets stabilized for some values of delay. Limit cycles are also observed in some cases.

We can observe striking resemblence between these phase portraits and those obtained for the original system by Chen [33]. We see a period-doubling type route to chaos. Thus, structural properties of fractional and delayed system vary smoothly from the system of ordinary differential equations without delay. However, for smaller values of the derivative, system loses its chaotic character.

As is evident from phase portraits, introducing these variations seem to change the dynamics in a rather smooth manner.

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