PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 79, No. 1
July 2012
pp. 41–60

# Coupled Higgs field equation and Hamiltonian amplitude equation: Lie classical approach and $(G'/G)$-expansion method

SACHIN KUMAR[1], K SINGH[2] and R K GUPTA[1,*]

[1]School of Mathematics and Computer Applications, Thapar University, Patiala 147 004, India
[2]Department of Mathematics, Jaypee University of Information Technology, Waknaghat, Distt. Solan 173 234, India
*Corresponding author. E-mail: rajeshgupta@thapar.edu

MS received 6 September 2011; revised 25 December 2011; accepted 25 January 2012

**Abstract.** In this paper, coupled Higgs field equation and Hamiltonian amplitude equation are studied using the Lie classical method. Symmetry reductions and exact solutions are reported for Higgs equation and Hamiltonian amplitude equation. We also establish the travelling wave solutions involving parameters of the coupled Higgs equation and Hamiltonian amplitude equation using $(G'/G)$-expansion method, where $G = G(\xi)$ satisfies a second-order linear ordinary differential equation (ODE). The travelling wave solutions expressed by hyperbolic, trigonometric and the rational functions are obtained.

**Keywords.** Lie classical method; the $(G'/G)$-expansion method; travelling wave solutions; coupled Higgs equation; Hamiltonian amplitude equation.

**PACS Nos 03.65.−w; 11.30.−j; 02.70.Wz; 94.05.Fg**

## 1. Introduction

Nonlinear evolution equations (NEEs) are widely used as models to describe complex physical phenomena in various fields of science, especially in fluid mechanics, solid-state physics, plasma physics, plasma wave and chemical physics. Various methods have been utilized to explore different kinds of solutions of physical models described by nonlinear partial differential equations (PDEs). One of the basic physical problems for those models is to obtain their exact solutions. In recent years, exact solutions of nonlinear PDEs have been investigated by many authors who are interested in nonlinear physical phenomena. Various methods for obtaining exact travelling wave solutions to nonlinear equations, such as the homogeneous balance method [1], the tanh function method [2,3], the Jacobi elliptic function method [4,5] and the F-expansion method [6,7] have been presented. Among various methods, the Lie symmetry method, also called as Lie group method, is one of the

**DOI: 10.1007/s12043-012-0284-7**; ***e*Publication: 25 May 2012** 

most powerful methods to determine exact solutions of nonlinear PDEs. It is based on the study of the invariance under one-parameter Lie group of point transformations [8–10]. In the recent past, there have been considerable developments in symmetry methods for differential equations as is evident by the number of research papers, books and a new symbolic software devoted to the subject. Some recent and important contributions are in [11–13].

The Higgs equation [14]

$$u_{tt} - u_{xx} - \alpha u + \beta |u|^2 u - 2uv = 0,$$

$$v_{tt} + v_{xx} - \beta(|u|^2)_{xx} = 0, \tag{1.1}$$

describes a system of conserved scalar nucleons interacting with neutral scalar mesons. Equation (1.1) is the coupled nonlinear Klein–Gorden equation for $\alpha < 0,\ \beta < 0$ and the Coupled Higgs field equation for $\alpha > 0, \beta > 0$. Tajiri [14] obtained N-soliton solutions to system (1.1). Zhao [15] constructed more general travelling wave solutions of system (1.1).

A new Hamiltonian amplitude equation

$$\iota u_x + u_{tt} - 2\eta |u|^2 u - \beta u_{xt} = 0, \tag{1.2}$$

where $\eta = \pm 1, \beta \ll 1$, was introduced by Wadati *et al* [16]. This equation governs certain instabilities of modulated wave trains; the addition of the term $-\beta u_{xt}$ overcomes the ill-posedness of the unstable nonlinear Schrödinger equation. The equation is apparently not integrable, but a Hamiltonian analogue of the Kuramoto–Sivashinsky equation, which arises in dissipative system. Yan [17] found solitary wave solutions for a Hamiltonian amplitude equation using a simple transformation.

The paper has been structured as follows. In §§2.1 and 2.2, we applied Lie classical method to the system (1.1) and eq. (1.2) respectively, to reduce them to ordinary differential equations (ODEs) and some exact solutions are derived. In §3.1, we applied $(G'/G)$-expansion method for finding exact travelling wave solutions of Higgs field equation. Section 3.2 is devoted to find travelling wave solutions of Hamiltonian amplitude equation. In §4, some conclusions are given.

## 2. Lie symmetry analysis

Lie's method [8–10] is an effective method and is the simplest among group theoretic techniques and many equations are solved using this method. The Lie group method is also called the symmetry analysis. Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. Once one has determined the symmetry group of a system of differential equations, a number of applications become available. To start with, one can directly use the defining property of such a group and construct new solutions to the system from known ones.

### 2.1 *Higgs field equation*

Here, we shall perform Lie symmetry analysis for the Higgs field equation. As $u$ is a complex variable, to separate the real and imaginary parts of $u$, we shall consider two cases.

*Case* i. Here, on setting

$$u(x,t) = \psi_1(x,t) + \iota\psi_2(x,t), \tag{2.1}$$

system (1.1) decomposes into the following system of equations:

$$\begin{aligned} &\psi_{1tt} - \psi_{1xx} + \alpha\psi_1 + \beta(\psi_1^2 + \psi_2^2)\psi_1 - 2\psi_1 v = 0, \\ &\psi_{2tt} - \psi_{2xx} + \alpha\psi_2 + \beta(\psi_1^2 + \psi_2^2)\psi_2 - 2\psi_2 v = 0, \\ &v_{tt} + v_{xx} - \beta(\psi_1^2 + \psi_2^2)_{xx} = 0. \end{aligned} \tag{2.2}$$

Let us consider the Lie group of point transformations

$$\begin{aligned} &t^* = t + \epsilon\tau(x,t,\psi_1,\psi_2,v) + O(\epsilon^2), \\ &x^* = x + \epsilon\xi(x,t,\psi_1,\psi_2,v) + O(\epsilon^2), \\ &\psi_1^* = \psi_1 + \epsilon\eta(x,t,\psi_1,\psi_2,v) + O(\epsilon^2), \\ &\psi_2^* = \psi_2 + \epsilon\phi(x,t,\psi_1,\psi_2,v) + O(\epsilon^2), \\ &v^* = v + \epsilon\zeta(x,t,\psi_1,\psi_2,v) + O(\epsilon^2), \end{aligned} \tag{2.3}$$

with $\epsilon \ll 1$. For determining the symmetry group of (2.2), one has to find the infinitesimals $\xi, \tau, \eta, \phi$ and $\zeta$, which are functions of $x, t, \psi_1, \psi_2$ and $v$.

Assuming that eqs (2.2) are invariant under the transformations (2.3), the infinitesimals $\xi, \tau, \eta, \phi$ and $\zeta$ must satisfy the symmetry conditions

$$\begin{aligned} &\eta^{tt} - \eta^{xx} + \alpha\eta + 3\beta\psi_1^2\eta + 2\beta\phi\psi_1\psi_2 - 2\psi_1\zeta + \beta\psi_2^2\eta - 2\eta v = 0, \\ &\phi^{tt} - \phi^{xx} + \alpha\phi + 3\beta\psi_2^2\phi + 2\beta\eta\psi_1\psi_2 - 2\psi_2\zeta + \beta\psi_1^2\phi - 2\phi v = 0, \\ &\zeta^{tt} + \zeta^{xx} - 4\beta\psi_{1x}\eta^x - 4\beta\psi_{2x}\phi^x - 2\beta\psi_1\eta^{xx} - 2\beta\eta\psi_{1xx} - 2\beta\psi_2\phi^{xx} \\ &\quad - 2\beta\phi\psi_{2xx} = 0, \end{aligned} \tag{2.4}$$

where $\eta^x, \eta^{xx}, \eta^{tt}, \phi^x, \phi^{xx}, \phi^{tt}, \zeta^{xx}$ and $\zeta^{tt}$ are extended (prolonged) infinitesimals acting on an enlarged space (jet space) that includes all derivatives of the dependent variables (for more details, the readers can refer to [9]). Substituting the values of $\eta^x, \eta^{xx}, \eta^{tt}, \phi^x, \phi^{xx}, \phi^{tt}, \zeta^{xx}$ and $\zeta^{tt}$ into (2.4), then equating the coefficients of the various monomials in the first, second and the other order partial derivatives of $\psi_1, \psi_2, v$ and their powers, we can find the determining equations for the symmetry group of the Higgs field equation. Solving these equations, we get the following forms of the coefficient functions:

$$\xi = a_1, \quad \tau = a_2, \quad \eta = -\psi_2 a_3, \quad \phi = \psi_1 a_3, \quad \zeta = 0, \tag{2.5}$$

where $a_1, a_2$ and $a_3$ are arbitrary constants.

Now we can reduce system (1.1) to a system of ODEs using the characteristic equation

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}\psi_1}{\eta} = \frac{\mathrm{d}\psi_2}{\phi} = \frac{\mathrm{d}v}{\zeta}. \tag{2.6}$$

Solving characteristic equation and using (2.1) we have the following similarity variables for system (1.1):

$$\rho = x - b_0 t, \quad u = P(\rho)\mathrm{e}^{\iota a_0 t}\mathrm{e}^{\iota Q(\rho)}, \quad v = R(\rho), \tag{2.7}$$

where $a_0 = a_3/a_2$ and $b_0 = a_1/a_2$. Here $\rho(x,t)$ is the new independent variable and $P(\rho)$, $Q(\rho)$ and $R(\rho)$ are the new dependent variables.

Using (2.7) in (1.1) and separating real and imaginary parts, we have

$$(b_0^2-1)P''+\beta P^3+(1-b_0^2)PQ'^2+2a_0b_0PQ'-2PR-(a_0^2+\alpha)P=0, \tag{2.8}$$

$$2(b_0^2-1)P'Q'-2a_0b_0P'+(b_0^2-1)PQ''=0, \tag{2.9}$$

$$(b_0^2+1)R''-2\beta PP''-2\beta P'^2=0, \tag{2.10}$$

where ($'$) denotes derivative with respect to $\rho$.

Integrating (2.10) twice we get

$$R = \frac{\beta}{b_0^2+1}P^2 + \frac{C_1}{b_0^2+1}\rho + \frac{C_2}{b_0^2+1}, \tag{2.11}$$

where $C_1$ and $C_2$ are arbitrary constants.

Let

$$Q' = Z(P) \tag{2.12}$$

so that

$$Q'' = P'\frac{\mathrm{d}Z}{\mathrm{d}P}. \tag{2.13}$$

Using (2.12) and (2.13) in (2.9) and integrating we have

$$Q' = Z = \frac{a_0b_0}{b_0^2-1} + \frac{C_0}{P^2}, \tag{2.14}$$

where $C_0$ is an arbitrary constant.

Now using (2.11), (2.14) in (2.8) with $C_1 = 0$ and integrating once we get

$$P'^2 = -\frac{\beta}{2(b_0^2+1)}P^4 + \frac{1}{b_0^2-1}\left(\alpha + a_0^2 + \frac{2C_2}{b_0^2+1} - \frac{a_0^2b_0^2}{b_0^2-1}\right)P^2 - \frac{C_0}{P^2} + C_3, \tag{2.15}$$

where $C_3$ is an arbitrary constant.

*Case* i(a). When $C_0 = 0,\ C_3 \neq 0$

Integrating (2.15) we get

$$P(\rho) = \frac{\mathrm{sn}\left(\frac{1}{2}\sqrt{-2b_2+2\sqrt{b_2{}^2+4b_1C_3}}\rho + C_4, m\right)C_3\sqrt{2}}{\sqrt{C_3\left(-b_2+\sqrt{b_2{}^2+4b_1C_3}\right)}}, \tag{2.16}$$

where

$$m = \frac{\sqrt{\left(-2b_1C_3 - b_2{}^2 + b_2\sqrt{b_2{}^2 + 4b_1b_3}\right)2b_1C_3}}{-2b_1C_3 - b_2{}^2 + b_2\sqrt{b_2{}^2 + 4b_1C_3}},$$

$$b_1 = \frac{\beta}{2(b_0^2+1)},$$

$$b_2 = \frac{1}{b_0^2-1}\left(\alpha + a_0^2 + \frac{2C_2}{b_0^2+1} - \frac{a_0^2b_0^2}{b_0^2-1}\right), \tag{2.17}$$

and sn is the Jacobi elliptic sine function.

Using (2.16) in (2.11), we have

$$R(\rho) = \frac{\beta}{b_0^2+1}\left(\frac{\operatorname{sn}\left(\frac{1}{2}\sqrt{-2b_2 + 2\sqrt{b_2{}^2 + 4b_1C_3}}\rho + C_4, m\right)C_3\sqrt{2}}{\sqrt{C_3\left(-b_2 + \sqrt{b_2^2 + 4b_1C_3}\right)}}\right)^2 + \frac{C_2}{b_0^2+1}, \tag{2.18}$$

where $m$, $b_1$ and $b_2$ are given by eq. (2.17).

Integrating (2.14) we have

$$Q(\rho) = \frac{a_0b_0}{b_0^2-1}\rho + C_5, \tag{2.19}$$

where $C_5$ is an arbitrary constant.

Using (2.16), (2.18) and (2.19) in (2.7), solution of the main system (1.1) is given as

$$u(x,t) = \left(\frac{\operatorname{sn}\left(\frac{1}{2}\sqrt{-2b_2 + 2\sqrt{b_2{}^2 + 4b_1C_3}}\rho + C_4, m\right)C_3\sqrt{2}}{\sqrt{C_3\left(-b_2 + \sqrt{b_2{}^2 + 4b_1C_3}\right)}}\right) \times \mathrm{e}^{\iota a_0 t}\mathrm{e}^{\iota\left(\frac{a_0b_0}{b_0^2-1}\rho + C_5\right)},$$

$$v(x,t) = \frac{\beta}{b_0^2+1}\left(\frac{\operatorname{sn}\left(\frac{1}{2}\sqrt{-2b_2 + 2\sqrt{b_2{}^2 + 4b_1C_3}}\rho + C_4, m\right)C_3\sqrt{2}}{\sqrt{C_3\left(-b_2 + \sqrt{b_2{}^2 + 4b_1C_3}\right)}}\right)^2 + \frac{C_2}{b_0^2+1}, \tag{2.20}$$

where $m$, $b_1$, $b_2$ are given by eq. (2.17) and $\rho = x - b_0t$.

*Case* i(b). When $C_0 = 0,\ C_3 = 0$

In this case eq. (2.15) becomes

$$P'^2 = -b_1 P^4 + b_2 P^2, \tag{2.21}$$

where $b_1$ and $b_2$ are given by eq. (2.17).

Using transformation

$$X(\rho) = P(\rho)^2 \tag{2.22}$$

in (2.21), we have

$$X'^2 = -4b_1 X^3 + 4b_2 X^2. \tag{2.23}$$

Equation (2.23) admits the following solutions

$$X = P^2 = \begin{cases} \dfrac{b_2}{b_1}\operatorname{sech}\left(\sqrt{b_2}\,(\pm\rho\ + C_4\,)\right)^2, & \text{when } b_2 > 0 \\[2ex] \dfrac{b_2}{b_1}\sec\left(\sqrt{-b_2}\,(\pm\rho\ + C_4\,)\right)^2, & \text{when } b_2 < 0 \end{cases} \tag{2.24}$$

where $C_4$ is an arbitrary constant. Corresponding solutions of the main system (1.1) are given as:

When $b_2 > 0$

$$\begin{aligned} u(x,t) &= \sqrt{\frac{b_2}{b_1}}\operatorname{sech}\left(\sqrt{b_2}\,(\pm\rho\ + C_4\,)\right) \mathrm{e}^{\iota a_0} \mathrm{e}^{\iota\left(\frac{a_0 b_0}{b_0^2-1}\rho + C_5\right)}, \\ v(x,t) &= \frac{b_2\beta}{b_1(b_0^2+1)}\operatorname{sech}\left(\sqrt{b_2}\,(\pm\rho\ + C_4\,)\right)^2 + \frac{C_2}{b_0^2+1}, \end{aligned} \tag{2.25}$$

where $b_1, b_2$ are given by eq. (2.17) and $\rho = x - b_0 t$.

When $b_2 < 0$

$$\begin{aligned} u(x,t) &= \sqrt{\frac{b_2}{b_1}}\sec\left(\sqrt{-b_2}\,(\pm\rho\ + C_4\,)\right) \mathrm{e}^{\iota a_0} \mathrm{e}^{\iota\left(\frac{a_0 b_0}{b_0^2-1}\rho + C_5\right)} \\ v(x,t) &= \frac{b_2\beta}{b_1(b_0^2+1)}\sec\left(\sqrt{-b_2}\,(\pm\rho\ + C_4)\right)^2 + \frac{C_2}{b_0^2+1}, \end{aligned} \tag{2.26}$$

where $b_1, b_2$ are given by eq. (2.17) and $\rho = x - b_0 t$.

In this case, we get trivial Lie symmetries of system (1.1).

*Case* ii. Now on setting

$$u(x,t) = \psi_1(x,t)\mathrm{e}^{\iota\psi_2(x,t)}, \tag{2.27}$$

system (1.1) decomposes to the following system:

$$\begin{aligned} &\psi_{1tt} - \psi_1\psi_{2t}^2 - \psi_{1xx} + \psi_1\psi_{2x}^2 - \alpha\psi_1 + \beta\psi_1^3 - 2\psi_1 v = 0 \\ &2\psi_{1t}\psi_{2t} + \psi_1\psi_{2tt} - 2\psi_{1x}\psi_{2x} - \psi_1\psi_{2xx} = 0 \\ &v_{tt} + v_{xx} - \beta(\psi_1)_{xx}^2 = 0. \end{aligned} \tag{2.28}$$

Apply the Lie classical method on system (2.28) as mentioned in Case i, we get the following symmetries:

$$\eta = a_1\psi_1, \quad \phi = a_2, \quad \zeta = a_1(2v+\alpha),$$
$$\tau = -a_1 t + a_4, \quad \xi = -a_1 x + a_3, \tag{2.29}$$

where $a_1$, $a_2$, $a_3$ and $a_4$ are arbitrary constants.

Now we shall give symmetry reductions and corresponding exact solutions of system (1.1).

*Case* ii(a). When $a_1 \neq 0$ and $a_2 = a_3 = a_4 = 0$

Solving the characteristic equation (2.6) and using (2.27), we have the following similarity variables:

$$\rho = \frac{x}{t}, \quad u = \frac{F(\rho)}{t}\mathrm{e}^{G(\rho)}, \quad v = \frac{1}{2}\left(\frac{H(\rho)}{t^2} - \alpha\right). \tag{2.30}$$

Here $\rho(x,t)$ is the new independent variable and $F(\rho)$, $G(\rho)$, $H(\rho)$ are the new dependent variables.

Using (2.30) in (1.1), we have

$$(\rho^2-1)F'' - (\rho^2-1)FG'^2 + 4\rho F' + \beta F^3 - FH + 2F = 0, \tag{2.31}$$
$$(1-\rho^2)FG'' + 2(1-\rho^2)F'G' - 4\rho FG' = 0, \tag{2.32}$$
$$(1+\rho^2)H'' + 6\rho H' + 6H - 2\beta(F^2)'' = 0, \tag{2.33}$$

where $(')$ denotes derivative with respect to $\rho$.

From eq. (2.32), we have

$$G = \int \frac{C_1}{F^2(1-\rho^2)^2}\mathrm{d}\rho. \tag{2.34}$$

Now using eq. (2.34) in (2.31), we have

$$H = \frac{((\rho^2-1)F)''}{F} + \frac{1}{(1-\rho^2)^3F^4} + \beta F^2. \tag{2.35}$$

Substituting $H$ from (2.35) in (2.33), we have

$$\begin{aligned}(1+\rho^2)&\left(\frac{((\rho^2-1)F)''}{F} + \frac{1}{(1-\rho^2)^3F^4} + \beta F^2\right)'' \\ &+ 6\rho\left(\frac{((\rho^2-1)F)''}{F} + \frac{1}{(1-\rho^2)^3F^4} + \beta F^2\right)' \\ &+ 6\left(\frac{((\rho^2-1)F)''}{F} + \frac{1}{(1-\rho^2)^3F^4} + \beta F^2\right) \\ &- 2\beta(F^2)'' = 0.\end{aligned} \tag{2.36}$$

The solution of the main system (1.1) is given as

$$u = \frac{F}{t}\mathrm{e}^{\int \frac{C_1}{F^2(1-\rho^2)^2}\mathrm{d}\rho}, \quad v = \frac{1}{2}\left(\frac{\frac{((\rho^2-1)F)''}{F} + \frac{1}{(1-\rho^2)^3F^4} + \beta F^2}{t^2} - \alpha\right), \tag{2.37}$$

where $F$ is given by eq. (2.36) and $\rho$ is given by (2.30).

*Case* ii(b). When $a_1 = a_3 = 0, a_2 \neq 0$ and $a_4 \neq 0$

For simplicity let $a_2 = a_3 = 1$ and solving characteristic equation, we get the following similarity variables:

$$\rho = x, \quad u = F(\rho)\mathrm{e}^{\iota t + G(\rho)}, \quad w = H(\rho). \tag{2.38}$$

Here $\rho(x,t)$ is the new independent variable and $F(\rho),\ G(\rho),\ H(\rho)$ are the new dependent variables.

Using (2.38) in (1.1), we have

$$H'' - 2\beta F'^2 - 2\beta F F'' = 0, \tag{2.39}$$

$$-F - F'' + FG'^2 - \alpha F + \beta F^3 - 2FH = 0, \tag{2.40}$$

$$-2F'G' - FG'' = 0, \tag{2.41}$$

where ($'$) denotes derivative with respect to $\rho$.

From (2.39), we have

$$H = \beta F^2 + C_1\rho + C_2, \tag{2.42}$$

where $C_1$ and $C_2$ are arbitrary constants.

Now integrating (2.41), we have

$$G' = \frac{C_3}{F^2}, \tag{2.43}$$

where $C_3$ is an arbitrary constant.

Using (2.42) and (2.43) in (2.40), we have

$$F'' - \frac{C_3^2}{F^3} + (1+\alpha)F - \beta F^3 + 2F(\beta F^2 + C_1\rho + C_2) = 0. \tag{2.44}$$

Here choosing $C_3 = C_1 = 0$ and using (2.38), corresponding to ODE (2.44), solutions of the main system (1.1) can be given as

$$\begin{aligned}
&\text{(i)} \quad u = -\frac{\sqrt{-\beta\,(1+\alpha+2C_2)}\tanh\left(C_5 - 1/2\sqrt{2+2\,\alpha+4C_2}x\right)\mathrm{e}^{\iota(t+C_4)}}{\beta},\\
&\qquad v = -\,(1+\alpha+2C_2)\tanh\left(C_5 - 1/2\sqrt{2+2\,\alpha+4C_2}x\right)^2 + C_2,\\
&\text{(ii)} \quad u = \frac{1}{\beta}\sqrt{-\beta\left(2C_2+\alpha+1-C_6{}^2\right)}\\
&\qquad \times \operatorname{cn}\left(C_5 + C_6\,x,\ 1/2\,\frac{\sqrt{-4C_2-2\,\alpha-2+2C_6{}^2}}{C_6}\right)\mathrm{e}^{\iota(t+C_4)}\\
&\qquad v = -\left(2C_2+\alpha+1-C_6{}^2\right)\\
&\qquad \times \operatorname{cn}\left(C_5 + C_6\,x,\ 1/2\,\frac{\sqrt{-4C_2-2\,\alpha-2+2C_6{}^2}}{C_6}\right)^2 + C_2\\
&\text{(iii)} \quad u = \frac{1}{\sqrt{\beta}}\sqrt{2}C_6\operatorname{dn}\left(C_5 + C_6\,x,\ \frac{\sqrt{2C_6{}^2+1+\alpha+2C_2}}{C_6}\right)\mathrm{e}^{\iota(t+C_4)}\\
&\qquad v = 2\,C_6{}^2\operatorname{dn}\left(C_5 + C_6\,x,\ \frac{\sqrt{2C_6{}^2+1+\alpha+2C_2}}{C_6}\right)^2 + C_2,
\end{aligned} \tag{2.45}$$

where $C_4$ and $C_5$ are arbitrary constants.

### 2.2 *Hamiltonian amplitude equation*

In this section, we shall find the symmetries and exact solution of Hamiltonian amplitude equation (1.2). Here too, we shall consider two cases.

*Case* i. On setting

$$u = \psi_1 + \iota\psi_2 \tag{2.46}$$

eq. (1.2) decomposes into the following system:

$$\begin{aligned} &-\psi_{2x} + \psi_{1tt} + 2\eta(\psi_1^2 + \psi_2^2)\psi_1 - \beta\psi_{1xt} = 0 \\ &\psi_{1x} + \psi_{2tt} + 2\eta(\psi_1^2 + \psi_2^2)\psi_2 - \beta\psi_{2xt} = 0. \end{aligned} \tag{2.47}$$

Proceeding in similar manner as mentioned earlier, to find the Lie symmetries of system (2.47), we get

$$\xi = a_1, \quad \tau = a_2, \quad \phi_1 = a_3 v, \quad \phi_2 = -a_3 u, \tag{2.48}$$

where $\xi$, $\tau$, $\phi_1$ and $\phi_2$ are infinitesimals corresponding to $x, t, \psi_1$ and $\psi_2$, respectively.

Solving the characteristic equation we get the following similarity variables of system (1.2):

$$\rho = x - b_0 t, \quad u = P(\rho)\mathrm{e}^{\iota a_0 t}\mathrm{e}^{\iota Q(\rho)}, \tag{2.49}$$

where $a_0 = a_3/a_2$ and $b_0 = a_1/a_2$.

Using (2.49) in (1.2) we have

$$\begin{aligned} &b_0(b_0+\beta)P'' + 2\eta P^3 - Pa_0^2 - b_0(b_0+\beta)PQ'^2 \\ &\quad + (\beta a_0 + 2a_0 b_0 - 1)PQ' = 0, \end{aligned} \tag{2.50}$$

$$b_0(b_0+\beta)PQ'' + (1 - \beta a_0 - 2a_0 b_0)P' + 2b_0(b_0+\beta)P'Q' = 0, \tag{2.51}$$

where ($'$) denotes derivative with respect to $\rho$.

Using

$$Q' = Z(P) \tag{2.52}$$

in (2.51) and integrating, we get

$$Q' = Z = \frac{2a_0 b_0 + \beta a_0 - 1}{2b_0(\beta + b_0)} + \frac{C_0}{P^2}, \tag{2.53}$$

where $C_0$ is an arbitrary constant.

Using (2.53) in (2.50) and integrating, we have

$$\begin{aligned} P'^2 = &-\frac{\eta}{b_0(\beta+b_0)}P^4 \\ &- \frac{1}{b_0(\beta+b_0)}\left(\frac{(\beta a_0 + 2a_0 b_0 - 1)^2}{4b_0(\beta+b_0)} - a_0^2\right)P^2 - \frac{C_0}{P^2} + C_3, \end{aligned} \tag{2.54}$$

where $C_3$ is an arbitrary constant.

Here we shall consider two cases as follows:

*Case* i(a). When $C_0 = 0$, $C_3 \neq 0$

Integrating (2.54) we get solution in the form of Jacobi elliptical sine function

$$P(\rho) = \frac{\sqrt{2C_3}\,\mathrm{sn}\left(\frac{1}{2}\sqrt{2b_2 - 2\sqrt{b_2{}^2 - 4b_1C_3}}\rho + C_4, m\right)}{\left(-b_2 + \sqrt{b_2{}^2 - 4b_1C_3}\right)}, \tag{2.55}$$

where

$$m = \frac{\sqrt{-2\left(2b_1C_3 - b_2{}^2 + b_2\sqrt{b_2{}^2 - 4b_1C_3}\right)b_1C_3}}{2b_1C_3 - b_2{}^2 + b_2\sqrt{b_2{}^2 - 4b_1C_3}},$$

$$b_1 = \frac{\eta}{b_0\,(\beta + b_0)},$$

$$b_2 = \frac{1}{b_0(\beta + b_0)}\left(\frac{(\beta a_0 + 2a_0b_0 - 1)^2}{4b_0(\beta + b_0)} - a_0^2\right) \tag{2.56}$$

and $C_3$ and $C_4$ are arbitrary constants.

Using (2.53) and (2.55) in (2.49), solution of Hamiltonian amplitude equation (1.2) is given as

$$u(x,t) = \frac{\sqrt{2C_3}\,\mathrm{sn}\left(\frac{1}{2}\sqrt{2b_2 - 2\sqrt{b_2{}^2 - 4b_1C_3}}(x - b_0t) + C_4, m\right)}{\left(-b_2 + \sqrt{b_2{}^2 - 4b_1C_3}\right)}$$

$$\times\, \mathrm{e}^{\iota a_0}\mathrm{e}^{\iota\left(\frac{2a_0b_0 + \beta a_0 - 1}{2b_0(\beta + b_0)}(x - b_0t) + C_5\right)}, \tag{2.57}$$

where $m$, $b_1$, $b_2$ are given by (2.56) and $C_4$, $C_5$ are arbitrary constants.

*Case* i(b). When $C_0 = 0,\ C_3 = 0$

In this case, eq. (2.54) reduces to

$$P'^2 = -b_1P^4 - b_2P^2, \tag{2.58}$$

where $b_1$ and $b_2$ are given by (2.56).

Using the transformation (2.22), eq. (2.58) reduces to

$$X'^2 = -4b_1X^3 - 4b_2X^2. \tag{2.59}$$

Equation (2.59) admits the following solutions:

$$X = P^2 = \begin{cases} -\dfrac{b_2}{b_1}\mathrm{sech}\left(\sqrt{-b_2}\,(\pm\rho\ + C_4\,)\right)^2, & \text{when } b_2 < 0 \\[2ex] -\dfrac{b_2}{b_1}\sec\left(\sqrt{b_2}\,(\pm\rho\ + C_4\,)\right)^2, & \text{when } b_2 > 0 \end{cases}. \tag{2.60}$$

Corresponding solutions of the main eq. (1.2) are given as

$$u(x,t) = \begin{cases} \sqrt{-\dfrac{b_2}{b_1}}\operatorname{sech}\left(\sqrt{-b_2}\,(\pm(x-b_0t)\,+C_4\,)\right) \\ \quad\times\, \mathrm{e}^{\iota a_0}\mathrm{e}^{\iota\left(\frac{2a_0b_0+\beta a_0-1}{2b_0(\beta+b_0)}(x-b_0t)+C_5\right)}, \quad \text{when } b_2<0,\ b_1>0 \\ \sqrt{-\dfrac{b_2}{b_1}}\sec\left(\sqrt{b_2}\,(\pm(x-b_0t)\,+C_4\,)\right) \\ \quad\times\, \mathrm{e}^{\iota a_0}\mathrm{e}^{\iota\left(\frac{2a_0b_0+\beta a_0-1}{2b_0(\beta+b_0)}(x-b_0t)+C_5\right)}, \quad \text{when } b_2>0,\ b_1<0 \end{cases}, \tag{2.61}$$

where $b_1$ and $b_2$ are given by (2.56) and $C_4$, $C_5$ are arbitrary constants.

*Case* ii. Now on setting

$$u(x,t) = \psi_1(x,t)\mathrm{e}^{\iota\psi_2(x,t)}, \tag{2.62}$$

system (1.2) decomposes to the following system:

$$\begin{aligned} &\psi_{1tt} - \psi_1\psi_{2x} - \psi_1\psi_{2t}^2 - \beta\psi_{xt} + \beta\psi_1\psi_{2x}\psi_{2t} - 2\eta\psi_1^3 = 0 \\ &\psi_1\psi_{2tt} + \psi_{1x} + 2\psi_{1t}\psi_{2t} - \beta(\psi_{1t}\psi_{2x} + \psi_{1x}\psi_{2t}) - \beta\psi_1\psi_{2xt} = 0. \end{aligned} \tag{2.63}$$

Now applying Lie classical method on (2.63), we get the following symmetries:

$$\begin{aligned} &\xi = a_1x + a_2, \quad \tau = -a_1\left(t + \frac{2x}{\beta}\right) + a_3, \\ &\phi_1 = 0, \quad \phi_2 = -a_1\left(\frac{t}{\beta}\right) + a_4, \end{aligned} \tag{2.64}$$

where $\xi, \tau, \phi_1, \phi_2$ are infinitesimals corresponding to $x, t, \psi_1, \psi_2$, respectively and $a_1,\ a_2,\ a_3,\ a_4$ are arbitrary constants.

Now we shall give similarity reduction and corresponding exact solutions of eq. (1.2).

*Case* ii(a). $a_1 \neq 0$ and $a_2 = a_3 = a_4 = 0$

Solving characteristic equation, we have the following similarity variables:

$$\rho = tx + \frac{x^2}{\beta}, \quad \psi_1 = F(\rho), \quad \psi_2 = \frac{2x}{\beta^2} + \frac{t}{\beta} + G(\rho), \tag{2.65}$$

where $\rho$ is the new independent variable and $F(\rho),\ G(\rho)$ are the new dependent variables.

Using (2.65) in (2.63), we have

$$\beta^3\rho F'' - \beta^3\rho F G'^2 + \beta^3 F' + 2\eta\beta^2F^3 + F = 0, \tag{2.66}$$

$$\rho F G'' + 2\rho F'G' + FG' = 0, \tag{2.67}$$

where ($'$) denotes derivative with respect to $\rho$.

Integrating (2.67) once, we have

$$G' = \frac{C_1}{\rho F^2}, \tag{2.68}$$

where $C_1$ is an arbitrary constant.

Substituting (2.68) in (2.66), we have

$$\beta^3\rho^2F^3F'' + \beta^3\rho F^3F' + 2\eta\beta^2\rho F^6 + \rho F^4 - \beta^3C_1^2 = 0. \tag{2.69}$$

Using (2.65) and (2.68) in (2.62), solution of the main eq. (1.2) can be given as

$$u = F(\rho)\mathrm{e}^{\iota\left(\frac{2x}{\beta^2}+\frac{t}{\beta}+\int\frac{C_1}{\rho F^2}d\rho+C_2\right)}, \tag{2.70}$$

where $\rho$ is given by (2.65) and $F$ is given by (2.69).

*Case* ii(b). $a_2 \neq 0, a_4 \neq 0$ and $a_1 = a_3 = 0$

Let $a_2 = a_4 = 1$ and solving characteristic equation, we have following similarity variables:

$$\rho = t, \quad \psi_1 = F(\rho), \quad \psi_2 = x + G(\rho), \tag{2.71}$$

where $\rho$ is the new independent variable and $F$, $G$ are the new dependent variables.

Using (2.71) into system (2.63), we have

$$F'' - FG'^2 - 2\eta F^3 + \beta FG' - F = 0, \tag{2.72}$$

$$FG'' + 2F'G' - \beta F' = 0, \tag{2.73}$$

where $(')$ denotes derivative with respect to $\rho$.

Corresponding to ODEs (2.72)–(2.73), solutions of Hamiltonian eq. (1.2) are given as

$$\text{(i)} \quad u = -\frac{\sqrt{2\eta(\beta^2-4)}\tanh\left(C_2 - 1/4\sqrt{2\beta^2-8}t\right)}{4\eta}\mathrm{e}^{i\left(x+\frac{\beta t}{2}+C_1\right)},$$

$$\text{(ii)} \quad u = \frac{1}{4\eta}\sqrt{2\eta(\beta^2 - 4C_3{}^2 - 4)} \times \mathrm{cn}\left(C_2 + C_3t,\ \frac{\sqrt{-2\beta^2+8C_3{}^2+8}}{4C_3}\right)\mathrm{e}^{i\left(x+\frac{\beta t}{2}+C_1\right)},$$

$$\text{(iii)} \quad u = \frac{1}{2\eta}\sqrt{\eta(4C_3{}^2 - 4 + \beta^2)} \times \mathrm{nd}\left(C_2 + C_3t,\ \frac{\sqrt{8C_3{}^2+\beta^2-4}}{2C_3}\right)\mathrm{e}^{i\left(x+\frac{\beta t}{2}+C_1\right)}, \tag{2.74}$$

where $C_1$, $C_2$ and $C_3$ are arbitrary functions.

## 3. $(G'/G)$-expansion method and travelling wave solutions

In this section, we shall use $(G'/G)$-expansion method [18,19] to obtain some new exact solutions of eqs (1.1) and (1.2). The main idea of this method is that the travelling wave solutions of nonlinear equations can be expressed by a polynomial in $(G'/G)$, where $G = G(\xi)$ satisfies the second-order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$. Here $\xi = x + ct$, where $\lambda, \mu$ and $c$ are arbitrary constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest-order derivatives and the nonlinear term appearing in the given nonlinear equations. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the proposed method.

Suppose that we have a complex nonlinear PDE in the following form:

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...) = 0, \tag{3.1}$$

where $u = u(x, t)$ is an unknown function, $P$ is a polynomial in $u = u(x, t)$ and its partial derivatives, in which the highest-order derivatives and nonlinear terms are involved. Now we are giving the main steps [18] for solving eq. (3.1) using the $(G'/G)$-expansion method.

*Step* i. Seek travelling wave solutions of (3.1) by taking $u(x, t) = \phi(\xi)\mathrm{e}^{\iota(px+rt)}$, $\xi = x + ct$, and transform (3.1) to the ordinary differential equation (ODE)

$$Q(\phi, \phi', \phi'', ...) = 0, \tag{3.2}$$

where prime ($'$) denotes the derivative with respect to $\xi$.

*Step* ii. If possible, integrate (3.2) term by term as many times as possible to yield constant(s) of integration. For simplicity the integration constant(s) can be set to zero.

*Step* iii. Suppose that the solution $u(\xi)$ of ODE (3.2) can be expressed as a finite series in the form

$$u(\xi) = a_m \left(\frac{G'}{G}\right)^m + a_{m-1}\left(\frac{G'}{G}\right)^{m-1} + \cdots + a_0, \quad a_m \neq 0, \tag{3.3}$$

where $G = G(\xi)$ satisfies the second-order linear ODE in the form

$$G'' + \lambda G' + \mu G = 0. \tag{3.4}$$

$a_0, a_1, ..., a_m, \lambda$ and $\mu$ are constants to be determined later. $m$ is a positive integer, which is determined by the homogeneous balancing method.

*Step* iv. Substituting (3.3) together with (3.4) into (3.2) yields an algebraic equation involving powers of $(G'/G)$. Equate the coefficients of each power of $(G'/G)$ to zero, to obtain a system of algebraic equations for $a_i, \lambda, \mu$ and $c$. Then, to determine these constants we solve the system with the aid of softwares, such as Maple. Since we know the solutions of (3.4), depending on the sign of the discriminant $\triangle = \lambda^2 - 4\mu$, the exact solutions of the given (3.1) can be obtained.

### 3.1 *Coupled Higgs equation*

To find the explicit exact solutions of the coupled Higgs eq. (1.1), we proceed using the methodology explained above. First we make the transformation

$$u = \mathrm{e}^{\iota\theta}U(\xi), \quad v = V(\xi), \tag{3.5}$$

where $\theta = px + rt$, $\xi = x + ct$, we have a relation $p = rc$ and system (1.1) reduces to the following system of ODEs:

$$(c^2 - 1)U'' + [r^2(c^2 - 1) - \alpha]U - 2UV + \beta U^3 = 0,$$
$$(c^2 + 1)V'' - \beta(U^2)'' = 0. \tag{3.6}$$

Integrating the second equation in system (3.6), we find

$$(c^2 + 1)V' - 2\beta UU' + C = 0, \tag{3.7}$$

where $C$ is a constant of integration.

Suppose that the solution of the system can be expressed by polynomials in $(G'/G)$ as follows:

$$U(\xi) = a_m \left(\frac{G'}{G}\right)^m + \cdots,$$
$$V(\xi) = b_n \left(\frac{G'}{G}\right)^n + \cdots, \tag{3.8}$$

where $G = G(\xi)$ satisfies the second-order linear ODE (3.4).

Balancing the highest-order derivatives in linear terms with the nonlinear terms, we get $m = 1$ and $n = 2$. We can suppose that the solution of systems (3.6)–(3.7) is of the form

$$U(\xi) = a_1 \left(\frac{G'}{G}\right) + a_0, \quad a_1 \neq 0, \tag{3.9}$$

$$V(\xi) = b_2 \left(\frac{G'}{G}\right)^2 + b_1 \left(\frac{G'}{G}\right) + b_0, \quad b_2 \neq 0. \tag{3.10}$$

Substituting (3.9)–(3.10) together with (3.4) into eq. (3.6) and then in eq. (3.7), collecting all terms with equal power of $(G'/G)$ and setting each coefficient to zero, we obtain the following set of algebraic equations:

$$\beta {a_1}^3 - 2a_1 - 2a_1b_2 + 2c^2a_1 = 0$$
$$3\beta {a_1}^2a_0 - 2a_0b_2 - 2a_1b_1 + 3c^2\lambda a_1 - 3\lambda a_1 = 0$$
$$c^2\lambda^2a_1 + 2c^2a_1\mu + r^2c^2a_1 - \alpha a_1 + 3\beta a_1{a_0}^2 - 2a_1b_0$$
$$\quad - \lambda^2a_1 - 2a_1\mu - r^2a_1 - 2a_0b_1 = 0$$
$$- 2a_0b_0 + r^2c^2a_0 - r^2a_0 + c^2\lambda\mu a_1 + \beta {a_0}^3 - \lambda\mu a_1 - \alpha a_0 = 0$$
$$- 2b_2 + 2\beta {a_1}^2 - 2c^2b_2 = 0$$
$$- 2c^2\lambda b_2 - 2\lambda b_2 - b_1 + 2\beta a_1a_0 - c^2b_1 + 2\beta {a_1}^2\lambda = 0$$
$$2\beta {a_1}^2\mu + 2\beta a_1a_0\lambda - c^2b_1\lambda - 2b_2\mu - b_1\lambda - 2c^2b_2\mu = 0$$
$$- c^2b_1\mu + 2\beta a_1a_0\mu - b_1\mu = 0. \tag{3.11}$$

Solving these algebraic equations yields two different cases as follows:

*Case* i.

$$\left\{a_0 = a_0, a_1 = a_1, b_0 = -\frac{\alpha}{2} + \frac{{a_0}^2\beta}{2},\right.$$

$$\left. b_1 = \beta a_1a_0, b_2 = \frac{\beta {a_1}^2}{2}, c = \pm 1\right\}. \tag{3.12}$$

*Case* ii.

$$\left\{a_0 = \pm\frac{\lambda}{2}\sqrt{-\frac{2\,c^2+2}{\beta}}, a_1 = \pm\sqrt{-\frac{2\,c^2+2}{\beta}},\right.$$

$$b_0 = -\frac{c^2\lambda^2}{4} + c^2\mu + \frac{r^2c^2}{2} - \frac{\alpha}{2} - \frac{\lambda^2}{4} - \mu - \frac{r^2}{2},$$

$$\left. b_1 = -2\,\lambda, b_2 = -2, c = c\right\} \tag{3.13}$$

For Case i, substituting (3.12) into (3.9)–(3.10), we have

$$U(\xi) = a_1\left(\frac{G'}{G}\right) + a_0,$$

$$V(\xi) = \frac{\beta {a_1}^2}{2}\left(\frac{G'}{G}\right)^2 + \beta a_1 a_0\left(\frac{G'}{G}\right) - \frac{\alpha}{2} + \frac{{a_0}^2\beta}{2},$$

$$\xi = x \pm t. \tag{3.14}$$

Substituting the general solution of (3.4), we have three types of travelling wave solutions of the coupled Higgs eq. (1.1) as follows:

*Case* i(a). When $\lambda^2 - 4\mu > 0$

$$u=\left(a_1\left(-\frac{1}{2}\lambda+\frac{1}{2}\frac{\left(C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)\right)\sqrt{\lambda^2-4\mu}}{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}\right)+a_0\right)\mathrm{e}^{\iota\theta},$$

$$\begin{aligned}
v &= \frac{\beta {a_1}^2}{2} \\
&\quad\times\left(-\frac{1}{2}\lambda+\frac{1}{2}\frac{\left(C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)\right)\sqrt{\lambda^2-4\mu}}{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}\right)^2 \\
&\quad+\beta a_1 a_0 \\
&\quad\times\left(-\frac{1}{2}\lambda+\frac{1}{2}\frac{\left(C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)\right)\sqrt{\lambda^2-4\mu}}{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}\right) \\
&\quad-\frac{\alpha}{2}+\frac{{a_0}^2\beta}{2},
\end{aligned} \tag{3.15}$$

where $\xi = x \pm t$ and $\theta = px + rt$, and $C_1$ and $C_2$ are two arbitrary constants.

*Case* i(b). When $\lambda^2 - 4\mu < 0$

$$u=\left(a_1\left(-\frac{1}{2}\lambda+\frac{1}{2}\frac{\left(-C_1\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)+C_2\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)\right)\sqrt{-\lambda^2+4\mu}}{C_1\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)+C_2\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)}\right)+a_0\right)e^{\iota\theta},$$

$$v=\frac{\beta a_1{}^2}{2}\left(-\frac{1}{2}\lambda+\frac{1}{2}\frac{\left(-C_1\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)+C_2\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)\right)\sqrt{-\lambda^2+4\mu}}{C_1\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)+C_2\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)}\right)^2$$
$$+\beta a_1 a_0\left(-\frac{1}{2}\lambda+\frac{1}{2}\frac{\left(-C_1\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)+C_2\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)\right)\sqrt{-\lambda^2+4\mu}}{C_1\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)+C_2\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)}\right)-\frac{\alpha}{2}+\frac{a_0{}^2\beta}{2}, \tag{3.16}$$

where $\xi = x \pm t$, $\theta = px + rt$, and $C_1$ and $C_2$ are two arbitrary constants.

*Case* i(c). When $\lambda^2 - 4\mu = 0$

$$u=\left(a_1\left(\frac{2C_2-C_1\lambda-C_2\lambda\xi}{2(C_1+C_2\xi)}\right)+a_0\right)e^{\iota\theta},$$
$$v=\frac{\beta a_1{}^2}{2}\left(\frac{2C_2-C_1\lambda-C_2\lambda\xi}{2(C_1+C_2\xi)}\right)^2+\beta a_1 a_0\left(\frac{2C_2-C_1\lambda-C_2\lambda\xi}{2(C_1+C_2\xi)}\right)-\frac{\alpha}{2}+\frac{a_0{}^2\beta}{2}, \tag{3.17}$$

where $\xi = x \pm t$, $\theta = px + rt$, and $C_1$ and $C_2$ are two arbitrary constants.

For Case ii, substituting (3.13) into (3.9)–(3.10), we have

$$U(\xi)=\pm\sqrt{-\frac{2c^2+2}{\beta}}\left(\frac{G'}{G}\right)\pm\frac{\lambda}{2}\sqrt{-\frac{2c^2+2}{\beta}},$$
$$V(\xi)=-2\left(\frac{G'}{G}\right)^2-2\lambda\left(\frac{G'}{G}\right)-\frac{c^2\lambda^2}{4}+c^2\mu+\frac{r^2c^2}{2}-\frac{\alpha}{2}-\frac{\lambda^2}{4}-\mu-\frac{r^2}{2},$$
$$\xi=x+ct. \tag{3.18}$$

Substituting the general solution of (3.4), again we have three types of solutions as given below:

*Case* ii(a). When $\lambda^2 - 4\mu > 0$

$$u(x,t) = \pm\frac{1}{2}\sqrt{-\frac{2c^2+2}{\beta}} \times \left(\frac{\left(C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)\right)\sqrt{\lambda^2-4\mu}}{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}\right)\mathrm{e}^{\iota\theta}$$

$$\begin{aligned} v(x,t) = & -2\left(-\frac{1}{2}\lambda+\frac{1}{2}\frac{\left(C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)\right)\sqrt{\lambda^2-4\mu}}{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}\right)^2 \\ & -2\lambda\left(-\frac{1}{2}\lambda+\frac{1}{2}\frac{\left(C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)\right)\sqrt{\lambda^2-4\mu}}{C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)}\right) \\ & -\frac{c^2\lambda^2}{4}+c^2\mu+\frac{r^2c^2}{2}-\frac{\alpha}{2}-\frac{\lambda^2}{4}-\mu-\frac{r^2}{2}, \end{aligned} \tag{3.19}$$

where $\xi = x + ct$, $\theta = px + rt$, and $C_1$ and $C_2$ are two arbitrary constants.

*Case* ii(b). When $\lambda^2 - 4\mu < 0$

$$u(x,t) = \pm\frac{1}{2}\sqrt{-\frac{2c^2+2}{\beta}} \times \left(\frac{\left(-C_1\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)+C_2\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)\right)\sqrt{-\lambda^2+4\mu}}{C_1\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)+C_2\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)}\right)\mathrm{e}^{\iota\theta}$$

$$\begin{aligned} v(x,t) & \\ = & -2\left(-\frac{1}{2}\lambda+\frac{1}{2}\frac{\left(-C_1\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)+C_2\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)\right)\sqrt{-\lambda^2+4\mu}}{C_1\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)+C_2\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)}\right)^2 \\ & -2\lambda\left(-\frac{1}{2}\lambda+\frac{1}{2}\frac{\left(-C_1\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)+C_2\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)\right)\sqrt{-\lambda^2+4\mu}}{C_1\cos\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)+C_2\sin\left(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi\right)}\right) \\ & -\frac{c^2\lambda^2}{4}+c^2\mu+\frac{r^2c^2}{2}-\frac{\alpha}{2}-\frac{\lambda^2}{4}-\mu-\frac{r^2}{2}, \end{aligned} \tag{3.20}$$

where $\xi = x + ct$, $\theta = px + rt$, and $C_1$ and $C_2$ are two arbitrary constants.

*Case* ii(c). When $\lambda^2 - 4\mu = 0$

$$u(x,t) = \pm\sqrt{-\frac{2c^2+2}{\beta}}\left(\frac{2C_2 - C_1\lambda - C_2\lambda\xi}{2(C_1+C_2\xi)}\right) \pm \frac{\lambda}{2}\sqrt{-\frac{2c^2+2}{\beta}},$$

$$v(x,t) = -2\left(\frac{2C_2 - C_1\lambda - C_2\lambda\xi}{2(C_1+C_2\xi)}\right)^2 - 2\lambda\left(\frac{2C_2 - C_1\lambda - C_2\lambda\xi}{2(C_1+C_2\xi)}\right)$$
$$- \frac{c^2\lambda^2}{4} + c^2\mu + \frac{r^2c^2}{2} - \frac{\alpha}{2} - \frac{\lambda^2}{4} - \mu - \frac{r^2}{2}, \tag{3.21}$$

where $\xi = x + ct$, $\theta = px + rt$, and $C_1$ and $C_2$ are two arbitrary constants.

### 3.2 *Hamiltonian amplitude equation*

To find explicit exact travelling wave solutions of Hamiltonian amplitude eq. (1.2), let

$$u = e^{\iota\theta}\phi(\xi), \quad \theta = px + rt, \quad \xi = x + ct. \tag{3.22}$$

Substituting (3.22) into (1.2), we have

$$(c^2 - \beta c)\phi''(\xi) + \iota(1 + 2rc - \beta cp - \beta r)\phi'(\xi)$$
$$-(p + r^2 - \beta pr)\phi(\xi) + 2\eta\phi^3(\xi) = 0. \tag{3.23}$$

Employing the condition

$$1 + 2rc - \beta cp - \beta r = 0, \tag{3.24}$$

eq. (3.23) transforms into the following equation:

$$(c^2 - \beta c)\phi''(\xi) - (p + r^2 - \beta pr)\phi(\xi) + 2\eta\phi^3(\xi) = 0. \tag{3.25}$$

By balancing the highest-order derivative term $\phi''$ with the nonlinear term $\phi^3$, we get $m = 1$. Therefore we suppose that (3.25) has the following formal solution with $a_1 \neq 0$:

$$\phi(\xi) = a_1\left(\frac{G'}{G}\right) + a_0, \tag{3.26}$$

where $a_0$ and $a_1$ are constants to be determined later.

Substituting (3.26) together with (3.4) into (3.25), collecting all terms with equal power of $(G'/G)$ and setting each coefficient to zero, we obtain the following set of algebraic equations:

$$2(c^2 - \beta c)a_1 + 2\eta {a_1}^3 = 0$$
$$3(c^2 - \beta c)\lambda a_1 + 6\eta a_0 {a_1}^2 = 0$$
$$(c^2 - \beta c)(\lambda^2 a_1 + 2a_1\mu) + 6\eta {a_0}^2 a_1 - (p + r^2 - \beta pr)a_1 = 0$$
$$(c^2 - \beta c)\,\lambda\mu a_1 + 2\eta {a_0}^3 - (p + r^2 - \beta pr)a_0 = 0. \tag{3.27}$$

Solving these algebraic equations yields

$$a_0 = \pm \frac{\lambda\left(-p - r^2 + \beta pr\right)}{\sqrt{\frac{2(-p-r^2+\beta pr)}{(-\lambda^2+4\mu)\eta}}\left(-\lambda^2 + 4\mu\right)\eta}, \quad a_1 = \pm\sqrt{\frac{2(-p - r^2 + \beta pr)}{\eta(-\lambda^2 + 4\mu)}},$$

$$c = \frac{-\beta\lambda^2 + 4\beta\mu \pm \sqrt{(-\lambda^2 + 4\mu)(4\beta^2\mu + 8p - \lambda^2\beta^2 + 8r^2 - 8\beta pr)}}{2(-\lambda^2 + 4\mu)}. \tag{3.28}$$

Substituting (3.28) into (3.26), we have

$$\phi(\xi) = \pm \frac{\lambda\left(-p - r^2 + \beta pr\right)}{\sqrt{\frac{2(-p-r^2+\beta pr)}{(-\lambda^2+4\mu)\eta}}\left(-\lambda^2 + 4\mu\right)\eta}\left(\frac{G'(\xi)}{G(\xi)}\right) \pm \sqrt{\frac{2(-p - r^2 + \beta pr)}{\eta(-\lambda^2 + 4\mu)}}. \tag{3.29}$$

Substituting the general solution of (3.4) into (3.29), we have the solution of eq. (1.2) with relation (3.24) as follows:

*Case* i. When $\lambda^2 - 4\mu > 0$

$$u = \pm \frac{\left(-p - r^2 + \beta pr\right)\left(C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)\right)\sqrt{2}}{2\left(C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)\right)\eta\sqrt{-\frac{-p-r^2+\beta pr}{\eta}}}\mathrm{e}^{\iota\theta}, \tag{3.30}$$

where

$$\xi = x + \left(\frac{-\beta\lambda^2 + 4\beta\mu \pm \sqrt{(-\lambda^2 + 4\mu)(4\beta^2\mu + 8p - \lambda^2\beta^2 + 8r^2 - 8\beta pr)}}{2(-\lambda^2 + 4\mu)}\right)t$$

and

$$\theta = px + rt.$$

$C_1$ and $C_2$ are arbitrary constants.

*Case* ii. When $\lambda^2 - 4\mu < 0$

$$u = \pm \frac{\left(-p - r^2 + \beta pr\right)\left(C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) - C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)\right)\sqrt{2}}{2\left(C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)\right)\eta\sqrt{\frac{-p-r^2+\beta pr}{\eta}}}\mathrm{e}^{\iota\theta}, \tag{3.31}$$

where

$$\xi = x + \left(\frac{-\beta\lambda^2 + 4\beta\mu \pm \sqrt{(-\lambda^2 + 4\mu)(4\beta^2\mu + 8p - \lambda^2\beta^2 + 8r^2 - 8\beta pr)}}{2(-\lambda^2 + 4\mu)}\right)t$$

and

$$\theta = px + rt.$$

$C_1$ and $C_2$ are arbitrary constants.

## 4. Conclusion

In this paper, we have investigated the symmetries and invariant solutions of Higgs field equation and Hamiltonian amplitude equation. Using the symmetries, we have reduced eqs (1.1) and (1.2) into a system of ODEs and then certain exact solutions of Higgs field and Hamiltonian amplitude equations are obtained. Along with the Lie symmetry method, the travelling wave solutions of the coupled Higgs equation and Hamiltonian amplitude equation are successfully found by using the $(G'/G)$-expansion method, which include hyperbolic function solutions, trigonometry function solutions and rational solutions.

## Acknowledgements

The authors wish to thank the referees for several valuable suggestions which have considerably improved the presentation of the paper.

## References

[1] M Wang, *Phys. Lett.* **A216**, 67 (1996)
[2] W Malfliet, *Phys. Scr.* **54**, 563 (1996)
[3] M A Abdou, *Appl. Math. Comput*. **190**, 988 (2007)
[4] D Lu, *Chaos, Solitons and Fractals* **24**, 1373 (2005)
[5] Z Yan, *Chaos, Solitons and Fractals* **18**, 299 (2003)
[6] M Wang and X Li, *Phys. Lett.* **A343**, 48 (2005)
[7] M Wang and X Li, *Chaos, Solitons and Fractals* **24**, 283 (2005)
[8] P J Olver, *Applications of Lie groups to differential equations* (Springer-Verlag, New York, 1993)
[9] G W Bluman and S Kumei, *Symmetries and differential equations* (Springer, Berlin, 1989)
[10] J M Hill, A J Avagliano and M P Edwards, *IMA J. Appl. Math.* **48**, 283 (1992)
[11] K Singh, R K Gupta and S Kumar, *Appl. Math. Comput.* **217**, 7021 (2011)
[12] K Singh and R K Gupta, *Commun. Nonlin. Sci. Numer. Simul.* **16**, 4189 (2011)
[13] G Bluman, P Broadbridge, J R King and M J Ward, *J. Engg. Math.* **66**, 4189 (2010)
[14] M Tajiri, *J. Phys. Soc. Jpn.* **52**, 2277 (1983)
[15] H Zhao, *Chaos, Solitons and Fractals* **36**, 359 (2008)
[16] M Wadati, H Segur and M J Ablowitz, *J. Phys. Soc. Jpn.* **61**, 1187 (1992)
[17] Z Yan, *Z. Angew. Math. Phys.* **53**, 533 (2002)
[18] M L Wang, X Z Li and J L Zhang, *Phys. Lett.* **A372**, 417 (2008)
[19] E M E Zayed and K A Gepreel, *J. Math. Phys.* **50**, 013502 (2009)