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Exact complex integrals in two dimensions for shifted harmonic oscillators

JASVINDER SINGH VIRDI[1,*] and S C MISHRA[2]
[1]Department of Physics, Panjab University, Chandigarh 160 014, India
[2]Department of Physics, Kurukshetra University, Kurukshetra 136 119, India
*Corresponding author. E-mail: jpsvirdi@gmail.com

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Abstract. We use rationalization method to study two-dimensional complex dynamical systems (shifted harmonic oscillator in complex plane) on the extended complex phase space (ECPS). The role and scope of the derived invariants in the context of various physical problems are highlighted.

Keywords. Complex Hamiltonian; exact complex integrals; $\mathcal{PT}$-symmetry.

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The role of integrals of motion in physics can hardly be overestimated. They help one to analyse and classify the behaviour of various classical and quantum systems. In particular, knowledge of integrals of motion simplifies significantly the process of solving dynamical equations governing the system evolution. The usual definition of (Liouville) integrability of an N-dimensional Hamiltonian system requires the existence of N-integrals of motion (the Hamiltonian being one of them) in involution with operators that should commute [1–3]. Complex Hamiltonians have been in practice for a long time for studying a physical system such as the optical model of nucleus [4]. Also non-Hermitian $\mathcal{PT}$-symmetric complex Hamiltonians are used to study delocalization of transitions in condensed matter systems such as vortex flux line depinning in type-II superconductors [5], population biology [6], complex trajectories particularly in laser physics [7], resonance scattering in atomic, molecular and nuclear physics and also in some chemical reactions. While a complex Hamiltonian is no longer Hermitian, it is $\mathcal{PT}$-symmetric, and the system is found to exhibit real eigenvalues [8]. Hollowood through the Hamiltonian of a complex Toda lattice and Zinn-Justin showed that, though the Hamiltonian is non-Hermitian, the energy levels are real [9]. Complex Hamiltonian are found in the context of condensed matter physics. Consider the complex crystal lattice whose potential $V(x) = i \sin x$. While the Hamiltonian $H = p^2 + i \sin x$ is not Hermitian, it is $\mathcal{PT}$-symmetric and all of its energy bands are real. It is mentioned that real Hamiltonians

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are also found to admit complex integrals. For example, for simple harmonic oscillator system, there exists a complex integral, namely $u = \ln(p + im\omega x) - i\omega t$ [10].

In recent years, another class of complex Hamiltonians (called the $\mathcal{PT}$-symmetric Hamiltonians) have been discussed [8,11], in which, despite the lack of conventional hermiticity of H, the eigenvalue spectrum for certain domains of underlying parameter of the system turns out to be real. Then it is argued that the reality of the spectrum is a consequence of the combined action of parity and time reversal invariance of H. The parity operator $\mathcal{P}$ and the time reversal operator $\mathcal{T}$ are defined by their action on position and momentum operators (in quantum mechanics) as

$$\mathcal{P}\colon x \to -x; \quad p \to -p;$$

$$\mathcal{T}\colon x \to x; \quad p \to -p; \quad i \to -i.$$

Here, while operators x and p are real, the commutator $[x, p] = i$ is required to be integrals under both $\mathcal{P}$ and $\mathcal{T}$. It is interesting to note that this commutation relation still remains invariant even if x and p become complex, provided that the above transformations hold. In fact, in terms of real and imaginary parts of x and p, introduced as $x = \mathrm{Re}\, x + i\, \mathrm{Im}\, x$, $p = \mathrm{Re}\, p + i\, \mathrm{Im}\, p$, one should have

$$\mathcal{P}\colon \mathrm{Re}\, x \to -\mathrm{Re}\, x; \quad \mathrm{Im}\, x \to -\mathrm{Im}\, x : \quad \mathrm{Re}\, p \to -\mathrm{Re}\, p; \quad \mathrm{Im}\, p \to -\mathrm{Im}\, p;$$

$$\mathcal{T}\colon \mathrm{Re}\, x \to \mathrm{Re}\, x; \quad \mathrm{Im}\, x \to -\mathrm{Im}\, x : \quad \mathrm{Re}\, p \to -\mathrm{Re}\, p; \quad \mathrm{Im}\, p \to \mathrm{Im}\, p.$$

We define $x_1 \equiv \mathrm{Re}\, x$, $p_2 \equiv \mathrm{Im}\, p$, $p_1 \equiv \mathrm{Re}\, p$, $x_2 \equiv \mathrm{Im}\, p$, then the complex versions of x and p, written as

$$x = x_1 + ip_2; \qquad p = p_1 + ix_2,$$

which have been used by Xavier and Aguiar to develop an algorithm for the computation of the semi-classical coherent-state propagator [7]. Very recently, Kaushal and his co-workers have investigated the construction of complex integrals of one-dimensional complex Hamiltonian systems on the extended complex phase plane (ECPS), characterized by $x = x_1 + ip_2$ and $p = p_1 + ix_2$ [12,13]. In this approach both x and p are separately made complex by extending each of them to the corresponding complex planes, i.e. inserting an imaginary component in each. From physics point of view p_2 and x_2 can be regarded as fictitious/spurious components of momentum and coordinate respectively and their presence in the above transformation equations as such allow the introduction of some sort of coordinate–momentum coupling of the dynamical system. $\mathcal{PT}$-symmetric Hamiltonians appear to be special cases of such transformations.

In the present work we carry out the ECPS approach to obtain exact complex integrals of a two-dimensional classical dynamical system [14,15]. Rationalization method and Lie-algebraic method are explored for such constructions as these have been widely used in literature for the construction of exact real and complex integrals.

1. Rationalization method

Consider a two-dimensional real phase space (x, y, p_x, p_y, t), which may be transformed into a complex space $(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t)$, by defining position and momenta variables as

$$x = x_1 + ip_3; \qquad y = x_2 + ip_4;$$
$$p_x = p_1 + ix_3; \qquad p_y = p_2 + ix_4. \tag{1}$$

Of course, the presence of variables (x_3, x_4, p_3, p_4) in the above transformation eq. (1), can be regarded as some sort of coordinate–momentum interaction of the dynamical system.

The Hamiltonian $H(x, y, p_x, p_y, t)$ of a two-dimensional system in complex space can be expressed, using eq. (1), as

$$H = H_1(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t)$$
$$+ iH_2(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t). \tag{2}$$

Note that (x_1, p_1), (x_2, p_2), (x_3, p_3) and (x_4, p_4) constitute canonical pairs. With regard to the physical insight into the complexified version of the Hamiltonian, the following observation is worth mentioning. In fact, it is found [16] that if H_1 can be identified with the (real) Hamiltonian of a physical system, then in several cases (studied in ref. [16]) the analyticity property of H suggests that H_2 is the second integral of the motion of the system in the sense that $[H_1, H_2] = 0$, and H_1 and H_2 are linearly independent with respect to the canonical pairs (x_1, p_1), (x_2, p_2), (x_3, p_3) and (x_4, p_4).

Now consider a complex phase space function $I(x, y, p_x, p_y, t)$ as

$$I = I_1(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t)$$
$$+ iI_2(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t). \tag{3}$$

Thus, for function I to be the time-dependent (TD) dynamical integral of the system in complex phase space, this must conform the following invariance condition:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + [I, H] = 0, \tag{4}$$

where $[.,.]$ is the Poisson bracket. Now using $I = I_1 + iI_2$, $H = H_1 + iH_2$ in eq. (4), where I_1, I_2, H_1 and H_2 are real functions of real variables $(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t)$, eq. (4) will yield

$$\frac{\partial}{\partial t}(I_1 + iI_2) + \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_3}\right)(I_1 + iI_2)\left(\frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3}\right)(H_1 + iH_2)$$
$$- \left(\frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3}\right)(I_1 + iI_2)\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_3}\right)(H_1 + iH_2)$$
$$+ \left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4}\right)(I_1 + iI_2)\left(\frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}\right)(H_1 + iH_2)$$
$$- \left(\frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}\right)(I_1 + iI_2)\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4}\right)(H_1 + iH_2) = 0. \tag{5}$$

After equating real and imaginary parts separately to zero, one obtains the following pair of equations:

$$\begin{aligned}\frac{\partial I_1}{\partial t}&+\left(\frac{\partial I_1}{\partial x_1}+\frac{\partial I_2}{\partial p_3}\right)\left(\frac{\partial H_1}{\partial p_1}+\frac{\partial H_2}{\partial x_3}\right)-\left(\frac{\partial I_2}{\partial x_1}-\frac{\partial I_1}{\partial p_3}\right)\left(\frac{\partial H_2}{\partial p_1}-\frac{\partial H_1}{\partial x_3}\right)\\&-\left(\frac{\partial I_1}{\partial p_1}+\frac{\partial I_2}{\partial x_3}\right)\left(\frac{\partial H_1}{\partial x_1}+\frac{\partial H_2}{\partial p_3}\right)+\left(\frac{\partial I_2}{\partial p_1}-\frac{\partial I_1}{\partial x_3}\right)\left(\frac{\partial H_2}{\partial x_1}-\frac{\partial H_1}{\partial p_3}\right)\\&+\left(\frac{\partial I_1}{\partial x_2}+\frac{\partial I_2}{\partial p_4}\right)\left(\frac{\partial H_1}{\partial p_2}+\frac{\partial H_2}{\partial x_4}\right)-\left(\frac{\partial I_2}{\partial x_2}-\frac{\partial I_1}{\partial p_4}\right)\left(\frac{\partial H_2}{\partial p_2}-\frac{\partial H_1}{\partial x_4}\right)\\&-\left(\frac{\partial I_1}{\partial p_2}+\frac{\partial I_2}{\partial x_4}\right)\left(\frac{\partial H_1}{\partial x_2}+\frac{\partial H_2}{\partial p_4}\right)+\left(\frac{\partial I_2}{\partial p_2}-\frac{\partial I_1}{\partial x_4}\right)\left(\frac{\partial H_2}{\partial x_2}-\frac{\partial H_1}{\partial p_4}\right)=0\end{aligned} \tag{6}$$

and imaginary part is

$$\begin{aligned}\frac{\partial I_2}{\partial t}&+\left(\frac{\partial I_2}{\partial x_1}-\frac{\partial I_1}{\partial p_3}\right)\left(\frac{\partial H_1}{\partial p_1}+\frac{\partial H_2}{\partial x_3}\right)+\left(\frac{\partial I_1}{\partial x_1}+\frac{\partial I_2}{\partial p_3}\right)\left(\frac{\partial H_2}{\partial p_1}-\frac{\partial H_1}{\partial x_3}\right)\\&-\left(\frac{\partial I_2}{\partial p_1}-\frac{\partial I_1}{\partial x_3}\right)\left(\frac{\partial H_1}{\partial x_1}+\frac{\partial H_2}{\partial p_3}\right)-\left(\frac{\partial I_1}{\partial p_1}+\frac{\partial I_2}{\partial x_3}\right)\left(\frac{\partial H_2}{\partial x_1}-\frac{\partial H_1}{\partial p_3}\right)\\&+\left(\frac{\partial I_2}{\partial x_2}-\frac{\partial I_1}{\partial p_4}\right)\left(\frac{\partial H_1}{\partial p_2}+\frac{\partial H_2}{\partial x_4}\right)+\left(\frac{\partial I_1}{\partial x_2}+\frac{\partial I_2}{\partial p_4}\right)\left(\frac{\partial H_2}{\partial p_2}-\frac{\partial H_1}{\partial x_4}\right)\\&-\left(\frac{\partial I_2}{\partial p_2}-\frac{\partial I_1}{\partial x_4}\right)\left(\frac{\partial H_1}{\partial x_2}+\frac{\partial H_2}{\partial p_4}\right)-\left(\frac{\partial I_1}{\partial p_2}+\frac{\partial I_2}{\partial x_4}\right)\left(\frac{\partial H_2}{\partial x_2}-\frac{\partial H_1}{\partial p_4}\right)=0.\end{aligned} \tag{7}$$

If the system does not involve explicit dependence on time t, then the term $\partial I_1/\partial t$ and $\partial I_2/\partial t$ in eqs (6) and (7) can be set equal to zero. Further, if imaginary parts I_2 and H_2 in I and H, respectively are absent, then eqs (6) and (7) reduce to the following simpler form:

$$\begin{aligned}\frac{\partial I_1}{\partial t}&+\left(\frac{\partial I_1}{\partial x_1}\right)\left(\frac{\partial H_1}{\partial p_1}\right)-\left(\frac{\partial I_2}{\partial x_1}\right)\left(\frac{\partial H_2}{\partial p_1}\right)-\left(\frac{\partial I_1}{\partial p_1}\right)\left(\frac{\partial H_1}{\partial x_1}\right)\\&-\left(\frac{\partial I_2}{\partial p_1}\right)\left(\frac{\partial H_2}{\partial x_1}\right)+\left(\frac{\partial I_1}{\partial x_2}\right)\left(\frac{\partial H_1}{\partial p_2}\right)-\left(\frac{\partial I_1}{\partial p_4}\right)\left(\frac{\partial H_2}{\partial p_2}\right)\\&-\left(\frac{\partial I_1}{\partial p_2}\right)\left(\frac{\partial H_1}{\partial x_2}\right)-\left(\frac{\partial I_2}{\partial p_2}\right)\left(\frac{\partial H_2}{\partial x_2}\right)=0\end{aligned} \tag{8}$$

and

$$\begin{aligned}\frac{\partial I_2}{\partial t}&+\left(\frac{\partial I_2}{\partial x_1}\right)\left(\frac{\partial H_1}{\partial p_1}\right)+\left(\frac{\partial I_1}{\partial x_1}\right)\left(\frac{\partial H_2}{\partial p_1}\right)-\left(\frac{\partial I_2}{\partial p_1}\right)\left(\frac{\partial H_1}{\partial x_1}\right)\\&+\left(\frac{\partial I_1}{\partial p_1}\right)\left(\frac{\partial H_2}{\partial x_1}\right)+\left(\frac{\partial I_2}{\partial x_2}\right)\left(\frac{\partial H_1}{\partial p_2}\right)+\left(\frac{\partial I_2}{\partial x_2}\right)\left(\frac{\partial H_2}{\partial p_2}\right)\\&-\left(\frac{\partial I_2}{\partial p_2}\right)\left(\frac{\partial H_1}{\partial x_2}\right)+\left(\frac{\partial I_1}{\partial p_2}\right)\left(\frac{\partial H_2}{\partial x_2}\right)=0.\end{aligned} \tag{9}$$

The prescription for the construction of complex integral I is the same as that used earlier for the real Hamiltonian system. In brief, it can be listed as follows. For a given $H(x, p, t)$, make an ansatz for I preferably in power of momentum p, using $\mathcal{PT}$-symmetry. Both H and I reduce to the form $I = I_1 + iI_2$, $H = H_1 + iH_2$, and then substitute the resultant I_1, I_2, H_1 and H_2 in eqs (8) and (9) with respect to power of p_1 and x_2 and their combination will yield coupled partial differential equations for the arbitrary complex coefficient functions appearing in the ansatz for I. The substitution of the solutions of these partial equations (if the solutions exist and are unique) in the ansatz for I then yields the final form of integral I.

1.1 *Illustrative example*

With a view of constructing complex integral for some cases here, in this section we use methods discussed in the previous section. To start with, we first consider the case of shifted harmonic oscillator systems within the framework of rationalization method. Note that for shifted harmonic oscillator in complex plane

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}w^2\left\{\left(x + \frac{1}{2}i\gamma\right)^2 + \left(y + \frac{1}{2}i\gamma\right)^2\right\}, \tag{10}$$

one possible complex TD harmonic oscillator system, (there exist a complex integral, namely $u = \ln(p + im\omega x) - i\omega t$ [6]) has already been known in literature, where ω and γ are real constants. This form of H, after appropriate scaling of x and p (with $\omega = 1$) can be expressed as

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + i\gamma(x + y).$$

Here we demonstrate that the complex version of (10), namely the $\mathcal{PT}$-symmetric one obtained by using (1) in (10), as $H = H_1 + iH_2$ with

$$H_1 = \frac{1}{2}\left(p_1^2 + p_2^2 + x_1^2 + x_2^2 - p_3^2 - p_4^2 - x_3^2 - x_4^2\right) - \gamma(x_1p_4 + x_2p_3),$$

$$H_2 = p_1x_3 + p_2x_4 + x_1p_3 + x_2p_4 + \gamma(x_1x_2 - p_3p_4) \tag{11}$$

admits complex integral I in the form

$$I = a_{01}(x, y) + a_{02}(x, y)\left(p_x^2 + p_y^2\right) + a_{11}(x, y)p_xp_y \tag{12}$$

and write its complex version in the form $I = I_1 + iI_2$ where

$$\begin{aligned} I_1 = {} & a_{01xr} + a_{01yr} + (a_{02xr} + a_{02yr})\left(p_1^2 - x_3^2\right) - (a_{02xi} + a_{02yi})2p_1x_3 \\ & + (a_{02xr} + a_{02yr})\left(p_2^2 - x_4^2\right) - (a_{02xi} + a_{02yi})2p_2x_4 \\ & + (a_{11xr} + a_{11yr})(p_1p_2 - x_1x_2) - (a_{11xi} + a_{11yi})(p_1x_4 - p_2x_3) \end{aligned} \tag{13}$$

and

$$\begin{aligned} I_2 = {} & a_{01xi} + a_{01yi} + (a_{02xi} + a_{02yi})\left(p_1^2 - x_3^2\right) + (a_{02xr} + a_{02yr})2p_1x_3 \\ & + (a_{02yi} + a_{02xi})\left(p_2^2 - x_4^2\right) + (a_{02xr} + a_{02yr})2p_2x_4 \\ & + (a_{11xr} + a_{11yr})(p_1x_4 - p_2x_3) + (a_{11xi} + a_{11yi})(p_1p_2 - x_4x_3) \end{aligned} \tag{14}$$

and the complex coefficient functions $a_{01}(x, y), a_{02}(x, y), a_{11}(x, y)$ are the real functions of their real arguments (x_1, p_3, x_2, p_4). Substitution of (11), (13), (14) in (6) yields the expression

$$\Bigg[\Bigg[\left(\frac{\partial a_{01xr}}{\partial x_1}+\frac{\partial a_{01xi}}{\partial p_3}\right)+\left(\frac{\partial a_{02xr}}{\partial x_1}+\frac{\partial a_{02xi}}{\partial p_3}\right)(p_1^2-x_3^2)$$

$$-(2p_1x_3)\left(\frac{\partial a_{02xi}}{\partial x_1}-\frac{\partial a_{02xr}}{\partial p_3}\right)+\left(\frac{\partial a_{02xr}}{\partial x_1}+\frac{\partial a_{02xi}}{\partial p_3}\right)(p_2^2-x_4^2)$$

$$-(2p_2x_4)\left(\frac{\partial a_{02xi}}{\partial x_1}-\frac{\partial a_{02xr}}{\partial p_3}\right)-(p_1x_4-p_2x_3)\left(\frac{\partial a_{11xi}}{\partial x_1}-\frac{\partial a_{11xr}}{\partial p_3}\right)$$

$$+(p_1p_2-x_4x_3)\left(\frac{\partial a_{11xr}}{\partial x_1}+\frac{\partial a_{11xi}}{\partial p_3}\right)\Bigg]2p_1$$

$$-\Bigg[\left(\frac{\partial a_{01xi}}{\partial x_1}-\frac{\partial a_{01xr}}{\partial p_3}\right)+\left(\frac{\partial a_{02xi}}{\partial x_1}+\frac{\partial a_{02xr}}{\partial p_3}\right)(p_1^2-x_3^2)$$

$$+(2p_1x_3)\left(\frac{\partial a_{02xr}}{\partial x_1}+\frac{\partial a_{02xi}}{\partial p_3}\right)+\left(\frac{\partial a_{02xi}}{\partial x_1}-\frac{\partial a_{02xr}}{\partial p_3}\right)(p_2^2-x_4^2)$$

$$+(2p_2x_4)\left(\frac{\partial a_{02xr}}{\partial x_1}-\frac{\partial a_{02xi}}{\partial p_3}\right)+(p_1x_4+p_2x_3)\left(\frac{\partial a_{11xr}}{\partial x_1}-\frac{\partial a_{11xi}}{\partial p_3}\right)$$

$$+(p_1p_2-x_4x_3)\left(\frac{\partial a_{11xi}}{\partial x_1}+\frac{\partial a_{11xr}}{\partial p_3}\right)\Bigg](2x_3)\Bigg]$$

$$-\big[4p_1(a_{02xr}+a_{02yr})-4x_3(a_{02xi}+a_{02yi})+2p_2(a_{11xr}+a_{11yr})$$

$$-2x_4(a_{11xi}+a_{11yi})2(x_1-\gamma p_4)-4p_1(a_{02xi}+a_{02yi})+4x_3(a_{02xr}+a_{02yr})$$

$$+2p_2(a_{11xi}+a_{11yi})+2x_4(a_{11xr}+a_{11yr})2(p_3+\gamma x_2)\big]$$

$$+\Bigg[\Bigg[\left(\frac{\partial a_{01yr}}{\partial x_2}+\frac{\partial a_{01yi}}{\partial p_4}\right)+\left(\frac{\partial a_{02yr}}{\partial x_2}+\frac{\partial a_{02yi}}{\partial p_4}\right)(p_1^2-x_3^2)$$

$$-(2p_1x_3)\left(\frac{\partial a_{02yi}}{\partial x_2}-\frac{\partial a_{02yr}}{\partial p_4}\right)+\left(\frac{\partial a_{02yr}}{\partial x_2}+\frac{\partial a_{02yi}}{\partial p_4}\right)(p_2^2-x_4^2)$$

$$-(2p_2x_4)\left(\frac{\partial a_{02yi}}{\partial x_2}-\frac{\partial a_{02yr}}{\partial p_4}\right)-(p_1x_4-p_2x_3)\left(\frac{\partial a_{11yi}}{\partial x_2}-\frac{\partial a_{11yr}}{\partial p_4}\right)$$

$$+(p_1p_2-x_4x_3)\left(\frac{\partial a_{11yr}}{\partial x_2}+\frac{\partial a_{11yi}}{\partial p_4}\right)\Bigg]2p_2$$

$$
\begin{aligned}
&- \Bigg[\Bigg(\left(\frac{\partial a_{01yi}}{\partial x_2} - \frac{\partial a_{01yr}}{\partial p_4}\right) + \left(\frac{\partial a_{02yi}}{\partial x_2} + \frac{\partial a_{02yr}}{\partial p_4}\right)(p_1^2 - x_3^2) \\
&\quad + (2p_1x_3)\left(\frac{\partial a_{02yr}}{\partial x_2} + \frac{\partial a_{02yi}}{\partial p_4}\right) + \left(\frac{\partial a_{02yi}}{\partial x_2} - \frac{\partial a_{02yr}}{\partial p_4}\right)(p_2^2 - x_4^2) \\
&\quad + (2p_2x_4)\left(\frac{\partial a_{02yr}}{\partial x_2} - \frac{\partial a_{02yi}}{\partial p_4}\right) + (p_1x_4 + p_2x_3)\left(\frac{\partial a_{11yr}}{\partial x_2} - \frac{\partial a_{11yi}}{\partial p_4}\right) \\
&\quad + (p_1p_2 - x_4x_3)\left(\frac{\partial a_{11yi}}{\partial x_2} + \frac{\partial a_{11yr}}{\partial p_4}\right)\Bigg] 2x_4\Bigg] \\
&- \big[4p_2(a_{02xr} + a_{02yr}) - 4x_4(a_{02xi} + a_{02yi}) + 2p_1(a_{11xr} + a_{11yr}) \\
&\quad - 2x_3(a_{11xi} + a_{11yi})2(x_2 - \gamma p_3) - 4p_2(a_{02xi} + a_{02yi}) \\
&\quad + 4x_4(a_{02xr} + a_{02yr}) + 2p_1(a_{11xi} + a_{11yi}) \\
&\quad + 2x_3(a_{11xr} + a_{11yr})(p_4 + \gamma x_1)\big] = 0 \qquad (15)
\end{aligned}
$$

which can be rationalized with respect to the power of p_1, x_3, p_2, x_4 and their combinations to give the following set of twelve coupled partial differential equations:

$$
\begin{aligned}
&\left(\frac{\partial a_{01xr}}{\partial x_1} + \frac{\partial a_{01xi}}{\partial p_3}\right) - 4(x_1 - \gamma p_4)(a_{02xr} + a_{02yr}) \\
&\qquad + 4(p_3 + \gamma x_2)(a_{02xi} + a_{02yi}) = 0, \\
&\left(\frac{\partial a_{02xr}}{\partial x_1} + \frac{\partial a_{02xi}}{\partial p_3}\right) = 0, \qquad \left(\frac{\partial a_{11xr}}{\partial p_3} + \frac{\partial a_{11xi}}{\partial x_1}\right) = 0. \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{\partial a_{01xi}}{\partial x_1} + \frac{\partial a_{01xr}}{\partial p_3}\right) + 4(x_1 - \gamma p_4)(a_{02xi} + a_{02yi}) \\
&\qquad + 4(p_3 + \gamma x_2)(a_{02xr} + a_{02yr}) = 0, \\
&-\left(\frac{\partial a_{02xi}}{\partial x_1} + \frac{\partial a_{02xr}}{\partial p_3}\right) = 0, \qquad \left(\frac{\partial a_{11xr}}{\partial x_1} - \frac{\partial a_{11xi}}{\partial p_3}\right) = 0. \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{\partial a_{01yr}}{\partial x_2} + \frac{\partial a_{01yi}}{\partial p_4}\right) - 4(x_2 - \gamma p_3)(a_{02xr} + a_{02yr}) \\
&\qquad + 4(p_4 + \gamma x_1)(a_{02xi} + a_{02yi}) = 0, \\
&\left(\frac{\partial a_{02yr}}{\partial x_2} + \frac{\partial a_{02yi}}{\partial p_4}\right) = 0, \qquad \left(\frac{\partial a_{11yr}}{\partial x_2} - \frac{\partial a_{11yi}}{\partial p_4}\right) = 0. \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{\partial a_{01yi}}{\partial x_2} - \frac{\partial a_{01yr}}{\partial p_4}\right) + 4(x_2 - \gamma p_3)(a_{02xi} + a_{02yi}) \\
&\qquad + 4(p_4 + \gamma x_1)(a_{02xr} + a_{02yr}) = 0, \\
&-\left(\frac{\partial a_{02yi}}{\partial x_2} - \frac{\partial a_{02yr}}{\partial p_4}\right) = 0, \qquad \left(\frac{\partial a_{11yr}}{\partial x_2} + \frac{\partial a_{11yi}}{\partial p_4}\right) = 0. \qquad (19)
\end{aligned}
$$

So for constructing complex integrals in two dimensions one has to find out solutions for the following unknown parameters a_{01xr}, a_{01yr}, a_{01xi}, a_{01yi}, a_{02xr}, a_{02yr}, a_{02xi}, a_{02yi}, a_{11xr}, a_{11yr}, a_{11xi}, a_{11yi} which are functions of (x_1, p_3, x_2, p_4).

(A) Solutions for a_{11xr}, a_{11xi}: eqs (16) and (17) can be reduced to similar second-order forms for the functions a_{11xr}, a_{11xi}, respectively as

$$\frac{\partial^2 a_{11xr}}{\partial x_1^2} + \frac{\partial^2 a_{11xr}}{\partial p_3^2} = 0, \qquad \frac{\partial^2 a_{11xi}}{\partial x_1^2} + \frac{\partial^2 a_{11xi}}{\partial p_3^2} = 0. \tag{20}$$

Assuming separability of a_{11xr} and a_{11xi} under addition as $a_{11xr} = X_{11xr}(x_1) + P_{11xr}(p_3)$, $a_{11xi} = X_{11xi}(x_1) + P_{11xi}(p_3)$ it is not difficult to obtain the solution of (20) in the form

$$a_{11xr} = \frac{\nu}{2}(x_1^2 - p_3^2) + \nu_1 x_1 + \nu_2 p_3 + \delta_1, \quad a_{11xi} = \frac{\rho}{2}(x_1^2 - p_3^2) + \rho_1 x_1 + \rho_2 p_3 + \delta_2, \tag{21}$$

where $\nu, \rho, \nu_1, \nu_2, \rho_1, \rho_2, \delta_1, \delta_2$ are arbitrary constants of integration to be determined later.

(B) Solutions for a_{02xr} and a_{02xi}: eqs (16) and (17) can be reduced to similar second-order forms for the functions a_{02xr} and a_{02xi}, respectively as

$$\frac{\partial^2 a_{02xr}}{\partial x_1^2} + \frac{\partial^2 a_{02xr}}{\partial p_3^2} = 0, \qquad \frac{\partial^2 a_{02xi}}{\partial x_1^2} + \frac{\partial^2 a_{02xi}}{\partial p_3^2} = 0. \tag{22}$$

It is not difficult to obtain the solution of (22) in the form

$$a_{02xr} = \frac{\alpha}{2}(x_1^2 - p_3^2) + \alpha_1 x_1 + \alpha_2 p_3 + \delta_3, \quad a_{02xi} = \frac{\beta}{2}(x_1^2 - p_3^2) + \beta_1 x_1 + \beta_2 p_3 + \delta_4, \tag{23}$$

where $\alpha, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \delta_3, \delta_4$ are arbitrary constants of integration to be determined later.

(C) Solutions for a_{01xr}, a_{01xi}: On differentiating (16) with respect to x_1 and again (17) with respect to p_3

$$\begin{aligned}\frac{\partial^2 a_{01xr}}{\partial x_1^2} + \frac{\partial^2 a_{01xr}}{\partial p_3^2} &= 4(x_1 - \gamma p_4)\left(-2\frac{\partial a_{02xi}}{\partial p_3}\right) - \left(2\frac{\partial a_{02xi}}{\partial x_1}\right)4(p_4 + \gamma x_1), \\ &= -8[\beta_2(x_1 - \gamma p_3) + \beta_1(p_3 + \gamma x_1)],\end{aligned} \tag{24}$$

where we have used eq. (16) and then expression (23) (using constraints) to simplify the right-hand side. For the solution of (24) we again assume a separable form $a_{01xr}(x_1, p_3) = X_{01xr}(x_1) + P_{01xr}(p_3)$, $a_{01xi} = X_{01xi}(x_1) + P_{11xi}(p_3)$ and this leads to a pair of ordinary partial differential equations whose solution immediately will yield

$$a_{01xr}(x_1, p_3) = -\frac{4}{3}\left(\beta_2 x_1^3 + \beta_1 p_3^3\right) - \beta_1\gamma\left(p_3^2 + x_1^2\right) + c_1 x_1 + d_2 p_3 + c_3, \tag{25}$$

where c_1, c_2 and c_3 are the arbitrary constants of integration. For determining a_{01xi} one follows the same procedure as followed for a_{01xr} and obtains the coefficient function $a_{01xi}(x_1, p_3)$ in the form

$$a_{01xi}(x_1, p_3) = \frac{4}{3}\left(\beta_1 x_1^3 - \beta_2 p_3^3\right) - \beta_2\gamma\left(x_1^2 + p_3^2\right) + d_1 x_1 - c_1 p_3 + d_3, \tag{26}$$

where d_1, d_2 and d_3 are the arbitrary constants of integration. With the inclusion of constraints with $\rho = 0$, $\nu = 0$ obtained by constraint relations for the arbitrary constants the expressions a_{11xr} and a_{11xi} now take the form

$$a_{11xr} = \beta_1 x_1 + \beta_2 p_3 + \delta_1, \quad a_{11xi} = \beta_1 x_1 + \beta_2 p_3 + \delta_2. \tag{27}$$

(D) Solutions for a_{11yr}, a_{11yi}: eqs (18) and (19) can be reduced to similar second-order forms for the functions as

$$\frac{\partial^2 a_{11yr}}{\partial x_2^2} + \frac{\partial^2 a_{11yr}}{\partial p_4^2} = 0, \quad \frac{\partial^2 a_{11yi}}{\partial x_2^2} + \frac{\partial^2 a_{11yi}}{\partial p_4^2} = 0. \tag{28}$$

Solution of (28) is in the form

$$\begin{aligned} a_{11yr} &= \frac{\nu}{2}\left(x_2^2 - p_4^2\right) + \nu_1 x_2 + \nu_2 p_4 + \delta_5, \\ a_{11yi} &= \frac{\rho}{2}\left(x_2^2 - p_4^2\right) + \rho_1 x_2 + \rho_2 p_4 + \delta_6, \end{aligned} \tag{29}$$

where $\nu, \rho, \nu_1, \nu_2, \rho_1, \rho_2, \delta_3, \delta_4$ are arbitrary constants of integration to be determined later.

(E) Solutions for a_{02yr}, a_{02yi}: eqs (18) and (19) can be reduced to

$$\frac{\partial^2 a_{02yr}}{\partial x_2^2} + \frac{\partial^2 a_{02yr}}{\partial p_4^2} = 0, \quad \frac{\partial^2 a_{02yi}}{\partial x_2^2} + \frac{\partial^2 a_{02yi}}{\partial p_4^2} = 0. \tag{30}$$

Solution of (30) is in the form

$$\begin{aligned} a_{02yr} &= \frac{\alpha}{2}\left(x_1^2 - p_3^2\right) + \alpha_1 x_1 + \alpha_2 p_3 + \delta_7, \\ a_{02yi} &= \frac{\beta}{2}\left(x_1^2 - p_3^2\right) + \beta_1 x_1 + \beta_2 p_3 + \delta_8, \end{aligned} \tag{31}$$

where $\alpha, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \delta_5, \delta_6$ are arbitrary constants of integration to be determined later.

(F) Similarly to solve a_{01yr}, a_{01yi}. On differentiating (18) with respect to x_2 and (19) with respect to p_4

$$\begin{aligned} \frac{\partial^2 a_{01yr}}{\partial x_2^2} + \frac{\partial^2 a_{01yr}}{\partial p_4^2} &= 4(p_3 + \gamma)\left(-2\frac{\partial a_{02yi}}{\partial p_4}\right) - \left(2\frac{\partial a_{02yi}}{\partial x_2}\right)4(p_4 + \gamma) \\ &= -8[(p_3 + \gamma)\beta_2 + \beta_1(p_4 + \gamma)], \end{aligned} \tag{32}$$

where we have used eqs (18) and (19) and then expression (31) (using constraints) to simplify the right-hand side. For solution of (32) we again assume a separable form $a_{01yr}(x_2, p_4) = X_{01yr}(x_2) + P_{01yr}(p_4)$, $a_{01yi} = X_{01yi}(x_2) + P_{11yi}(p_4)$. This leads to a pair of ordinary partial differential equations whose solution immediately will yield

$$a_{01yr}(x_2, p_4) = -\frac{4}{3}\left(\beta_2 x_2^3 + \beta_1 p_4^3\right) - \beta_1\gamma\left(x_2^2 + p_4^2\right) + c_1 x_2 + d_2 p_4 + c_3, \tag{33}$$

where c_1, c_2 and c_3 are the arbitrary constants of integration. For determining a_{01yi} one follows the same procedure as followed for a_{01xr} and obtains the coefficient function $a_{01yr}(x_2, p_4)$ in the form

$$a_{01yi}(x_2, p_4) = \frac{4}{3}w^2\left(\beta_1 x_2^3 - \beta_2 p_4^3\right) - \beta_2\gamma\left(x_2^2 + p_4^2\right) + d_1 x_2 - c_1 p_4 + d_3, \quad (34)$$

where d_1, d_2 and d_3 are arbitrary constants of integration. With the inclusion of constraint relations $\rho = 0$, $\nu = 0$ for the arbitrary constants, the expressions a_{11yr} and a_{11yi} now take the form

$$a_{11yr} = \rho_1 x_2 + \rho_2 p_4 + \delta_5, \quad a_{11yi} = \rho_1 x_2 + \rho_2 p_4 + \delta_6. \quad (35)$$

Note that the forms (20)–(35) of $a_{01xr}, a_{01yr}, a_{01xi}, a_{01yi}, a_{02xr}, a_{02yr}, a_{02xi}, a_{02yi}, a_{11xr}, a_{11yr}, a_{11xi}$ and a_{11yi} are determined only from (6). With these expressions for the coefficient function when (7) is rationalized, one obtained several constrained relations among the arbitrary constants appearing in eqs (20)–(35), thereby reducing the number of arbitrary constants in the final solutions, and the constrained relations so obtained are $c_1 = -d_2$, $d_1 = c_2$, $\rho_1 = -\nu_2$, $\rho_2 = \nu_1$, $\beta_2 = -\alpha_1$, $\beta_1 = \alpha_2$ and all δ's $= 0$ which gives rise to forms of coefficient functions as

$$\begin{aligned}
a_{11xr} &= -\rho_1 x_1 + \rho_2 p_3, \quad a_{11xi} = \rho_1 x_1 + \rho_2 p_3,\\
a_{02xr} &= -\beta_2 x_1 + \beta_1 p_3, \quad a_{02xi} = \beta_1 x_1 + \beta_2 p_3,\\
a_{01xr} &= -\frac{4}{3}\left(\beta_2 x_1^3 + \beta_1 p_3^3\right) - \beta_1\gamma\left(p_3^2 + x_1^2\right) + c_1 x_1 + d_2 p_3,\\
a_{01xi} &= \frac{4}{3}\left(\beta_1 x_1^3 - \beta_2 p_3^3\right) - \beta_2\gamma\left(x_1^2 + p_3^2\right) + d_1 x_1 - c_1 p_3,\\
a_{11yr} &= -\rho_2 x_2 + \rho_1 p_4, \quad a_{11yi} = \rho_2 x_2 + \rho_1 p_4,\\
a_{02yr} &= -\beta_2 x_2 + \beta_1 p_4, \quad a_{02yi} = \beta_1 x_2 + \beta_2 p_4,\\
a_{01yr} &= -\frac{4}{3}\left(\beta_2 x_2^3 + \beta_1 p_4^3\right) - \beta_1\gamma\left(x_2^2 + p_4^2\right) + c_1 x_2 + d_2 p_4,\\
a_{01yi} &= \frac{4}{3}w^2\left(\beta_1 x_2^3 - \beta_2 p_4^3\right) - \beta_2\gamma\left(x_2^2 + p_4^2\right) + d_1 x_2 - c_1 p_4. \quad (36)
\end{aligned}$$

From the above set of equations, we have

$$\begin{aligned}
a_{11r} &= a_{11xr} + a_{11yr} = -\rho_2(x_1 + x_2) + \rho_1(p_3 + p_4),\\
a_{11i} &= a_{11xi} + a_{11yi} = \rho_1(x_1 + x_2) + \rho_2(p_3 + p_4),\\
a_{02r} &= a_{02xr} + a_{02yr} = -\beta_2(x_1 + x_2) + \beta_1(p_3 + p_4),\\
a_{02i} &= a_{02xi} + a_{02yi} = \beta_1(x_1 + x_2) + \beta_2(p_3 + p_4),\\
a_{01r} &= a_{01xr} + a_{01yr} = -\frac{4}{3}\left[\beta_2\left(x_1^3 + x_2^3\right) + \beta_1\left(p_3^3 + p_4^3\right)\right]\\
&\quad - \beta_1\gamma\left(x_1^2 + x_2^2 + p_3^2 + p_4^2\right)\\
&\quad + c_1(x_1 + x_2) + d_2(p_3 + p_4)
\end{aligned}$$

$$a_{01i} = a_{01xi} + a_{01yi} = \frac{4}{3}\left[\beta_1\left(x_1^3 + x_2^3\right) - \beta_2\left(p_3^3 + p_4^3\right)\right] - \beta_2\gamma\left(x_1^2 + x_2^2 + p_3^2 + p_4^2\right) + d_1(x_1 + x_2) - c_1(p_3 + p_4). \quad (37)$$

Construction of integrals. Using the results (37) one can obtain the complex integral I from (13) and (14), in which the real and imaginary parts I_1 and I_2 are given by

$$\begin{aligned} I_1 = & -\frac{4}{3}\left[\beta_2\left(x_1^3 + x_2^3\right) + \beta_1\left(p_3^3 + p_4^3\right)\right] - \beta_1\gamma\left(x_1^2 + x_2^2 + p_3^2 + p_4^2\right) \\ & + c_1(x_1 + x_2) + d_2(p_3 + p_4) - 2(p_1x_3 + p_4x_4) \\ & \times\left\{\beta_1(x_1 + x_2) + \beta_2(p_3 + p_4)\right\} + \left(p_1^2 + p_2^2 - x_3^2 - x_4^2\right) \\ & \times\left\{-\beta_2(x_1 + x_2) + \beta_1(p_3 + p_4)\right\} + (p_1p_2 - x_3x_4) \\ & \times\left\{-\rho_2(x_1 + x_2) + \rho_1(p_3 + p_4)\right\} - (p_1x_4 + p_2x_3) \\ & \times\left\{\rho_1(x_1 + x_2) + \rho_2(p_3 + p_4)\right\} \end{aligned} \quad (38)$$

and

$$\begin{aligned} I_2 = & \frac{4}{3}\left[\beta_1\left(x_1^3 + x_2^3\right) - \beta_2\left(p_3^3 + p_4^3\right)\right] - \beta_2\gamma\left(x_1^2 + x_2^2 + p_3^2 + p_4^2\right) \\ & + d_1(x_1 + x_2) - c_1(p_3 + p_4) + 2(p_1x_3 + p_4x_4) \\ & \times\left\{-\beta_2(x_1 + x_2) + \beta_1(p_3 + p_4)\right\} + \left(p_1^2 + p_2^2 - x_3^2 - x_4^2\right) \\ & \times\left\{\beta_1(x_1 + x_2) + \beta_2(p_3 + p_4)\right\} + (p_1p_2 - x_3x_4)\left\{\rho_1(x_1 + x_2)\right. \\ & \left. + \rho_2(p_3 + p_4)\right\} + (p_1x_4 + p_2x_3)\left\{-\rho_2(x_1 + x_2) + \rho_1(p_3 + p_4)\right\}. \end{aligned} \quad (39)$$

Finally, the complex integrals $I = I_1 + iI_2$ can be written as

$$\begin{aligned} I = & \frac{b}{3}[x^*(3x^2 + x^*) + y^*(3y^2 + y^*)] + ib\gamma(xx^* + yy^*) + \sigma(x^* + y^*) \\ & + b\left[(x^* + y^*)\right](p_x^2 + p_y^2) + e(x^* + y^*)p_xp_y, \end{aligned}$$

where $x^* = x_1 - ip_3$, $y^* = x_2 - ip_4$, $p_x^* = p_1 - ix_3$, $p_y^* = p_2 - ix_4$, and $e = -\rho_2 + i\rho_1$; $\sigma_1 = c_1 + id_2$ and $b = -\beta_2 + i\beta_1$ are the arbitrary constants, which conforms the integrability condition.

2. Lie-algebraic method

Consider a two-dimensional real phase space (x, y, p_x, p_y, t), which may be transformed into a complex space $(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4, t)$, by defining position and momenta variables as

$$\begin{aligned} & x = x_1 + ip_3; \quad y = x_2 + ip_4; \\ & p_x = p_1 + ix_3; \quad p_y = p_2 + ix_4. \end{aligned} \quad (40)$$

The Hamiltonian $H(x, y, p_x, p_y, t)$ of a two-dimensional system in complex space can be expressed as $H = H_1 + iH_2$. From eq. (40) one can easily obtain

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_3}; \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4};$$
$$\frac{\partial}{\partial p_x} = \frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3}; \qquad \frac{\partial}{\partial p_y} = \frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}. \tag{41}$$

The Hamiltonian equations of motion for complex H using eq. (41), can be written as

$$\begin{aligned}
\dot{x}_1 &= \frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_3}; \quad \dot{p}_3 = \frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_3};\\
\dot{x}_2 &= \frac{\partial H_1}{\partial p_2} + \frac{\partial H_2}{\partial x_4}; \quad \dot{p}_4 = \frac{\partial H_2}{\partial p_2} - \frac{\partial H_1}{\partial x_4};\\
\dot{p}_1 &= -\left(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_3}\right); \quad \dot{x}_3 = -\left(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_3}\right);\\
\dot{p}_2 &= -\left(\frac{\partial H_1}{\partial x_2} + \frac{\partial H_2}{\partial p_4}\right); \quad \dot{x}_4 = -\left(\frac{\partial H_2}{\partial x_2} - \frac{\partial H_1}{\partial p_4}\right).
\end{aligned} \tag{42}$$

If H is an analytic function of complex variables, then H_1 and H_2 satisfy the Cauchy–Riemann conditions and after invoking such analyticity conditions, eq. (42) reduces to

$$\begin{aligned}
\dot{x}_1 &= 2\frac{\partial H_1}{\partial p_1}; \quad \dot{p}_1 = -2\frac{\partial H_1}{\partial x_1}; \quad \dot{x}_2 = 2\frac{\partial H_1}{\partial p_2}; \quad \dot{p}_2 = -2\frac{\partial H_1}{\partial x_2};\\
\dot{x}_3 &= 2\frac{\partial H_1}{\partial p_3}; \quad \dot{p}_3 = -2\frac{\partial H_1}{\partial x_3}; \quad \dot{x}_4 = 2\frac{\partial H_1}{\partial p_4}; \quad \dot{p}_4 = -2\frac{\partial H_1}{\partial x_4}.
\end{aligned} \tag{43}$$

Note that (x_1, p_1), (x_2, p_2), (x_3, p_3) and (x_4, p_4) constitute canonical pairs. Now consider a complex phase space function $I(x, y, p_x, p_y, t)$ as $I = I_1 + iI_2$. Thus, for the function I to be the TD (time-dependent) dynamical integral of the system in complex phase space, this must conform the following invariance condition:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + [I, H] = 0, \tag{44}$$

where $[\cdot, \cdot]$ is the Poisson bracket, which in view of the definition, eq. (40), turns out to be

$$\begin{aligned}
[A, B]_{(x,p)} &= [A, B]_{(x_1,p_1)} - i[A, B]_{(x_1,x_3)} - i[A, B]_{(p_3,p_1)} - [A, B]_{(p_3,x_3)}\\
&\quad + [A, B]_{(x_2,p_2)} - i[A, B]_{(x_2,x_4)} - i[A, B]_{(p_4,p_2)} - [A, B]_{(p_4,x_4)}.
\end{aligned} \tag{45}$$

In order to demonstrate the underlying elegance of the Lie-algebraic approach, we briefly describe the method to construct complex integrals of classical dynamical systems. In the Lie-algebraic approach, one can express the complex Hamiltonian $H(x, y, p_x, p_y, t)$ of the system as

$$H = \sum_n h_n(t)\Gamma_n(x, y, p_x, p_y), \tag{46}$$

where the set of functions $\{\Gamma_1, ..., \Gamma_n\}$ are not explicitly TD and $h_n(t)$'s are complex coefficient functions of time. The Γ_ns in eq. (46) generate a closed dynamical algebra, which implies

$$[\Gamma_n, \Gamma_m] = \sum_l C^l_{nm}\Gamma_l, \tag{47}$$

where C^l_{nm} are the complex structure constants of the algebra. If the Γ_n,s in eq. (46) are not sufficient to close the algebra, then the set of Γ_n's must be extended by adding new Γ_ls, such that $\Gamma_l = [\Gamma_n, \Gamma_m]$, until the closure is obtained. The additional $h_l(t)$'s are taken to be zero.

Since the complex dynamical integral I is also a part of Lie algebra, then one can write this as

$$I(t) = \sum_k \lambda_k(t)\Gamma_k(x, y, p_x, p_y), \tag{48}$$

where $\lambda_k(t)$'s are TD complex coefficients. Thus, by using eqs (46) and (48) for H and I respectively in eq. (44), we get a system of linear, first-order differential equations, namely

$$\dot{\lambda}_r + \sum_n \left[\sum_m C^r_{nm}h_m(t)\right]\lambda_n = 0, \tag{49}$$

in λ_n's. Thus, the solutions of these differential equations in turn provide classical complex integrals of a given system. In the next subsection we shall use the prescription given above to obtain complex integral of a classical complex Hamiltonian system.

2.1 *Example*

Consider a shifted harmonic oscillator systems in two dimensions, whose Hamiltonian is given by

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\kappa_1^2(t)(x^2 + y^2) + \kappa_2(t)xy. \tag{50}$$

Using eq. (40), the above Hamiltonian can be expressed as

$$\begin{aligned} H &= \frac{1}{2}p_1^2 - \frac{1}{2}x_3^2 + ip_1x_3 + \frac{1}{2}p_2^2 - \frac{1}{2}x_4^2 + ip_2x_4 \\ &\quad +\kappa_1^2(t)\left(\frac{1}{2}x_1^2 - \frac{1}{2}p_3^2 + ip_3x_1 + \frac{1}{2}x_2^2 - \frac{1}{2}p_4^2 + ix_2p_4\right) \\ &\quad +\kappa_2(t)(x_1x_2 + ip_3x_2 + ip_4x_1 - p_3p_4) \\ &= \sum_{m=1}^{16} h_m(t)\Gamma_m(x_1, p_3, x_2, p_4, p_1, x_3, p_2, x_4), \end{aligned} \tag{51}$$

and the various Γ's and $h(t)$'s for the above complex H are given as

$$\Gamma_1 = \frac{1}{2}p_1^2; \qquad \Gamma_2 = \frac{1}{2}x_3^2; \qquad \Gamma_3 = \frac{1}{2}p_2^2; \qquad \Gamma_4 = \frac{1}{2}x_4^2;$$

$$\Gamma_5 = p_1x_3; \qquad \Gamma_6 = p_2x_4; \qquad \Gamma_7 = p_3x_1; \qquad \Gamma_8 = p_4x_2;$$

$$\Gamma_9 = \frac{1}{2}x_1^2; \qquad \Gamma_{10} = \frac{1}{2}p_3^2; \qquad \Gamma_{11} = \frac{1}{2}x_2^2; \qquad \Gamma_{12} = \frac{1}{2}p_4^2;$$
$$\Gamma_{13} = x_1x_2; \qquad \Gamma_{14} = p_3p_4; \qquad \Gamma_{15} = p_3x_2; \qquad \Gamma_{16} = p_4x_1; \tag{52}$$

with

$$h_1 = h_3 = 1, \qquad h_2 = h_4 = -1, \qquad h_5 = h_6 = i,$$
$$h_7 = h_8 = i\kappa_1^2, \qquad h_9 = h_{11} = \kappa_1^2, \qquad h_{10} = h_{12} = -\kappa_1^2,$$
$$h_{13} = \kappa_2, \qquad h_{14} = -\kappa_2, \qquad h_{15} = i\kappa_2, \qquad h_{16} = i\kappa_2. \tag{53}$$

The dynamical algebra in this case is not closed. To find the closure property for the above system, we have to add 20 more phase space functions Γ_l's. The additional (Γ_l)'s are as follows:

$$\Gamma_{17} = p_1p_3; \quad \Gamma_{18} = p_1x_1; \quad \Gamma_{19} = p_4p_1; \quad \Gamma_{20} = p_3x_3;$$
$$\Gamma_{21} = x_1x_3; \quad \Gamma_{22} = x_3x_2; \quad \Gamma_{23} = x_2p_2; \quad \Gamma_{24} = p_2p_4;$$
$$\Gamma_{25} = p_3p_2; \quad \Gamma_{26} = x_2x_4; \quad \Gamma_{27} = p_4x_4; \quad \Gamma_{28} = x_1x_4;$$
$$\Gamma_{29} = x_3p_4; \quad \Gamma_{30} = x_1p_2; \quad \Gamma_{31} = p_1x_2; \quad \Gamma_{32} = p_3x_4,$$
$$\Gamma_{33} = p_1x_4; \quad \Gamma_{34} = p_1p_2; \quad \Gamma_{35} = x_3x_4; \quad \Gamma_{36} = p_2x_3; \tag{54}$$

with corresponding

$$h_l(t) = 0. \tag{55}$$

Now in the light of Poisson bracket for complex systems, eq. (45), we get a large number of nonvanishing Poisson brackets, namely

$$[\Gamma_1, \Gamma_7] = -\Gamma_{17} + i\Gamma_{18}; \quad [\Gamma_1, \Gamma_9] = -\Gamma_{18}; \quad [\Gamma_1, \Gamma_{10}] = i\Gamma_{17};$$
$$[\Gamma_1, \Gamma_{13}] = -\Gamma_{31}; \quad [\Gamma_1, \Gamma_{14}] = i\Gamma_{19};$$
$$[\Gamma_1, \Gamma_{15}] = i\Gamma_{31}; \quad [\Gamma_1, \Gamma_{16}] = -\Gamma_{19}; \quad [\Gamma_1, \Gamma_{17}] = 2i\Gamma_1;$$
$$[\Gamma_1, \Gamma_{18}] = -2\Gamma_1; \quad [\Gamma_1, \Gamma_{20}] = i\Gamma_5;$$
$$[\Gamma_1, \Gamma_{21}] = -\Gamma_5; \quad [\Gamma_1, \Gamma_{25}] = i\Gamma_{34};$$
$$[\Gamma_1, \Gamma_{28}] = -\Gamma_{33}; \quad [\Gamma_1, \Gamma_{30}] = -\Gamma_{34}; \quad [\Gamma_1, \Gamma_{32}] = i\Gamma_{33},$$
$$[\Gamma_2, \Gamma_7] = -i\Gamma_{20} + \Gamma_{21}; \quad [\Gamma_2, \Gamma_9] = i\Gamma_{21}; \quad [\Gamma_2, \Gamma_{10}] = i\Gamma_{20};$$
$$[\Gamma_2, \Gamma_{13}] = i\Gamma_{22}; \quad [\Gamma_2, \Gamma_{14}] = \Gamma_{29}; \quad [\Gamma_2, \Gamma_{15}] = i\Gamma_{22}; [\Gamma_2, \Gamma_{16}] = i\Gamma_{29};$$
$$[\Gamma_2, \Gamma_{17}] = \Gamma_5; \quad [\Gamma_2, \Gamma_{18}] = i\Gamma_5; \quad [\Gamma_2, \Gamma_{20}] = 2\Gamma_2; \quad [\Gamma_2, \Gamma_{21}] = 2i\Gamma_2;$$
$$[\Gamma_2, \Gamma_{25}] = \Gamma_{36}; \quad [\Gamma_2, \Gamma_{28}] = i\Gamma_{35}; \quad [\Gamma_2, \Gamma_{30}] = i\Gamma_{36}; \quad [\Gamma_2, \Gamma_{32}] = \Gamma_{35},$$
$$[\Gamma_3, \Gamma_8] = i\Gamma_{23} + \Gamma_{24}; \quad [\Gamma_3, \Gamma_{11}] = -\Gamma_{23}; \quad [\Gamma_3, \Gamma_{12}] = i\Gamma_{24};$$
$$[\Gamma_3, \Gamma_{13}] = -\Gamma_{30}; \quad [\Gamma_3, \Gamma_{14}] = i\Gamma_{25}; \quad [\Gamma_3, \Gamma_{15}] = -\Gamma_{25};$$
$$[\Gamma_3, \Gamma_{16}] = i\Gamma_{30}; \quad [\Gamma_3, \Gamma_{19}] = i\Gamma_{34}; \quad [\Gamma_3, \Gamma_{22}] = -\Gamma_{36};$$
$$[\Gamma_3, \Gamma_{23}] = -2\Gamma_3; \quad [\Gamma_3, \Gamma_{24}] = 2i\Gamma_3; \quad [\Gamma_3, \Gamma_{26}] = -\Gamma_6;$$

$[\Gamma_3, \Gamma_{27}] = i\Gamma_6$; $[\Gamma_3, \Gamma_{29}] = i\Gamma_{36}$; $[\Gamma_3, \Gamma_{31}] = -\Gamma_{34}$,

$[\Gamma_4, \Gamma_8] = i\Gamma_{27} + \Gamma_{26}$; $[\Gamma_4, \Gamma_{11}] = i\Gamma_{26}$; $[\Gamma_4, \Gamma_{12}] = \Gamma_{27}$;

$[\Gamma_4, \Gamma_{13}] = i\Gamma_{28}$; $[\Gamma_4, \Gamma_{14}] = \Gamma_{32}$; $[\Gamma_4, \Gamma_{15}] = i\Gamma_{32}$; $[\Gamma_4, \Gamma_{16}] = \Gamma_{28}$;

$[\Gamma_4, \Gamma_{19}] = \Gamma_{33}$; $[\Gamma_4, \Gamma_{22}] = i\Gamma_{35}$; $[\Gamma_4, \Gamma_{23}] = i\Gamma_6$; $[\Gamma_4, \Gamma_{24}] = \Gamma_6$;

$[\Gamma_4, \Gamma_{26}] = 2i\Gamma_4$; $[\Gamma_4, \Gamma_{27}] = 2\Gamma_4$; $[\Gamma_4, \Gamma_{29}] = \Gamma_{35}$; $[\Gamma_4, \Gamma_{31}] = i\Gamma_{33}$,

$[\Gamma_5, \Gamma_7] = i\Gamma_{21} + i\Gamma_{17} + \Gamma_{18} + \Gamma_{20}$; $[\Gamma_5, \Gamma_9] = i\Gamma_{18} - \Gamma_{21}$;

$[\Gamma_5, \Gamma_{10}] = i\Gamma_{20} + \Gamma_{17}$; $[\Gamma_5, \Gamma_{13}] = i\Gamma_{31} - \Gamma_{22}$; $[\Gamma_5, \Gamma_{14}] = \Gamma_{19} + i\Gamma_{29}$;

$[\Gamma_5, \Gamma_{15}] = \Gamma_{31} + i\Gamma_{32}$; $[\Gamma_5, \Gamma_{16}] = -\Gamma_{29} + i\Gamma_{19}$; $[\Gamma_5, \Gamma_{17}] = i\Gamma_5 + \Gamma_1$;

$[\Gamma_5, \Gamma_{18}] = -\Gamma_5 + 2i\Gamma_1$; $[\Gamma_5, \Gamma_{20}] = 2i\Gamma_2 + \Gamma_5$; $[\Gamma_5, \Gamma_{21}] = \Gamma_{18}$;

$[\Gamma_5, \Gamma_{25}] = i\Gamma_{20}$, $[\Gamma_5, \Gamma_{28}] = \Gamma_{35} + i\Gamma_{33}$; $[\Gamma_5, \Gamma_{30}] = -\Gamma_{36} + i\Gamma_{34}$;

$[\Gamma_5, \Gamma_{32}] = \Gamma_{33} + i\Gamma_{33}$, $[\Gamma_6, \Gamma_8] = i\Gamma_{26} + \Gamma_{23} - \Gamma_{27} + i\Gamma_{24}$;

$[\Gamma_6, \Gamma_{11}] = i\Gamma_{23} - \Gamma_{26}$; $[\Gamma_6, \Gamma_{12}] = \Gamma_{24} + i\Gamma_{24}$; $[\Gamma_6, \Gamma_{13}] = -\Gamma_{28} + i\Gamma_{30}$;

$[\Gamma_6, \Gamma_{14}] = i\Gamma_{32} + \Gamma_{25}$, $[\Gamma_6, \Gamma_{15}] = -\Gamma_{32} + i\Gamma_{25}$; $[\Gamma_6, \Gamma_{16}] = i\Gamma_{28} + \Gamma_{30}$;

$[\Gamma_6, \Gamma_{19}] = i\Gamma_{33} + \Gamma_{34}$; $[\Gamma_6, \Gamma_{22}] = -\Gamma_{35} + i\Gamma_{36}$; $[\Gamma_6, \Gamma_{23}] = -\Gamma_6 + 2i\Gamma_3$;

$[\Gamma_6, \Gamma_{24}] = 2\Gamma_3 + i\Gamma_6$; $[\Gamma_6, \Gamma_{26}] = 2\Gamma_3 - i\Gamma_4$;

$[\Gamma_6, \Gamma_{27}] = \Gamma_6 + 2i\Gamma_4$; $[\Gamma_6, \Gamma_{29}] = i\Gamma_{35} + \Gamma_{33}$, $[\Gamma_6, \Gamma_{31}] = -\Gamma_{33} + i\Gamma_{34}$,

$[\Gamma_7, \Gamma_{17}] = -i\Gamma_7 + 2i\Gamma_{10}$; $[\Gamma_7, \Gamma_{18}] = \Gamma_7 - 2i\Gamma_9$;

$[\Gamma_7, \Gamma_{19}] = -i\Gamma_{16} - \Gamma_{14}$; $[\Gamma_7, \Gamma_{20}] = -\Gamma_7 - 2i\Gamma_{10}$;

$[\Gamma_7, \Gamma_{21}] = -i\Gamma_7 - 2\Gamma_9$; $[\Gamma_7, \Gamma_{22}] = i\Gamma_{15} - \Gamma_{13}$; $[\Gamma_7, \Gamma_{29}] = -i\Gamma_{14} - \Gamma_{16}$;

$[\Gamma_7, \Gamma_{31}] = \Gamma_{15} - i\Gamma_{13}$; $[\Gamma_7, \Gamma_{33}] = -i\Gamma_{28} + \Gamma_{32}$;

$[\Gamma_7, \Gamma_{34}] = -i\Gamma_{30} + \Gamma_{25}$; $[\Gamma_7, \Gamma_{35}] = -\Gamma_{32} - \Gamma_{28}$;

$[\Gamma_7, \Gamma_{36}] = -i\Gamma_{25} - \Gamma_{30}$,

$[\Gamma_8, \Gamma_{23}] = -2i\Gamma_{11} + \Gamma_8$; $[\Gamma_8, \Gamma_{24}] = -i\Gamma_8 + 2\Gamma_{12}$;

$[\Gamma_8, \Gamma_{25}] = \Gamma_{14} - i\Gamma_{15}$, $[\Gamma_8, \Gamma_{26}] = -2\Gamma_{11} - i\Gamma_8$;

$[\Gamma_8, \Gamma_{27}] = -\Gamma_8 - 2i\Gamma_{12}$; $[\Gamma_8, \Gamma_{28}] = -\Gamma_{13} - i\Gamma_{16}$;

$[\Gamma_8, \Gamma_{30}] = -i\Gamma_{13} + \Gamma_{16}$; $[\Gamma_8, \Gamma_{32}] = -\Gamma_{15} - i\Gamma_{14}$;

$[\Gamma_8, \Gamma_{33}] = -\Gamma_{31} - i\Gamma_{19}$, $[\Gamma_8, \Gamma_{34}] = -i\Gamma_{31} + \Gamma_{19}$;

$[\Gamma_8, \Gamma_{35}] = -\Gamma_{22} - i\Gamma_{29}$; $[\Gamma_8, \Gamma_{36}] = -\Gamma_{30} - i\Gamma_{25}$,

$[\Gamma_9, \Gamma_{17}] = \Gamma_7$; $[\Gamma_9, \Gamma_{18}] = 2\Gamma_9$; $[\Gamma_9, \Gamma_{19}] = -\Gamma_{16}$;

$[\Gamma_9, \Gamma_{20}] = -i\Gamma_7$; $[\Gamma_9, \Gamma_{21}] = -2i\Gamma_9$; $[\Gamma_9, \Gamma_{22}] = -i\Gamma_3$;

$[\Gamma_9, \Gamma_{29}] = -i\Gamma_{16}$; $[\Gamma_9, \Gamma_{31}] = \Gamma_{13}$; $[\Gamma_9, \Gamma_{33}] = \Gamma_{28}$;

$[\Gamma_9, \Gamma_{34}] = \Gamma_{30}$; $[\Gamma_9, \Gamma_{35}] = -i\Gamma_{28}$; $[\Gamma_9, \Gamma_{36}] = -i\Gamma_{30}$,

$[\Gamma_{10}, \Gamma_{17}] = -2i\Gamma_{10}$; $[\Gamma_{10}, \Gamma_{18}] = -i\Gamma_7$; $[\Gamma_{10}, \Gamma_{19}] = -i\Gamma_{14}$;

$[\Gamma_{10}, \Gamma_{20}] = -2\Gamma_{10}$; $[\Gamma_{10}, \Gamma_{21}] = -\Gamma_7$; $[\Gamma_{10}, \Gamma_{22}] = -\Gamma_{15}$;

$$
\begin{aligned}
&[\Gamma_{10}, \Gamma_{29}] = -\Gamma_{14}; \quad [\Gamma_{10}, \Gamma_{31}] = -i\Gamma_{15}; \quad [\Gamma_{10}, \Gamma_{33}] = -i\Gamma_{32};\\
&[\Gamma_{10}, \Gamma_{34}] = -i\Gamma_{25}; \quad [\Gamma_{10}, \Gamma_{35}] = -\Gamma_{32}; \quad [\Gamma_{10}, \Gamma_{36}] = -\Gamma_{25},\\
&[\Gamma_{11}, \Gamma_{23}] = 2\Gamma_{11}; \quad [\Gamma_{11}, \Gamma_{24}] = \Gamma_{8}; \quad [\Gamma_{11}, \Gamma_{25}] = \Gamma_{15};\\
&[\Gamma_{11}, \Gamma_{26}] = -2i\Gamma_{11}; \quad [\Gamma_{11}, \Gamma_{27}] = -i\Gamma_{8}; \quad [\Gamma_{11}, \Gamma_{28}] = -i\Gamma_{13};\\
&[\Gamma_{11}, \Gamma_{30}] = \Gamma_{13}; \quad [\Gamma_{11}, \Gamma_{32}] = -i\Gamma_{15}; \quad [\Gamma_{11}, \Gamma_{33}] = -\Gamma_{19};\\
&[\Gamma_{11}, \Gamma_{34}] = \Gamma_{31}; \quad [\Gamma_{11}, \Gamma_{35}] = -i\Gamma_{22}; \quad [\Gamma_{11}, \Gamma_{36}] = \Gamma_{22},\\
&[\Gamma_{12}, \Gamma_{23}] = -i\Gamma_{8}; \quad [\Gamma_{12}, \Gamma_{24}] = -2i\Gamma_{12}; \quad [\Gamma_{12}, \Gamma_{25}] = -i\Gamma_{14};\\
&[\Gamma_{12}, \Gamma_{26}] = -\Gamma_{8}; \quad [\Gamma_{12}, \Gamma_{27}] = -2\Gamma_{12}; \quad [\Gamma_{12}, \Gamma_{28}] = -\Gamma_{16};\\
&[\Gamma_{12}, \Gamma_{30}] = -i\Gamma_{16}; \quad [\Gamma_{12}, \Gamma_{32}] = -\Gamma_{14}; \quad [\Gamma_{12}, \Gamma_{33}] = -\Gamma_{19};\\
&[\Gamma_{12}, \Gamma_{34}] = -i\Gamma_{19}; \quad [\Gamma_{12}, \Gamma_{35}] = -\Gamma_{29}; \quad [\Gamma_{12}, \Gamma_{36}] = -i\Gamma_{29},\\
&[\Gamma_{13}, \Gamma_{17}] = \Gamma_{15}; \quad [\Gamma_{13}, \Gamma_{18}] = \Gamma_{13}; \quad [\Gamma_{13}, \Gamma_{19}] = \Gamma_{9};\\
&[\Gamma_{13}, \Gamma_{20}] = -i\Gamma_{15}; \quad [\Gamma_{13}, \Gamma_{21}] = -i\Gamma_{13}; \quad [\Gamma_{13}, \Gamma_{22}] = -2i\Gamma_{11};\\
&[\Gamma_{13}, \Gamma_{23}] = \Gamma_{13}; \quad [\Gamma_{13}, \Gamma_{24}] = \Gamma_{16}; \quad [\Gamma_{13}, \Gamma_{25}] = \Gamma_{7};\\
&[\Gamma_{13}, \Gamma_{26}] = -i\Gamma_{13}; \quad [\Gamma_{13}, \Gamma_{27}] = -i\Gamma_{16}; \quad [\Gamma_{13}, \Gamma_{28}] = -2i\Gamma_{9};\\
&[\Gamma_{13}, \Gamma_{29}] = -i\Gamma_{8}; \quad [\Gamma_{13}, \Gamma_{30}] = 2\Gamma_{9}; \quad [\Gamma_{13}, \Gamma_{31}] = 2\Gamma_{11};\\
&[\Gamma_{13}, \Gamma_{32}] = -i\Gamma_{7}; \quad [\Gamma_{13}, \Gamma_{33}] = \Gamma_{26} - i\Gamma_{18}; \quad [\Gamma_{13}, \Gamma_{34}] = \Gamma_{23} + \Gamma_{18};\\
&[\Gamma_{13}, \Gamma_{35}] = -i\Gamma_{26} - i\Gamma_{21}; \quad [\Gamma_{13}, \Gamma_{36}] = \Gamma_{21} - i\Gamma_{13},\\
&[\Gamma_{14}, \Gamma_{17}] = -\Gamma_{16}; \quad [\Gamma_{14}, \Gamma_{18}] = -i\Gamma_{16}; \quad [\Gamma_{14}, \Gamma_{19}] = -2i\Gamma_{12};\\
&[\Gamma_{14}, \Gamma_{20}] = -\Gamma_{14}; \quad [\Gamma_{14}, \Gamma_{21}] = -\Gamma_{16}; \quad [\Gamma_{14}, \Gamma_{22}] = -\Gamma_{8};\\
&[\Gamma_{14}, \Gamma_{23}] = -i\Gamma_{15}; \quad [\Gamma_{14}, \Gamma_{24}] = -i\Gamma_{14}; \quad [\Gamma_{14}, \Gamma_{25}] = -2i\Gamma_{10};\\
&[\Gamma_{14}, \Gamma_{26}] = -\Gamma_{15}; \quad [\Gamma_{14}, \Gamma_{27}] = -\Gamma_{14}; \quad [\Gamma_{14}, \Gamma_{28}] = -\Gamma_{7};\\
&[\Gamma_{14}, \Gamma_{29}] = -2\Gamma_{12}; \quad [\Gamma_{14}, \Gamma_{30}] = -i\Gamma_{7}; \quad [\Gamma_{14}, \Gamma_{31}] = -i\Gamma_{25};\\
&[\Gamma_{14}, \Gamma_{32}] = -2\Gamma_{10}; \quad [\Gamma_{14}, \Gamma_{33}] = -i\Gamma_{24} - \Gamma_{17}; \quad [\Gamma_{14}, \Gamma_{34}] = -i\Gamma_{17} - i\Gamma_{24};\\
&[\Gamma_{14}, \Gamma_{35}] = -2\Gamma_{20} - \Gamma_{27}; \quad [\Gamma_{14}, \Gamma_{36}] = -\Gamma_{20} - i\Gamma_{24},\\
&[\Gamma_{15}, \Gamma_{17}] = -i\Gamma_{15}; \quad [\Gamma_{15}, \Gamma_{18}] = -i\Gamma_{13}; \quad [\Gamma_{15}, \Gamma_{19}] = -i\Gamma_{8};\\
&[\Gamma_{15}, \Gamma_{20}] = -\Gamma_{20}; \quad [\Gamma_{15}, \Gamma_{21}] = -\Gamma_{13}; \quad [\Gamma_{15}, \Gamma_{22}] = -2\Gamma_{11};\\
&[\Gamma_{15}, \Gamma_{23}] = \Gamma_{15}; \quad [\Gamma_{15}, \Gamma_{24}] = \Gamma_{14}; \quad [\Gamma_{15}, \Gamma_{25}] = 2\Gamma_{10};\\
&[\Gamma_{15}, \Gamma_{26}] = -i\Gamma_{15}; \quad [\Gamma_{15}, \Gamma_{27}] = -i\Gamma_{14}; \quad [\Gamma_{15}, \Gamma_{28}] = -i\Gamma_{27};\\
&[\Gamma_{15}, \Gamma_{29}] = -\Gamma_{8}; \quad [\Gamma_{15}, \Gamma_{30}] = \Gamma_{7}; \quad [\Gamma_{15}, \Gamma_{31}] = -2i\Gamma_{11};\\
&[\Gamma_{15}, \Gamma_{32}] = -2i\Gamma_{10}; \quad [\Gamma_{15}, \Gamma_{33}] = -i\Gamma_{26} - i\Gamma_{17};\\
&[\Gamma_{15}, \Gamma_{34}] = \Gamma_{17} - i\Gamma_{14}; \quad [\Gamma_{15}, \Gamma_{35}] = -\Gamma_{26} - i\Gamma_{20}; \quad [\Gamma_{15}, \Gamma_{36}] = \Gamma_{20} - \Gamma_{23},\\
&[\Gamma_{16}, \Gamma_{17}] = \Gamma_{14}; \quad [\Gamma_{16}, \Gamma_{18}] = \Gamma_{16}; \quad [\Gamma_{16}, \Gamma_{19}] = 2\Gamma_{12};\\
&[\Gamma_{16}, \Gamma_{20}] = -i\Gamma_{14}; \quad [\Gamma_{16}, \Gamma_{21}] = -i\Gamma_{16};\\
&[\Gamma_{16}, \Gamma_{22}] = i\Gamma_{8}; \quad [\Gamma_{16}, \Gamma_{23}] = -i\Gamma_{13}; \quad [\Gamma_{16}, \Gamma_{24}] = -i\Gamma_{16};\\
&[\Gamma_{16}, \Gamma_{25}] = -i\Gamma_{7}; \quad [\Gamma_{16}, \Gamma_{26}] = -\Gamma_{13}; \quad [\Gamma_{16}, \Gamma_{27}] = -\Gamma_{17};\\
&[\Gamma_{16}, \Gamma_{28}] = -2\Gamma_{9}; \quad [\Gamma_{16}, \Gamma_{29}] = -2i\Gamma_{12}; \quad [\Gamma_{16}, \Gamma_{30}] = -2i\Gamma_{9};\\
&[\Gamma_{16}, \Gamma_{31}] = \Gamma_{8}; \quad [\Gamma_{16}, \Gamma_{32}] = -\Gamma_{7}; \quad [\Gamma_{16}, \Gamma_{33}] = -\Gamma_{18} + \Gamma_{27};\\
&[\Gamma_{16}, \Gamma_{34}] = \Gamma_{24} - i\Gamma_{18}; \quad [\Gamma_{16}, \Gamma_{35}] = -\Gamma_{21} - i\Gamma_{27};\\
&[\Gamma_{16}, \Gamma_{36}] = -i\Gamma_{24} - i\Gamma_{21}.
\end{aligned}
\tag{56}
$$

Therefore, their use in eq. (44) yields the following set of 36 PDEs in λ's:

$$\dot{\lambda}_1 = 4(i\lambda_{17} - \lambda_{18}), \tag{57}$$

$$\dot{\lambda}_2 = -4(\lambda_{20} + i\lambda_{21}), \tag{58}$$

$$\dot{\lambda}_3 = -4(\lambda_{23} - i\lambda_{24}), \tag{59}$$

$$\dot{\lambda}_4 = -4(i\lambda_{26} + \lambda_{27}), \tag{60}$$

$$\dot{\lambda}_5 = -2(\lambda_{17} + i\lambda_{18} + \lambda_{21} - i\lambda_{20}), \tag{61}$$

$$\dot{\lambda}_6 = -2(i\lambda_{23} - \lambda_{24} - i\lambda_{27} + \lambda_{26}), \tag{62}$$

$$\dot{\lambda}_7 = 2\kappa_2(\lambda_{17} + i\lambda_{18} + \lambda_{21} + i\lambda_{20}) + 2\kappa_1(\lambda_{32} + i\lambda_{30} + \lambda_{28} - i\lambda_{25}), \tag{63}$$

$$\dot{\lambda}_8 = 2\kappa_1(\lambda_{22} - i\lambda_{19} - i\lambda_{29} + i\lambda_{31}) + 2\kappa_2(i\lambda_{23} + \lambda_{24} - i\lambda_{27} + \lambda_{26}), \tag{64}$$

$$\dot{\lambda}_9 = 4\kappa_2(\lambda_{18} - i\lambda_{21}) + 4\kappa_1(-i\lambda_{28} + \lambda_{30}), \tag{65}$$

$$\dot{\lambda}_{10} = 4\kappa_2(i\lambda_{17} - \lambda_{20}) + 4\kappa_1(i\lambda_{32} - \lambda_{25}), \tag{66}$$

$$\dot{\lambda}_{11} = 4\kappa_2(\lambda_{23} - i\lambda_{26}) + 4\kappa_1(\lambda_{31} - i\lambda_{22}), \tag{67}$$

$$\dot{\lambda}_{12} = 4\kappa_2(i\lambda_{24} + \lambda_{27}) + 4\kappa_1(\lambda_{19} + \lambda_{29}), \tag{68}$$

$$\dot{\lambda}_{13} = 2\kappa_2(-i\lambda_{22} - i\lambda_{28} + \lambda_{30} + \lambda_{31}) + 2\kappa_1(\lambda_{18} - i\lambda_{21} + \lambda_{23} - i\lambda_{26}), \tag{69}$$

$$\dot{\lambda}_{14} = 2\kappa_2(\lambda_{19} - \lambda_{25} + \lambda_{27} + i\lambda_{32}) + 2\kappa_1(i\lambda_{17} - \lambda_{20} + \lambda_{27} + i\lambda_{24}), \tag{70}$$

$$\dot{\lambda}_{15} = 2\kappa_2(\lambda_{22} - i\lambda_{25} - \lambda_{32} + i\lambda_{31}) + 2\kappa_1(-\lambda_{17} - i\lambda_{20} + \lambda_{26} + i\lambda_{23}), \tag{71}$$

$$\dot{\lambda}_{16} = 2\kappa_2(i\lambda_{19} + \lambda_{28} + i\lambda_{29} + i\lambda_{30}) + 2\kappa_1(i\lambda_{18} + \lambda_{21} - \lambda_{24} + i\lambda_{27}), \tag{72}$$

$$\dot{\lambda}_{17} = 2\kappa_2(i\lambda_1 + \lambda_5) + 2(-\lambda_{10} - \lambda_7) + 2\kappa_1(\lambda_{33} + i\lambda_{34}), \tag{73}$$

$$\dot{\lambda}_{18} = 2\kappa_2(\lambda_1 - i\lambda_5) + 2(i\lambda_7 - i\lambda_9) + 2\kappa_1(-i\lambda_{33} + \lambda_{34}), \tag{74}$$

$$\dot{\lambda}_{19} = 2\kappa_2(\lambda_{33} + i\lambda_{34}) + 2(-i\lambda_{14} - \lambda_{16}) + 2\kappa_1(i\lambda_1 + \lambda_5), \tag{75}$$

$$\dot{\lambda}_{20} = 2\kappa_2(i\lambda_2 - i\lambda_5) + 2(i\lambda_7 - i\lambda_{10}) + 2\kappa_1(\lambda_{35} - i\lambda_{36}), \tag{76}$$

$$\dot{\lambda}_{21} = 2\kappa_2(-\lambda_2 + \lambda_5) + 2(-\lambda_7 - \lambda_9) + 2\kappa_1(i\lambda_{35} + \lambda_{36}), \tag{77}$$

$$\dot{\lambda}_{22} = 2\kappa_2(-i\lambda_{35} + \lambda_{36}) + 2(-i\lambda_{13} - \lambda_{15}) + 2\kappa_1(-i\lambda_2 + \lambda_5), \tag{78}$$

$$\dot{\lambda}_{23} = 2\kappa_2(\lambda_3 - i\lambda_6) + 2(-\lambda_{11} + i\lambda_8) + 2\kappa_1(\lambda_{34} - i\lambda_{36}), \tag{79}$$

$$\dot{\lambda}_{24} = 2\kappa_2(i\lambda_3 + \lambda_6) + 2(-\lambda_8 + i\lambda_{12}) + 2\kappa_1(\lambda_{36} + i\lambda_{34}), \tag{80}$$

$$\dot{\lambda}_{25} = 2\kappa_2(\lambda_{34} - i\lambda_{36}) + 2(-\lambda_{15} + i\lambda_{14}) + 2\kappa_1(\lambda_3 + \lambda_6), \tag{81}$$

$$\dot{\lambda}_{26} = 2\kappa_2(-i\lambda_4 + \lambda_6) + 2(-\lambda_8 - i\lambda_{11}) + 2\kappa_1(\lambda_{33} - i\lambda_{35}), \tag{82}$$

$$\dot{\lambda}_{27} = 2\kappa_2(\lambda_4 + i\lambda_6) + 2(-\lambda_{12} - i\lambda_8) + 2\kappa_1(\lambda_{35} + i\lambda_{33}), \tag{83}$$

$$\dot{\lambda}_{28} = 2\kappa_2(\lambda_{33} - i\lambda_{35}) + 2(-i\lambda_{13} - \lambda_{16}) + 2\kappa_1(\lambda_6 - i\lambda_4), \tag{84}$$

$$\dot{\lambda}_{29} = 2\kappa_2(\lambda_{35} + i\lambda_{36}) + 2(\lambda_{14} - i\lambda_{16}) + 2\kappa_1(\lambda_2 + i\lambda_5), \quad (85)$$

$$\dot{\lambda}_{30} = 2\kappa_2(\lambda_{34} - i\lambda_{36}) + 2(-\lambda_{13} + i\lambda_{16}) + 2\kappa_1(\lambda_3 + i\lambda_6), \quad (86)$$

$$\dot{\lambda}_{31} = 2\kappa_2(-i\lambda_{33} + \lambda_{34}) + 2(-\lambda_{13} + i\lambda_{15}) + 2\kappa_1(\lambda_1 - i\lambda_5), \quad (87)$$

$$\dot{\lambda}_{32} = 2\kappa_2(-i\lambda_{33} + \lambda_{35}) + 2(+\lambda_{14} - i\lambda_{15}) + 2\kappa_1(\lambda_4 + i\lambda_6), \quad (88)$$

$$\dot{\lambda}_{33} = 2(-\lambda_{19} - i\lambda_{31} - \lambda_{28} + i\lambda_{32}), \quad (89)$$

$$\dot{\lambda}_{34} = 2(i\lambda_{19} - \lambda_{30} - \lambda_{31} + \lambda_{25}), \quad (90)$$

$$\dot{\lambda}_{35} = 2(-i\lambda_{22} - i\lambda_{28} - \lambda_{32} - \lambda_{29}), \quad (91)$$

$$\dot{\lambda}_{36} = 2(-\lambda_{22} - \lambda_{25} + i\lambda_{29} - i\lambda_{30}). \quad (92)$$

In fact, it is difficult to solve these 36 coupled PDEs for complex λ's. Thus, here we make certain choices for λ's which facilitate the finding of solutions of the above equations.

From eqs (57), (58) and (61), we get $2\dot{\lambda}_5 = i\dot{\lambda}_1 - i\dot{\lambda}_2$, and if we consider $\lambda_5 = c_5$ (a constant), further $\lambda_1 = \lambda_2 = \eta_1(t)$, which immediately gives

$$\lambda_1 = \eta_1(t) + c_1, \qquad \lambda_2 = \eta_1(t) + c_2. \quad (93)$$

Again from eqs (59), (60) and (62), we get $2\dot{\lambda}_6 = i\dot{\lambda}_3 + i\dot{\lambda}_4$, and if we consider $\lambda_6 = c_6$ (a constant), further $\lambda_3 = \lambda_4 = \eta_2(t)$, which immediately gives

$$\lambda_3 = \eta_2(t) + c_3, \qquad \lambda_4 = -\eta_2(t) + c_4. \quad (94)$$

Now, to find solutions for λ_{11} and λ_{12}. From eqs (67), (68) and (64), we get $2\dot{\lambda}_8 = i\dot{\lambda}_{11} - i\dot{\lambda}_{12}$, and if we consider $\lambda_8 = c_8$ (a constant), further $\lambda_{11} = \lambda_{12} = \eta_3(t)$, which immediately gives

$$\lambda_{11} = \eta_3(t) + c_{11}, \qquad \lambda_{12} = \eta_3(t) + c_{12}. \quad (95)$$

From eqs (65), (66) and (63), we get $2\dot{\lambda}_7 = i\dot{\lambda}_9 - i\dot{\lambda}_{10}$, and if we consider $\lambda_7 = c_7$ (a constant), further $\lambda_9 = \lambda_{10} = \eta_4(t)$, which immediately gives

$$\lambda_9 = \eta_4(t) + c_9, \qquad \lambda_{10} = \eta_4(t) + c_{10}. \quad (96)$$

Now, to find solutions for λ_{13}. From eqs (65), (67) and (69), we get $2\dot{\lambda}_{13} = \dot{\lambda}_9 + \dot{\lambda}_{11}$, and if we consider $\lambda_{13} = c_{13}$ (a constant), and considering the relation (with $\lambda_9 = \eta_4(t) + c_9$, $\lambda_{11} = \eta_3(t) + c_{11}$) gives

$$\lambda_{13} = \eta(t) + c_{13}, \quad (97)$$

where $\eta(t) = \eta_4(t) + \eta_3(t)$ is another function of t and $c_{13} = c_3 + c_4$, a constant.

From eqs (66), (67) and (71), we get $2\dot{\lambda}_{15} = i\dot{\lambda}_{10} + i\dot{\lambda}_{11}$, and if we consider $\lambda_{15} = c_{15}$ (a constant), and considering the relation (with $\lambda_{10} = \eta_4(t) + c_{10}$, $\lambda_{11} = \eta_3(t) + c_{11}$) gives

$$\lambda_{15} = \eta(t) + c_{15}, \quad (98)$$

where $\eta(t) = \eta_4(t) + \eta_3(t)$ is another function of t and $c_{15} = c_{10} + c_{11}$, a constant.

Again, to find solutions for λ_{16}. From eqs (65), (68) and (72), we get $2\dot{\lambda}_{16} = i\dot{\lambda}_9 + i\dot{\lambda}_{12}$, and if we consider $\lambda_{16} = c_{16}$ (a constant), and considering the relation (with $\lambda_9 = \eta_4(t) + c_9$, $\lambda_{12} = \eta_3(t) + c_{12}$) gives

$$\lambda_{16} = \eta(t) + c_{16}, \tag{99}$$

where $\eta(t) = \eta_4(t) + \eta_3(t)$ is another function of t and $c_{16} = c_9 + c_{12}$, a constant.

From eqs (66), (68) and (70), we get $2\dot{\lambda}_{14} = \dot{\lambda}_{10} + \dot{\lambda}_{12}$, and if we consider $\lambda_{14} = c_{14}$ (a constant), and considering the relation (with $\lambda_{10} = \eta_4(t) + c_{10}$, $\lambda_{12} = \eta_3(t) + c_{12}$) gives

$$\lambda_{14} = \eta(t) + c_{14}, \tag{100}$$

where $\eta(t) = \eta_4(t) + \eta_3(t)$ is another function of t and $c_{16} = c_{10} + c_{12}$, a constant.

Now, for finding the solutions of λ_{17} and λ_{18}, add i times eq. (73) from eq. (74) and after substituting eq. (80), we get

$$\dot{\lambda}_{18} + i\dot{\lambda}_{17} = -i\lambda_9 - i\lambda_{10} = -i(\lambda_9 + \lambda_{10})$$

or

$$i\dot{\lambda}_{17} + \dot{\lambda}_{18} = -i(2\eta_4(t) + c_9 + c_{10}). \tag{101}$$

On the other hand, time derivative of eq. (57) is written as

$$\ddot{\lambda}_1 = 4(i\lambda_{17} - \lambda_{18}) = \ddot{\theta}. \tag{102}$$

Hence using eqs (101) and (102), one immediately get

$$\lambda_{17} = -\frac{i}{8}(\dot{\theta} - 8\xi) + c_{17}, \qquad \lambda_{18} = -\frac{1}{8}(\dot{\theta} + 8\xi) + c_{18}, \tag{103}$$

where $\xi = \int (2\eta_4(t) + c_9 + c_{10})\mathrm{d}t$.

Similarly, from eqs (76) and (77) with eq. (58), from eqs (79) and (80) with eq. (59), and from eqs (82) and (83) with eq. (60), we obtain solutions for $(\lambda_{20}, \lambda_{21})$, $(\lambda_{23}, \lambda_{24})$ and $(\lambda_{26}, \lambda_{27})$ respectively as

$$\lambda_{20} = -\frac{1}{8}(\dot{\theta} + 8\xi) + c_{20}, \qquad \lambda_{21} = \frac{i}{8}(\dot{\theta} - 8\xi) + c_{21}, \tag{104}$$

$$\lambda_{23} = -\frac{1}{8}(\dot{\rho} - 8i\sigma) + c_{23}, \qquad \lambda_{24} = -\frac{i}{8}(\dot{\rho} + 8i\sigma) + c_{24}, \tag{105}$$

$$\lambda_{26} = \frac{i}{8}(\dot{\rho} + 8i\sigma) + c_{26}, \qquad \lambda_{27} = -\frac{1}{8}(\dot{\rho} - 8i\sigma) + c_{27}, \tag{106}$$

where $\sigma = \int (2i\eta_3(t) + c_{11} + c_{12})\mathrm{d}t$. Now to obtain solutions for $(\lambda_{19}, \lambda_{22})$, $(\lambda_{25}, \lambda_{28})$, $(\lambda_{29}, \lambda_{30})$ and $(\lambda_{31}, \lambda_{32})$ respectively as, from eqs (75) and (87), from eqs (78) and (85), from eqs (81) and (86) and eqs (84) and (88), we obtain the following equations:

$$\dot{\lambda}_{19} - i\dot{\lambda}_{31} = (i\lambda_{13} - i\lambda_{14} + \lambda_{15} - \lambda_{16}) = 0 \tag{107}$$

$$\dot{\lambda}_{22} + i\dot{\lambda}_{29} = (-i\lambda_{13} + i\lambda_{14} - \lambda_{15} + \lambda_{16}) = 0 \tag{108}$$

$$\dot{\lambda}_{25} - \dot{\lambda}_{30} = (\lambda_{13} + i\lambda_{14} - \lambda_{15} - i\lambda_{16}) = 0 \tag{109}$$

$$\dot{\lambda}_{28} + i\dot{\lambda}_{32} = (-\lambda_{13} + i\lambda_{14} + \lambda_{15} - \lambda_{16}) = 0 \tag{110}$$

since

$$\lambda_{13} = \lambda_{14} = \lambda_{15} = \lambda_{16} = \eta(t)$$

or if we set

$$\dot{\lambda}_{19} = i\dot{\lambda}_{31} = \dot{\phi}(t), \quad \dot{\lambda}_{22} = -i\dot{\lambda}_{29} = \dot{\varphi}(t) \tag{111}$$

$$\dot{\lambda}_{25} = \dot{\lambda}_{30} = \dot{\chi}(t), \quad \dot{\lambda}_{28} = -i\dot{\lambda}_{32} = \dot{\psi}(t) \tag{112}$$

which immediately gives

$$\lambda_{19} = \phi(t) + c_{19}, \quad \lambda_{31} = -i\phi(t) + c_{31},$$

$$\lambda_{22} = \varphi(t) + c_{22}, \quad \lambda_{29} = i\varphi(t) + c_{29} \tag{113}$$

$$\lambda_{25} = \chi(t)c_{25}, \quad \lambda_{30} = \chi(t) + c_{30},$$

$$\lambda_{28} = \psi(t) + c_{28}, \quad \lambda_{32} = i\psi(t) + c_{32}. \tag{114}$$

Now, for obtaining solutions for $(\lambda_{33}, \lambda_{34})$, $(\lambda_{35}, \lambda_{36})$, simply put the values of $(\lambda_{19}, \lambda_{22})$, $(\lambda_{25}, \lambda_{28})$, $(\lambda_{29}, \lambda_{30})$, and $(\lambda_{31}, \lambda_{32})$ in (89), (90), (91) and (92), and we get

$$\lambda_{33} = \alpha_1(t) + c_{33}, \quad \lambda_{34} = \alpha_2(t) + c_{34},$$

$$\lambda_{35} = \alpha_3(t) + c_{35}, \quad \lambda_{36} = \alpha_4(t) + c_{36} \tag{115}$$

where the α's are as follows

$$\alpha_1 = -\int 2[2i\phi - 2i\psi + c_{19} + c_{28} + c_{31} - ic_{32}]\mathrm{d}t,$$

$$\alpha_2 = -\int 2[2\phi + [\chi - i\chi] - ic_{19} + c_{30} + c_{31} - ic_{25}]\mathrm{d}t$$

$$\alpha_3 = -\int 2[2i\varphi + 2\psi - ic_{22} - ic_{28} - c_{29} - c_{32}]\mathrm{d}t,$$

$$\alpha_4 = -\int 2[-2i\varphi + [\chi - i\chi] + c_{22} + c_{25} - ic_{29} - ic_{30}]\mathrm{d}t.$$

We have solved eqs (57)–(92) in terms of arbitrary functions η's, $\theta, \xi, \sigma, \psi, \phi, \varphi, \rho, \chi$ and α's and complex constants, c_i's ($i = 1, ..., 36$).

If one put back these solutions for λ_i ($i = 1, ..., 36$) in eqs (57) to (92), we obtain a number of constraint relations among c_i's and η's, $\theta, \phi, \xi, \sigma, \psi, \varphi, \rho, \chi$ and α's, which limit the choices of these arbitrary complex quantities. If we set all c_i's equal to zero, then these relations determining the arbitrary functions η's, $\theta, \phi, \varphi, \rho, \chi$ and α's are written as

$$\ddot{\eta}_4 + 4\dot{\theta} - 4\dot{\chi} + 2i(2\eta_4) = 0; \quad \ddot{\eta}_3 - 4\dot{\phi} - 4\dot{\rho} + 2i(2\eta_4) = 0;$$

$$\ddot{\eta} - 2(\dot{\varphi} + \dot{\phi} - \dot{\chi} - \dot{\phi}) = 0;$$

$$\ddot{\eta} - 2(\dot{\varphi} + \dot{\sigma} - \dot{\chi} - \dot{\xi}) = 0;\quad \ddot{\psi} + 16(i\dot{\alpha}_1 - \dot{\alpha}_2 - \dot{\eta}_1) = 0;$$
$$\ddot{\psi} + 16(\dot{\alpha}_3 - i\dot{\alpha}_4 + i\dot{\eta}_1) = 0;\quad \ddot{\varphi} + 16(\dot{\alpha}_2 - i\dot{\alpha}_4 + \dot{\eta}_2) = 0;$$
$$\ddot{\varphi} + 16(i\dot{\alpha}_1 + \dot{\alpha}_4 + \dot{\eta}_2) = 0;\quad \ddot{\psi} + 2i(i\dot{\alpha}_2 + \dot{\alpha}_1 - i\dot{\eta} - \dot{\eta} + i\dot{\eta}_1) = 0;$$
$$\ddot{\varphi} - 2(i\dot{\alpha}_4 + \dot{\alpha}_3 - i\dot{\eta} + \dot{\eta} + i\dot{\eta}_1) = 0;\quad \ddot{\sigma} - 2(\dot{\alpha}_2 - i\dot{\alpha}_4 + i\dot{\eta} - \dot{\eta} + \dot{\eta}_2) = 0;$$
$$i\varphi - \chi - i\chi = 0;\quad \ddot{\psi} - 2(i\dot{\alpha}_1 + \dot{\alpha}_4 - i\dot{\eta} + \dot{\eta} + i\dot{\eta}_2) = 0;$$
$$2i\phi + 2i\psi = 0;\quad \phi - \chi - i\chi = 0;\quad \varphi + \psi = 0. \tag{116}$$

Therefore, after substituting the solutions of λ_i's in eq. (50), the complex integral for a two-dimensional complex oscillator becomes

$$\begin{aligned} I = {} & \frac{1}{2}\eta_1\left(p_1^2 + x_3^2\right) + \frac{1}{2}\eta_2\left(p_2^2 + x_4^2\right) + \frac{1}{2}\eta_4\left(x_1^2 + p_3^2\right) + \frac{1}{2}\eta_3\left(x_2^2 + p_4^2\right) \\ & + \eta(x_1x_2 + p_3p_4 + x_2p_3 + x_1p_4) - \frac{i}{8}(\dot{\theta} - 8\xi)(p_1p_3 - x_1x_3) \\ & - \frac{1}{8}(\dot{\theta} + 8\xi)(x_1p_1 + x_3p_3) - \frac{i}{8}(\dot{\rho} - i8\sigma)(p_2p_4 - x_2x_4) - \frac{1}{8}(\dot{\rho} + i8\sigma) \\ & \times(p_2x_2 + p_4x_4) + \phi(p_1p_4 - ip_1x_2) + \varphi(x_2x_3 + ip_4x_3) \\ & + \psi(x_1x_4 + ip_3x_4) + \chi(p_2p_3 + p_2x_1) + \alpha_1p_1x_4 + \alpha_2p_1p_2 \\ & + \alpha_3x_3x_4 + \alpha_4p_2x_3 \end{aligned} \tag{117}$$

which conforms to condition eq. (46) in view of the Poisson bracket eq. (47).

3. Conclusion

As pointed out in §1, the existence of a real/complex integral for a dynamical system, if the integral exists, then its construction, in general, is a difficult task. Once it is constructed and becomes available then not only its physical interpretation(s) but also its viability with regard to a better theoretical understanding of a given phenomenon is often a problem. In spite of all this, the availability of a few or all [1–3,13–15] integrals for a dynamical system definitely offers insight into the finer details as far as an understanding of the phenomenon is concerned. Finally, a few remarks about the applicability of the systems investigated in this work are in order. The role of a linear integral designed, however, for a rotating TD harmonic oscillator in N dimensions is investigated by Malkin and Man'ko [17] in the context of coherent states. While the use of the quantum analogue of such TD systems in one dimension has been known [18] for more than three decades now in the fields of quantum optics and squeezed states, a discussion of two- and three-mode squeezing phenomena has been of current interest [19]. The integrals, although constructed here at the classical level, can however be of great help in finding alternative theoretical explanations of these phenomena mainly because the underlying approach used is easily amenable to extension [20] to the quantum domain. In this work, an attempt has been made to obtain exact complex second integrals of a two-dimensional complex shifted harmonic oscillators on an ECPS described by eq. (40). These complex integrals could be useful for a better understanding of the dynamical systems.

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