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Analytic treatment of nonlinear evolution equations using first integral method

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Abstract. In this paper, we show the applicability of the first integral method to combined KdV–mKdV equation, Pochhammer–Chree equation and coupled nonlinear evolution equations. The power of this manageable method is confirmed by applying it for three selected nonlinear evolution equations. This approach can also be applied to other nonlinear differential equations.

Keywords. Exact solutions; first integral method; combined KdV–mKdV equation; Pochhammer–Chree equation; coupled nonlinear evolution equations.

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1. Introduction

It is well known that searching for solitary solutions of nonlinear equations in mathematical physics and applied mathematics has become more and more attractive in solitary theory. A number of methods, such as the inverse scattering transformation [1,2], Hirota bilinear tranformation [3–5], the tanh–sech method [6–8], extended tanh method [9,10], sine–cosine method [11,12], homogeneous balance method [13,14], Lie symmetry method [15,16], Exp-function method [17,18] and (G'/G)-expansion method [19,20] have been proposed to obtain exact solutions. A common feature of all these methods is that when solving the solutions of nonlinear evolution equations, they all must need the help of a computer algebra system, such as *Maple* or *Mathematica*.

Among those approaches, the first integral method is a tool to generate the soliton and periodic solutions of the nonlinear partial differential equations. The advantage of the first integral method is that, the iterative algorithm is purely algebraic and computerizable using symbolic computation which can be found in refs [21–24].

The present paper investigates for the first time the applicability and effectiveness of the first integral method on nonlinear evolution equations. The paper is arranged as follows: in §2, we simply provide the mathematical framework of the first integral method. In

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§3–5, in order to illustrate the method, three nonlinear equations are investigated, and abundant exact solutions are obtained which include new solitary wave solutions and trigonometric function solutions. Finally, in §6 some conclusions are provided.

2. The first integral method

The pioneer, Feng [21] introduced the first integral method for a reliable treatment of the nonlinear PDEs. Raslan [25] proposed the first integral method to solve the Fisher equation. Abbasbandy and Shirzadi [26] solved the modified Benjamin–Bona–Mahony equation using the first integral method. Tascan and Bekir [27] used the first integral method to obtain exact solutions of the modified Zakharov–Kuznetsov equation and ZK–MEW equation. The useful first integral method is widely used by many such as in [28–31] and by the references therein.

Step 1. Consider a general nonlinear PDE in the form

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, ...) = 0. \tag{2.1}$$

Using a wave variable $\xi = x - ct$, we can write eq. (2.1) in the following nonlinear ODE:

$$Q(U, U', U'', U''', ...) = 0, \tag{2.2}$$

where the prime denotes the derivation with respect to ξ. If all terms contain derivatives, then eq. (2.2) is integrated where integration constants are considered zeros.

Step 2. Suppose that the solution of ODE (2.2) can be written as follows:

$$u(x, t) = f(\xi). \tag{2.3}$$

Step 3. We introduce a new independent variable

$$X(\xi) = f(\xi), \qquad Y = f_\xi(\xi), \tag{2.4}$$

which leads a system of

$$\begin{aligned} X_\xi(\xi) &= Y(\xi), \\ Y_\xi(\xi) &= F(X(\xi), Y(\xi)). \end{aligned} \tag{2.5}$$

Step 4. According to the qualitative theory of ordinary differential equations [32], if we can find the integrals to (2.5) under the same conditions, then the general solutions to (2.5) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is neither a systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We shall apply the Division Theorem to obtain one first integral to (2.5) which reduces (2.2) to a first-order integrable ordinary differential equation. An exact solution to (2.1) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Division Theorem: Suppose that $P(w,z)$, $Q(w,z)$ are polynomials in $C(w,z)$ and $P(w,z)$ is irreducible in $C(w,z)$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C(w,z)$ such that

$$Q[w,z] = P[w,z]G[w,z]. \tag{2.6}$$

The fact that the real field $\mathbb{R}$ is a subfield of the complex field $\mathbb{C}$ is well known. The extension of a given equation in $\mathbb{R}$ to an equation in $\mathbb{C}$ is always possible. If the extended equation has an algebraic curve solution in $\mathbb{C}$, then the intersection of the manifold of this solution and the real plane must be the algebraic curve solution of the original equation in $\mathbb{R}$. Thus, the Division Theorem stated in $\mathbb{C}$ can also be used in $\mathbb{R}$ [33].

Feng *et al* [34] pointed out that the Division Theorem follows immediately from Hilbert–Nullstellensatz Theorem [35] of commutative algebra.

3. The combined KdV–mKdV equation

Let us first consider the combined KdV–mKdV equation [36]

$$u_t + puu_x + ru^2u_x - \delta u_{xxx} = 0. \tag{3.1}$$

The KdV and mKdV equations are the most popular soliton equations and have been extensively investigated. But nonlinear terms of KdV and mKdV equations often simultaneously exist in practical problems such as fluid physics and quantum field theory. This equation may be described as the wave propagation of the bound particle, sound wave and thermal pulse [37,38]. Certain exact solutions of KdV–mKdV equation have been found by using Hirota bilinear method, inverse scattering and homogeneous balance method [39,40], and unified algebraic method [41]. The modified mapping method is developed to obtain new exact solutions to the combined KdV and mKdV equation in [42].

Using the transformation

$$u(x,t) = f(\xi), \quad \xi = c(x - \lambda t) \tag{3.2}$$

and substituting eq. (3.2) into eq. (3.1) yields

$$-c\lambda f'(\xi) + cpf(\xi)f'(\xi) + rcf^2(\xi)f'(\xi) - c^3\delta f'''(\xi) = 0, \tag{3.3}$$

where c and λ are constants and the prime denotes the derivation with respect to ξ. Integrating eq. (3.3), we obtain

$$-c\lambda f(\xi) + \frac{cp}{2}f^2(\xi) + \frac{rc}{3}f^3(\xi) - c^3\delta f''(\xi) = n, \tag{3.4}$$

where n is the integration constant.

Using (2.4) we get

$$X_{\xi}(\xi) = Y(\xi) \tag{3.5}$$

$$Y_{\xi}(\xi) = \frac{-6n - 6c\lambda X(\xi) + 3cpX^2(\xi) + 2rcX^3(\xi)}{6c^3\delta}. \tag{3.6}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (3.5), (3.6), and $q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{3.7}$$

where $a_i(X)$, $i = 0, 1, ..., m$, are polynomials of X and $a_m(X) \neq 0$. Equation (3.7) is called the first integral to (3.5)–(3.6). Due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X, Y]$ such that

$$\frac{\mathrm{d}q}{\mathrm{d}\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = [g(X) + h(X)Y]\sum_{i=0}^{m} a_i(X)Y^i. \tag{3.8}$$

In this example, we take two different cases, by assuming $m = 1$ and $m = 2$ in eq. (3.7).

Case I: Suppose $m = 1$. By equating the coefficients of Y^i, $i = 0, 1, 2$, on both sides of eq. (3.8), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{3.9}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X), \tag{3.10}$$

$$\begin{aligned} a_1(X)\dot{Y} &= g(X)a_0(X) \\ &= a_1(X)\left(\frac{-6n - 6c\lambda X + 3cpX^2 + 2rcX^3}{6c^3\delta}\right). \end{aligned} \tag{3.11}$$

Since $a_i(X)$, $i = 0, 1$, are polynomials, then from (3.9) we deduce that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, and $A_1 \neq 0$, then we find

$$a_0(X) = \frac{A_1}{2}X^2 + B_0X + A_0. \tag{3.12}$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in eq. (3.11) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$\begin{aligned} A_0 &= -\frac{(p^2 + 6r\lambda)}{2cr\sqrt{6r\delta}}, \qquad A_1 = \frac{1}{c}\sqrt{\frac{2r}{3\delta}}, \\ B_0 &= \sqrt{\frac{\delta}{6q}}\frac{p}{c\delta}, \qquad n = \frac{pc(p^2 + 6r\lambda)}{12r^2}. \end{aligned} \tag{3.13}$$

Substituting (3.13) into (3.7), we obtain

$$Y(\xi) = \frac{(p^2+6r\lambda)}{2cq\sqrt{6r\delta}} - \sqrt{\frac{\delta}{6r}}\frac{p}{c\delta}X(\xi) - \frac{1}{c}\sqrt{\frac{r}{6\delta}}X^2(\xi). \tag{3.14}$$

Combining (3.14) with (3.5), we obtain the exact solution to (3.4) and then the exact solution can be written as

$$X(\xi) = \frac{1}{2r^2}\left(-rp + \sqrt{3r^2p^2+12r^3\lambda} \times \tanh\left(\sqrt{\frac{\delta(2p^2+8r\lambda)}{r}}\frac{(\xi+C_1)}{4c\delta}\right)\right), \tag{3.15}$$

where C_1 is the integration constant. Thus the solitary wave solution to the combined KdV–mKdV eq. (3.1) can be written as

$$u(x,t) = \frac{1}{2r^2}\left(-rp + \sqrt{3r^2p^2+12r^3\lambda} \times \tanh\left(\sqrt{\frac{\delta(2p^2+8r\lambda)}{r}}\frac{(c(x-\lambda t)+C_1)}{4c\delta}\right)\right). \tag{3.16}$$

Case II: Suppose $m=2$. By equating the coefficients of Y^i, $i=0,1,2,3$, on both sides of eq. (3.8), we have

$$\dot{a}_2(X) = h(X)a_2(X), \tag{3.17}$$

$$\dot{a}_1(X) = g(X)a_2(X) + h(X)a_1(X), \tag{3.18}$$

$$\dot{a}_0(X) = -2a_2(X)\left(\frac{-6n-6c\lambda X+3cpX^2+2rcX^3}{6c^3\delta}\right) + g(X)a_1(X) + h(X)a_0(X), \tag{3.19}$$

$$a_1(X)\dot{Y} = g(X)a_0(X) = a_1(X)\left(\frac{-6n-6c\lambda X+3cpX^2+2rcX^3}{6c^3\delta}\right). \tag{3.20}$$

Since $a_2(X)$ is a polynomial of X, then from (3.17) we deduce that $a_2(X)$ is a constant and $h(X)=0$. For simplicity, take $a_2(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X))=1$ only. Suppose that $g(X)=A_1X+B_0$, and $A_1 \neq 0$, then we find $a_1(X)$ and $a_0(X)$ as

$$a_1(X) = \frac{A_1}{2}X^2 + B_0X + A_0. \tag{3.21}$$

$$a_0(X) = \left(\frac{A_1^2}{8} - \frac{r}{6c^2\delta}\right)X^4 + \left(-\frac{p}{3c^2\delta} + \frac{A_1B_0}{2}\right)X^3 + \left(\frac{\lambda}{c^2\delta} + \frac{A_1A_0}{2} + \frac{B_0^2}{2}\right)X^2 + \left(\frac{2n}{c^3\delta} + B_0A_0\right)X + d. \tag{3.22}$$

Substituting $a_0(X)$, $a_1(X)$, $a_2(X)$ and $g(X)$ in eq. (3.20) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = -\frac{(p^2+6r\lambda)}{\sqrt{6cr^2}}, \quad A_1 = \frac{2}{c}\sqrt{\frac{2r}{3\delta}},$$
$$B_0 = \frac{p}{c}\sqrt{\frac{2}{3\delta r}}, \quad n = \frac{pc(p^2+6r\lambda)}{12r^2}, \quad d = \frac{36\lambda^2r^2+12r\lambda p^2+p^4}{24c^2\delta r^3}. \tag{3.23}$$

Substituting (3.23) into (3.11), we obtain

$$Y(\xi) = -\sqrt{\frac{r}{6\delta}}\frac{(2rpX(\xi)+2r^2X^2(\xi)-6\lambda r-p^2)}{2r^2c}. \tag{3.24}$$

Combining (3.24) with (3.5), we obtain exact solution to (3.4) which can be written as

$$X(\xi) = -\frac{1}{2r^2}\left(rp - i\sqrt{3r^2p^2+12r^3\lambda} \times \tan\left(i\frac{(-\xi+C_2)}{2c}\sqrt{\frac{(p^2+4r\lambda)}{2r\delta}}\right)\right), \tag{3.25}$$

where C_2 is the integration constant. Thus the periodic wave solutions to the combined KdV–mKdV eq. (3.1) can be written as

$$u(x,t) = -\frac{1}{2r^2}\left(rp - i\sqrt{3r^2p^2+12r^3\lambda} \times \tan\left(i\frac{(-c(x-\lambda t)+C_2)}{2c}\sqrt{\frac{(p^2+4r\lambda)}{2r\delta}}\right)\right). \tag{3.26}$$

As a result, we find periodic and solitary wave solutions of the combined KdV–mKdV equation different from the soliton and Jacobi doubly periodic solutions, solitary wave solutions and travelling wave solutions which are found in [40,41,43], respectively.

4. The Pochhammer–Chree equation

Let us now consider the Pochhammer–Chree equation [44]

$$u_{tt} - u_{xxtt} - \left(\alpha u + \beta u^{n+1} + \gamma u^{2n+1}\right)_{xx} = 0, \tag{4.1}$$

where α, β and γ are real constants.

Equation (4.1) represents a non-linear model of the longitudinal wave propagation of elastic rods [45–50]. The model for $\alpha = 1$, $\beta = 1/(n+1)$ and $\gamma = 0$ was studied in [46,47] where solitary wave solutions for this model were obtained for $n = 1, 2$ and 4. Solitary wave solutions were obtained also for second model for $\alpha = 0$, $\beta = -\frac{1}{2}$ and $\gamma = 0$ which was studied by Parker [48].

We solved the equation for $n = 1$ in the form

$$u_{tt} - u_{xxtt} - \left(\alpha u + \beta u^2 + \gamma u^3\right)_{xx} = 0. \tag{4.2}$$

Using the transformation $U(x, t) = f(\xi)$, $\xi = k(x - \lambda t)$, eq. (4.2) is carried to a ODE

$$\begin{aligned} &k^2\lambda^2 f''(\xi) - k^4\lambda^2 f''''(\xi) - \alpha k^2 f''(\xi) - 2\beta k^2(f'(\xi)f'(\xi) + f(\xi)f''(\xi)) \\ &\quad -3\gamma k^2(2f(\xi)f'(\xi)f'(\xi) + f^2(\xi)f''(\xi)) = 0, \end{aligned} \tag{4.3}$$

where the prime denotes the derivation with respect to ξ. Integrating eq. (3.3) once and setting the constant of integration equal to zero we obtain

$$\begin{aligned} &k^2\lambda^2 f'(\xi) - k^4\lambda^2 f'''(\xi) - \alpha k^2 f'(\xi) - 2\beta k^2 f(\xi)f'(\xi) \\ &\quad -3\gamma k^2 f^2(\xi)f'(\xi) = 0. \end{aligned} \tag{4.4}$$

Then, integrating eq. (4.4) we obtain

$$k^4\lambda^2 f''(\xi) - k^2\lambda^2 f(\xi) + \alpha k^2 f(\xi) + \beta k^2 f^2(\xi) + \gamma k^2 f^3(\xi) = n, \tag{4.5}$$

where n is the integration constant.

Using (2.4) we get

$$X_\xi(\xi) = Y(\xi) \tag{4.6}$$

$$Y_\xi(\xi) = -\frac{-n + \alpha k^2 X(\xi) - k^2\lambda^2 X(\xi) + k^2\beta X^2(\xi) + k^2\gamma X^3(\xi)}{k^4\lambda^2}. \tag{4.7}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (4.6), (4.7), and $q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{4.8}$$

where $a_i(X)$, $i = 0, 1, ..., m$, are polynomials of X and $a_m(X) \neq 0$. Equation (4.8) is called the first integral to (4.6)–(4.7). Due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X, Y]$ such that

$$\frac{\mathrm{d}q}{\mathrm{d}\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = [g(X) + h(X)Y]\sum_{i=0}^{m} a_i(X)Y^i. \tag{4.9}$$

In this example, we take two different cases, by assuming $m = 1$ and $m = 2$ in eq. (4.8).

Case I: Suppose $m = 1$. By equating the coefficients of Y^i, $i = 0, 1, 2$, on both sides of eq. (4.9), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{4.10}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X), \tag{4.11}$$

$$a_1(X)\dot{Y} = g(X)a_0(X)$$
$$= a_1(X)\left(-\frac{-n+\alpha k^2 X - k^2\lambda^2 X + k^2\beta X^2 + k^2\gamma X^3}{k^4\lambda^2}\right). \quad (4.12)$$

Since $a_i(X)$, $i = 0, 1$, are polynomials, then from (4.10) we deduce that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1 X + B_0$, and $A_1 \neq 0$, then we find

$$a_0(X) = \frac{A_1}{2}X^2 + B_0 X + A_0. \quad (4.13)$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in eq. (4.12) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = -\frac{-2\beta^2 + 9\gamma\alpha - 9\gamma\lambda^2}{\sqrt{-2\gamma}9k\lambda\gamma}, \quad B_0 = -\frac{2\beta}{3k\lambda\sqrt{-2\gamma}},$$
$$A_1 = \frac{\sqrt{-2\gamma}}{\lambda k}, \quad n = -\frac{\beta k^2(-2\beta^2 + 9\gamma\alpha - 9\gamma\lambda^2)}{27\gamma^2}. \quad (4.14)$$

Substituting (4.14) into (4.8), we obtain

$$Y(\xi) = \frac{-2\beta^2 + 9\gamma\alpha - 9\gamma\lambda^2}{\sqrt{-2\gamma}9k\lambda\gamma} + \frac{2\beta}{3k\lambda\sqrt{-2\gamma}}X(\xi) - \frac{\sqrt{-2\gamma}}{2\lambda k}X^2(\xi). \quad (4.15)$$

Combining (4.15) with (4.6), we obtain the exact solution to (4.5) and then the exact solution to the Pochhammer–Chree equation can be written as

$$X(\xi) = \frac{1}{3\gamma^2}\left(-\beta\gamma + i\sqrt{3\beta^2\gamma^2 - 9\gamma^3\alpha + 9\gamma^3\lambda^2}\right.$$
$$\left.\times \tan\left(\frac{\sqrt{6\beta^2\gamma - 18\gamma^2\alpha + 18\gamma^2\lambda^2}(\xi + C_1)}{6k\lambda\gamma}\right)\right), \quad (4.16)$$

where C_1 is the integration constant. Thus the periodic wave solution to the Pochhammer–Chree equation (4.2) can be written as

$$u(x,t) = \frac{1}{3\gamma^2}\left(-\beta\gamma + i\sqrt{3\beta^2\gamma^2 - 9\gamma^3\alpha + 9\gamma^3\lambda^2}\right.$$
$$\left.\times \tan\left(\frac{\sqrt{6\beta^2\gamma - 18\gamma^2\alpha + 18\gamma^2\lambda^2}(k(x-\lambda t) + C_1)}{6k\lambda\gamma}\right)\right). \quad (4.17)$$

Case II: Suppose $m = 2$. By equating the coefficients of Y^i, $i = 0, 1, 2, 3$, on both sides of eq. (4.9), we have

$$\dot{a}_2(X) = h(X)a_2(X), \quad (4.18)$$

$$\dot{a}_1(X) = g(X)a_2(X) + h(X)a_1(X), \tag{4.19}$$

$$\dot{a}_0(X) = -2a_2(X)\left(-\frac{-n + \alpha k^2 X - k^2\lambda^2 X + k^2\beta X^2 + k^2\gamma X^3}{k^4\lambda^2}\right) + g(X)a_1(X) + h(X)a_0(X), \tag{4.20}$$

$$a_1(X)\dot{Y} = g(X)a_0(X) = a_1(X)\left(-\frac{-n + \alpha k^2 X - k^2\lambda^2 X + k^2\beta X^2 + k^2\gamma X^3}{k^4\lambda^2}\right). \tag{4.21}$$

Since $a_2(X)$ is a polynomial of X, then from (4.18) we deduce that $a_2(X)$ is constant and $h(X) = 0$. For simplicity, take $a_2(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1 X + B_0$, and $A_1 \neq 0$, then we find $a_1(X)$ and $a_0(X)$ as

$$a_1(X) = \frac{A_1}{2}X^2 + B_0 X + A_0, \tag{4.22}$$

$$a_0(X) = \left(\frac{A_1^2}{8} + \frac{\gamma}{2k^2\lambda^2}\right)X^4 + \left(\frac{2\beta}{3k^2\lambda^2} + \frac{A_1 B_0}{2}\right)X^3 + \left(\frac{\alpha}{k^2\lambda^2} - \frac{1}{k^2} + \frac{A_1 A_0}{2} + \frac{B_0^2}{2}\right)X^2 + \left(-\frac{2n}{k^4\lambda^2} + B_0 A_0\right)X + d. \tag{4.23}$$

Substituting $a_0(X)$, $a_1(X)$, $a_2(X)$ and $g(X)$ in eq. (4.21) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = -\frac{-2\beta^2 + 9\gamma\alpha - 9\gamma\lambda^2\sqrt{-2\gamma}}{9k\lambda\gamma^2}, \quad A_1 = \frac{2\sqrt{-2\gamma}}{\lambda k}, \quad B_0 = \frac{2\beta\sqrt{-2\gamma}}{3k\lambda\gamma},$$

$$d = -\frac{4\beta^4 - 36\alpha\gamma\beta^2 + 36\lambda^2\gamma\beta^2 + 81\alpha^2\gamma^2 - 162\alpha\gamma^2\lambda^2 + 81\lambda^4\gamma^2}{162k^2\lambda^2\gamma^3},$$

$$n = -\frac{\beta k^2(-2\beta^2 + 9\gamma\alpha - 9\gamma\lambda^2)}{27\gamma^2}. \tag{4.24}$$

Substituting (4.24) into (4.8), we obtain

$$Y(\xi) = -\frac{\sqrt{-2\gamma}(-9\lambda^2\gamma - 2\beta^2 + 9\alpha\gamma + 6\beta\gamma X(\xi) + 9\gamma^2 X^2(\xi))}{18\gamma^2\lambda k}. \tag{4.25}$$

Combining (4.25) with (4.6), we obtain exact solution to (4.5) and then the exact solutions to the Pochhammer–Chree equation can be written as

$$X(\xi) = -\frac{1}{3\gamma^2}\left(\beta\gamma - \sqrt{3\beta^2\gamma^2 - 9\gamma^3\alpha + 9\gamma^3\lambda^2} \times \tanh\left(-i\frac{\sqrt{2}\sqrt{3\beta^2\gamma^2 - 9\gamma^3\alpha + 9\gamma^3\lambda^2}(-\xi + C_2)}{6k\lambda\gamma^{(\frac{3}{2})}}\right)\right), \tag{4.26}$$

where C_2 is the integration constant. Thus the solitary wave solution to the Pochhammer–Chree equation (4.2) can be written as

$$u(x,t) = -\frac{1}{3\gamma^2}\left(\beta\gamma - \sqrt{3\beta^2\gamma^2 - 9\gamma^3\alpha + 9\gamma^3\lambda^2} \times \tanh\left(-i\frac{\sqrt{2}\sqrt{3\beta^2\gamma^2 - 9\gamma^3\alpha + 9\gamma^3\lambda^2}(-k(x-\lambda t)+C_2)}{6k\lambda\gamma^{(\frac{3}{2})}}\right)\right). \tag{4.27}$$

Thus we present a periodic and a solitary wave solution to Pochhammer–Chree equation in Case I and Case II respectively. These solutions are quite different from the travelling wave solutions found in [44].

5. The coupled nonlinear evolution equations

Let us now consider the coupled nonlinear evolution equations [51]

$$u_{xt} + v_x v_t = 0, \tag{5.1}$$

$$v_t + v_{xxx} + (v_x)^3 + 3u_{xx}v_x = 0. \tag{5.2}$$

Using the transformation

$$v(x,t) = v(\xi), \quad u(x,t) = u(\xi), \quad \xi = \alpha x - \lambda t \tag{5.3}$$

and substituting eq. (5.3) into eqs (5.1)–(5.2) yields

$$-\alpha\lambda u''(\xi) - \alpha\lambda(v'(\xi))^2 = 0, \tag{5.4}$$

$$-\lambda v'(\xi) + \alpha^3 v'''(\xi) + \alpha^3(v'(\xi))^3 + 3\alpha^3 v'(\xi)u''(\xi) = 0, \tag{5.5}$$

where the prime denotes the derivation with respect to ξ. From eq. (5.4) we obtain

$$u''(\xi) = -(v'(\xi))^2. \tag{5.6}$$

Substituting eq. (5.6) into eq. (5.5) we get

$$-\lambda v'(\xi) + \alpha^3 v'''(\xi) + \alpha^3(v'(\xi))^3 - 3\alpha^3(v'(\xi))^3 = 0. \tag{5.7}$$

Introducing $h(\xi) = v'(\xi)$ as a new dependent variable, we obtain

$$-\lambda h(\xi) + \alpha^3 h''(\xi) - 2\alpha^3 h^3(\xi) = 0. \tag{5.8}$$

Using (2.4) we get

$$X_\xi(\xi) = Y(\xi) \tag{5.9}$$

$$Y_\xi(\xi) = \frac{X(\xi)(\lambda + 2\alpha^3 X^2(\xi))}{\alpha^3}. \tag{5.10}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (5.9), (5.10), and $q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{5.11}$$

where $a_i(X)$, $i = 0, 1, ..., m$, are polynomials of X and $a_m(X) \neq 0$. Equation (5.11) is called the first integral to (5.9)–(5.10). Due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X, Y]$ such that

$$\frac{\mathrm{d}q}{\mathrm{d}\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = [g(X) + h(X)Y]\sum_{i=0}^{m} a_i(X)Y^i. \tag{5.12}$$

In this example, we take two different cases, by assuming $m = 1$ and $m = 2$ in eq. (5.11).

Case I: Suppose $m = 1$. By equating the coefficients of Y^i, $i = 0, 1, 2$, on both sides of eq. (5.12), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{5.13}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X), \tag{5.14}$$

$$a_1(X)\dot{Y} = g(X)a_0(X) = a_1(X)\left(\frac{X(\lambda + 2\alpha^3 X^2)}{\alpha^3}\right). \tag{5.15}$$

Since $a_i(X)$, $i = 0, 1$, are polynomials, then from (5.13) we deduce that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, and $A_1 \neq 0$, then we find

$$a_0(X) = \frac{A_1}{2}X^2 + B_0X + A_0. \tag{5.16}$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in eq. (5.15) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = \pm\frac{\lambda}{2\alpha^3}, \quad A_1 = \pm 2, \quad B_0 = 0. \tag{5.17}$$

Using (5.17) into (5.11), we obtain

$$Y(\xi) = \mp\frac{\lambda}{2\alpha^3} \mp X^2(\xi). \tag{5.18}$$

Combining (5.18) with (5.9), we obtain exact solutions to (5.4)–(5.5) and then the exact solutions can be written as

$$h(\xi) = \mp\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}\tan\left(\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(\xi + C_1)\right), \tag{5.19}$$

$$v(\xi) = \mp\frac{1}{2}\ln\left(1 + \tan\left(\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(\xi + C_1)\right)^2\right) + C_2, \tag{5.20}$$

$$u(\xi) = -\frac{1}{2}\ln\left(1 + \tan\left(\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(\xi + C_1)\right)^2\right) + \frac{1}{2}\left(\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(\xi + C_1)\right)^2 + C_3\xi + C_4, \tag{5.21}$$

where C_1, C_2, C_3, C_4 are integration constants. Thus the periodic wave solutions to the coupled nonlinear evolution equations (5.1), (5.2) can be written as

$$v(x,t) = \mp\frac{1}{2}\ln\left(1 + \tan\left(\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(\alpha x - \lambda t + C_1)\right)^2\right) + C_2, \tag{5.22}$$

$$u(x,t) = -\frac{1}{2}\ln\left(1 + \tan\left(\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(\alpha x - \lambda t + C_1)\right)^2\right) + \frac{1}{2}\left(\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(\alpha x - \lambda t + C_1)\right)^2 + C_3(\alpha x - \lambda t) + C_4. \tag{5.23}$$

Case II: Suppose $m = 2$. By equating the coefficients of Y^i, $i = 0, 1, 2, 3$, on both sides of eq. (5.12), we have

$$\dot{a}_2(X) = h(X)a_2(X), \tag{5.24}$$

$$\dot{a}_1(X) = g(X)a_2(X) + h(X)a_1(X), \tag{5.25}$$

$$\dot{a}_0(X) = -2a_2(X)\left(\frac{X(\lambda + 2\alpha^3X^2)}{\alpha^3}\right) + g(X)a_1(X) + h(X)a_0(X), \tag{5.26}$$

$$a_1(X)\dot{Y} = g(X)a_0(X) = a_1(X)\left(\frac{X(\lambda + 2\alpha^3X^2)}{\alpha^3}\right). \tag{5.27}$$

Since $a_2(X)$ is a polynomial of X, then from (5.24) we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, take $a_2(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1 X + B_0$, and $A_1 \neq 0$, then we find $a_1(X)$ and $a_0(X)$ as

$$a_1(X) = \frac{A_1}{2} X^2 + B_0 X + A_0, \tag{5.28}$$

$$a_0(X) = \left(\frac{A_1^2}{8} - 1\right) X^4 + \frac{A_1 B_0}{2} X^3 + \left(-\frac{\lambda}{\alpha^3} + \frac{A_1 A_0}{2} + \frac{B_0^2}{2}\right) X^2 + B_0 A_0 X + d. \tag{5.29}$$

Substituting $a_0(X)$, $a_1(X)$, $a_2(X)$ and $g(X)$ in eq. (5.27) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = \pm \frac{\lambda}{\alpha^3}, \quad A_1 = \pm 4, \quad B_0 = 0, \quad d = \frac{\lambda^2}{4\alpha^6}\,. \tag{5.30}$$

Using (5.30) into (5.11), we obtain

$$Y(\xi) = \mp \frac{\lambda + 2\alpha^3 X^2}{2\alpha^3}. \tag{5.31}$$

Combining (5.31) with (5.9), we obtain the exact solutions to (5.4)–(5.5) and then the exact solutions can be written as

$$h(\xi) = \pm i \frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}} \tanh\left(-i\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(-\xi + C_5)\right), \tag{5.32}$$

$$v(\xi) = \mp \frac{1}{2} \ln\left(1 + i \tanh\left(-i\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(-\xi + C_5)\right)^2\right) + C_6, \tag{5.33}$$

$$u(\xi) = -\frac{1}{2} \ln\left(1 + i \tanh\left(-i\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(-\xi + C_5)\right)^2\right) + \frac{1}{2}\left(\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(-\xi + C_5)\right)^2 + C_7 \xi + C_8, \tag{5.34}$$

where C_5, C_6, C_7, C_8 are integration constants. Thus the solitary wave solutions to the coupled nonlinear evolution equations (5.1), (5.2) can be written as

$$v(x,t) = \mp \frac{1}{2} \ln\left(1 + i \tanh\left(-i\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(\lambda t - \alpha x + C_5)\right)^2\right) + C_6, \tag{5.35}$$

$$u(x,t) = -\frac{1}{2} \ln\left(1 + i \tanh\left(-i\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(\lambda t - \alpha x + C_5)\right)^2\right) + \frac{1}{2}\left(\frac{1}{\alpha}\sqrt{\frac{\lambda}{2\alpha}}(\lambda t - \alpha x + C_5)\right)^2 + C_7(\lambda t - \alpha x) + C_8. \tag{5.36}$$

Thus, we show a periodic and a solitary wave solution to the coupled nonlinear evolution equations in Case I and Case II respectively. Travelling wave solutions of the coupled nonlinear evolution equations are shown in [51].

Remark 1. By assuming $m = 3, 4$ in eqs (3.7), (4.8) and (5.11), respectively, using similar arguments as earlier we obtain eqs (3.6), (4.7) and (5.10) that do not have any first integral in the form (3.7), (4.8) and (5.11). We do not need to consider the case $m \geq 5$ because an algebraic equation with the degree greater than or equal to 5 is generally not solvable.

Remark 2. With the aid of Maple, we have verified all solutions we obtained in §3–5, by putting them back into the original eqs (3.1), (4.1) and (5.1)–(5.2).

Remark 3. The obtained travelling wave solutions can be converted into periodic and solitary wave solutions.

6. Conclusion

In this paper, the first integral method was employed to obtain some new as well as some known solutions of a selected set of nonlinear equations. Thus, we conclude that the proposed method can be extended to solve the nonlinear problems which arise in the theory of solitons and other areas. We foresee that our results can be found potentially useful for applications in mathematical physics and engineering problems including numerical simulation. Though the obtained solutions represent only a small part of the large variety of possible solutions for the equations considered, they might serve as seeding solutions for a class of localized structures existing in the physical phenomena.

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