PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 78, No. 6
June 2012
pp. 907–925

Algebraic study of chiral anomalies†

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Abstract. The algebraic structure of chiral anomalies is made globally valid on non-trivial bundles by the introduction of a fixed background connection. Some of the techniques used in the study of the anomaly are improved or generalized, including a systematic way of generating towers of 'descent equations'.

Keywords. Chiral anomalies; gauge theories; bundles; connections; quantum field theory.

PACS Nos **11.10.−z; 11.10.Cd; 11.10.Ef; 11.10.Gh; 11.10.Hi; 11.15.Bt; 11.55.Hx; 12.38.Cy**

1. Introduction

Chiral anomalies have been studied at a slow pace over a period of almost 15 years during most of which the general lack of interest following the active pioneering period [1,2] (the pioneering period is extensively covered in [1]. From this period, we shall however select out [2], especially relevant to the subject of this paper) did not stimulate very active efforts [3–6]. Recent revival [7,8] of the subject has, however, encouraged us [9–13] to develop further some of the methods which slowly emerged and cast the results into a form suitable to make contact with the recent mathematical understanding of the connections between some of the algebraic structures which have been discovered and the topology of gauge field orbit spaces and of gauge groups [14–19].

In this paper, we shall limit ourselves to the algebraic aspects of the structure of chiral anomalies, but, by introduction of a background field (fixed connection), we shall extend

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Editor's Note:
†Reproduced with kind permission from Springer Science+Business Media: Algebraic study of chiral anomalies, Juan Mañes, Raymond Stora and Bruno Zumino, *Communications in Mathematical Physics* **102**, 157–174 (1985) Springer-Verlag. Even though at variance with normal *Pramana* policy, we are proud to reproduce this article of exceptional quality and lasting scientific value. It marks a milestone in the joint work of Raymond Stora and B Zumino, addressing issues on symmetries in gauge theories.

DOI: 10.1007/s12043-012-0316-3; ***e*Publication: 14 June 2012**

the local results so far obtained in such a way that they become globally valid on non-trivial bundles. This gives new insight into the problem and is also of physical interest, in particular in the gravitational case, when non-parallelizable manifolds are considered [20,21].

Section 2 is devoted to the description of the two main technical tools to be used in the sequel: the 'Russian' formula and the extended Cartan homotopy formula. It ends by the statement of a straightforward application, the 'triangle formula' which will be repeatedly used in the sequel.

Section 3 goes over the definition of anomalies through the Wess–Zumino (WZ) consistency conditions [2] which are stated in terms of cohomology. There follows the writing of the corresponding Wess–Zumino–Witten (WZW) action [2,7] in three equivalent forms.

Section 4 treats in detail the problem arising when the anomaly vanishes on a subalgebra Lie K of the structure Lie algebra Lie G [7,12] and the corresponding Bardeen action together with the covariant form of the vertex anomaly are exhibited. The chiral case, where the structure group G is a direct $G_R \times G_L$ of two isomorphic factors, and where the diagonal anomaly vanishes [22–26], is treated in detail.

2. Technical equipment

For a gauge theory with structure group G, a compact Lie group, we shall be concerned with a principal bundle $P(M, G)$ (an elementary exposition can be found in [6]), where M is of even dimension $d = 2n - 2$, compact, without boundary. Connections on $P(M, G)$ are represented locally by one-forms with values in the Lie algebra Lie G of G. Gauge transformations of $P(M, G)$ are locally represented on M by functions into G with suitable gluing properties. They form a group $\mathscr{G}$ which acts on the (affine) space of connections A:

$$g \in \mathscr{G}, \qquad A_g = g^{-1}Ag + g^{-1}\mathrm{d}g. \tag{1}$$

The curvature F of A is defined by (notice that in this paper the bracket is defined $[A, B] \equiv AB - (-)^{ab}BA$, where $a(b)$ is 1 if $A(B)$ is an anticommuting element and zero otherwise)

$$F(A) = \mathrm{d}A + \frac{1}{2}[A,\ A]. \tag{2}$$

Then

$$F\left(A_g\right) \equiv F_g(A) = g^{-1}\,F(A)\,g. \tag{3}$$

The Lie algebra Lie $\mathscr{G}$ of the gauge group is locally represented by functions to Lie G with the bracket law

$$\begin{matrix} u_1, u_2 \in \mathscr{G} \\ x \in M \end{matrix} \qquad [u_1, u_2](x) = [u_1(x), u_2(x)]. \tag{4}$$

Expressions involving connections, their curvature, gauge transformations and infinitesimal gauge transformations (elements of Lie $\mathscr{G}$), are globally defined provided they are locally gauge invariant, i.e., invariant under

$$\left.\begin{array}{l} A^i \to A^i_h \\ F(A^i) \to F_h(A^i) \\ g^i \to h^{-1} g^i h \\ u^i \to h^{-1} u^i h \end{array}\right. \quad h\text{: local map } (M \to G)\,. \tag{5}$$

We shall see in §3 that the definition of anomalies goes through the consideration of the cohomology algebra $H^*(\text{Lie}\,\mathscr{G},\ \Gamma_{\text{loc}})$ of Lie $\mathscr{G}$ with values in the representation space Γ_{loc} of $\mathscr{G}$ consisting in local functionals of a given set of connections, i.e., integrated locally gauge invariant polynomials of the coefficients of these connections and their derivatives. This is a graded commutative differential algebra [27] defined by the structure equations (this is the geometric part of the BRST algebra [4], which has an additional contractible piece involving the second Faddeev–Popov ghost $\bar{v}$ and the gauge fixing Lagrange multiplier γ: $\mathscr{S}\bar{v} = \gamma,\ \mathscr{S}\gamma = 0$)

$$\begin{aligned} &\mathscr{S} v = -\frac{1}{2}[v, v], \\ &\mathscr{S} A = -\mathrm{d}v - [A, v] = -D(A)v, \\ &\mathscr{S} d + d\mathscr{S} = 0, \qquad \mathscr{S}^2 = 0, \end{aligned} \tag{6}$$

where $\mathscr{S}$ is the appropriate coboundary operator and v, which generates $H^*(\text{Lie}\,\mathscr{G})$, is what physicists call the geometric Faddeev–Popov ghost, whereas A, or possibly several of them, generate Γ_{loc}.

The operator $\mathscr{S}$ can also be interpreted (under the homomorphism which maps H^* (Lie $\mathscr{G}$) into $H^*_{\text{deRham}}(\mathscr{G})$ [28]) as an antiderivation with respect to a set of parameters λ_1, λ_2,... upon which the group element $g(x, \lambda)$ may depend [11] since, if one considers that also, the connection A depends on these parameters through

$$A(x, \lambda) \equiv g^{-1}(x, \lambda) A(x) g(x, \lambda) + g^{-1}(x, \lambda)\, \mathrm{d}g\,(x, \lambda)\,, \tag{7}$$

and the Faddeev–Popov ghost is defined by

$$v = g^{-1} \mathscr{S} g, \tag{8}$$

then eqs (6) follow immediately.

A convenient change of generators is to go from (v, A) to $(v, A + v)$, and from d to the total differential $\mathrm{d} + \mathscr{S}$ (for an interpretation of $A + v$ as a connection on $P(M, G) \times \mathscr{G}$, see a forthcoming paper by R Coquereaux). Then, by virtue of the structure eqs (6) one has the 'Russian formula' [11,12,19]

$$\begin{aligned} \mathscr{F}(A + v) &\equiv (d + \mathscr{S})(A + v) + \frac{1}{2}[A + v, A + v] \\ &= \mathrm{d}A + \frac{1}{2}[A, A] = F(A). \end{aligned} \tag{9}$$

Of course, the Bianchi identity holds,

$$\mathrm{d}F(A) + [A, F(A)] = (d + \mathscr{S})\mathscr{F}(A + v) + [A + v, \mathscr{F}(A + v)] = 0. \tag{10}$$

This purely algebraic formulation easily extends to the consideration of the Lie algebra of vector fields on M needed to describe gravitational anomalies [8,13,20,21], whereas connection can be made with the topological considerations of [14–16] by identifying v with the Maurer–Cartan form of $\mathscr{G}$ and $\mathscr{S}$ with the differential on $\mathscr{G}$ as mentioned above.

Sometimes we shall consider the connected component of the identity in $\mathscr{G}$, which will be denoted as $\mathscr{G}_0$. It is exponentiable if G is simply connected, which we shall assume, when needed.

We now turn to the extended Cartan homotopy formula. Consider a family of connections smoothly parametrized by a set of variables t_l, t_2, ... which we shall denote $A_t(x)$. Besides the usual antiderivation d with respect to x, we introduce an antiderivation d_t with respect to the parameters $\{t\}$ and an even operator ℓ_t [11] defined in such a way that the following graded algebra is satisfied:

$$\begin{aligned} &d^2 = d_t^2 = dd_t + d_t d = 0, \\ &d_t = \ell_t d - \mathrm{d}\ell_t, \\ &d_t \ell_t - \ell_t d_t = 0. \end{aligned} \tag{11}$$

The operator ℓ_t is a homotopy derivation which increases the degree in dt by one and decreases the degree in dx by one. Its action on the algebra of polynomials generated by a particular set of forms will be defined so that (11) is satisfied and the algebra of polynomials is stable under the application of d, d_t and ℓ_t. It is easy to check that the unique action of ℓ_t on polynomials in $\{A_t, F_t \equiv dA_t + \frac{1}{2}[A_t, A_t], d_t A_t, d_t F_t\}$ satisfying these requirements is given by

$$\ell_t F_t = d_t A_t, \quad \ell_t A_t = \ell_t d_t A_t = \ell_t d_t F_t = 0. \tag{12}$$

The general problem of defining the action of ℓ_t on different algebras of polynomials will be considered in the Appendix.

From eqs (11) it follows immediately that

$$[f(\ell_t), d] = d_t f'(\ell_t) = f'(\ell_t)\, d_t \tag{13}$$

for $f(\ell_t)$, a polynomial in ℓ_t. Taking $f(\ell_t) = \mathrm{e}^{\ell_t}$, as given by its Taylor expansion, we obtain from eq. (13),

$$\mathrm{e}^{\ell_t} d - d\mathrm{e}^{\ell_t} = d_t \mathrm{e}^{\ell_t} = \mathrm{e}^{\ell_t} d_t. \tag{14}$$

If $\mathscr{Q}$ is a polynomial in the forms $\{A_t, F_t, d_t A_t, d_t F_t\}$ (or in any other set of forms on which the action of ℓ_t has been consistently defined), eq. (14) can be written as

$$(d + d_t)\mathrm{e}^{\ell_t} \mathscr{Q} = \mathrm{e}^{\ell_t} \mathrm{d}\mathscr{Q}. \tag{15}$$

Expanding both sides of this equation, we obtain

$$d_t \frac{\ell_t^p}{p!} \mathscr{Q} = \frac{\ell_t^{p+1}}{(p+1)!} \mathrm{d}\mathscr{Q} - d \frac{\ell_t^{p+1}}{(p+1)!} \mathscr{Q}. \tag{16}$$

This expression can be integrated (for fixed x) over a domain T in the space of parameters $\{t\}$ with boundary ∂T. Since the integrand is a form both in $\{x\}$ and $\{t\}$, we need to establish a convention for this 'incomplete' integration. For α a form of degrees (r, s) in $(\mathrm{d}x, \mathrm{d}t)$ we adopt the following definitions:

$$\int_{X_r} \alpha \equiv \int_{X_r} \alpha_{r,s} \mathrm{d}x^r \mathrm{d}t^s = \left(\int_{X_r} \alpha_{r,s} \mathrm{d}x^r \right) \mathrm{d}t^s,$$
$$\int_{T_s} \alpha \equiv \int_{T_s} \alpha_{r,s} \mathrm{d}x^r \mathrm{d}t^s = (-)^{rs} \left(\int_{T_s} \alpha_{r,s} \mathrm{d}t^s \right) \mathrm{d}x^r, \tag{17}$$

i.e., the non-integrated differentials are taken out of the integral to the right. It is easy to see that this convention implies the following rules:

$$d \int_{T_s} \alpha = (-)^s \int_{T_s} \mathrm{d}\alpha, \quad d_t \int_{X_r} \alpha = (-)^r \int_{X_r} d_t \alpha, \tag{18}$$

whereas Stokes theorem keeps its familiar form:

$$\int_{X_{r+1}} \mathrm{d}\alpha = \int_{\partial X_{r+1}} \alpha, \quad \int_{T_{s+1}} d_t \alpha = \int_{\partial T_{s+1}} \alpha. \tag{19}$$

Had we adopted the opposite convention (differentials out to the left), Stokes theorem would have picked up additional signs. Integration of eq. (16) with the above convention gives

$$\int_{\partial T} \frac{\ell_t^p}{p!} \mathscr{Q} = \int_T \frac{\ell_t^{p+1}}{(p+1)!} \mathrm{d}\mathscr{Q} + (-)^{p+q} \mathrm{d} \int_T \frac{\ell_t^{p+1}}{(p+1)!} \mathscr{Q}, \tag{20}$$

where q is the degree of $\mathscr{Q}$ in $\{\mathrm{d}t\}$.

Equations (16) and (20) are the extended Cartan homotopy formula in differential and integral forms respectively. They are valid for any $\mathscr{Q}$ belonging to an algebra of polynomials on which the action of ℓ_t has been consistently defined and for any parametrization, and they include as a special case the ordinary Cartan homotopy formula (eq. (24)) below. They also contain the following particular cases:

(1) If $\mathscr{Q}$ is a polynomial in A_t and F_t closed with respect to x, i.e., $\mathrm{d}\mathscr{Q} = 0$, then by eq. (15) we know that $\mathrm{e}^{\ell_t} \mathscr{Q}$ is closed with respect to the total differential operator $d + d_t$. Equation (20) reduces to

$$\int_{\partial T} \frac{\ell_t^p}{p!} \mathscr{Q} = (-)^p \, \mathrm{d} \int_T \frac{\ell_t^{p+1}}{(p+1)!} \mathscr{Q}. \tag{21}$$

This new set of descent equations has been studied in [28,29] in the case where $\mathscr{Q}$ is a symmetric invariant polynomial in F, and T a $(p+1)$-simplex with A_t given as a convex combination of connections A^i

$$A_t = \sum_{i=0}^{p+1} t_i A^i, \quad \sum_{i=0}^{p+1} t_i = 1. \tag{22}$$

(2) For A parametrized as in eq. (7) we have $d_\lambda \equiv \mathscr{S}$. If we take $\mathscr{Q} \equiv \omega_{2n-1}$, with $\mathrm{d}\omega_{2n-1}$ an invariant symmetric polynomial in F and we consider the action of ℓ_λ on the algebra of polynomials in $\{A, F, v, \mathrm{d}v\}$ (v is the geometric Faddeev–Popov

ghost), then eq. (16) becomes the ordinary 'descent equations' for the forms ω^p_{2n-1-p} that we shall consider in §3 (see also [11,12])

$$\mathscr{S}\omega^p_{2n-1-p} = -\mathrm{d}\omega^{p+1}_{2n-2-p}. \tag{23}$$

The integral eq. (20) gives the relations among the cocycles recently defined in [28, 31–34]. See the Appendix for the appropriate definition of ℓ_λ and a detailed derivation of these results.

(3) Since they will be used repeatedly in what follows, we consider in detail the first two equations in (21), with the restrictions indicated above. For $p = 0$, $\mathscr{Q} \equiv P(F^n_t)$ a symmetric invariant polynomial and $A_t = tA_2 + (1-t)\,A_1$, we get

$$\begin{aligned} P(F^n_2) - P(F^n_1) &= nd\int_{T_1} P(d_t A_t F^{n-1}_t) = nd\int_0^1 \mathrm{d}t\, P(A_2 - A_1, F^{n-1}_t) \\ &\equiv \mathrm{d}\omega_{2n-1}(A_2, A_1), \end{aligned} \tag{24}$$

where $\mathrm{d}t$ is an ordinary differential. Equation (24) is of course the ordinary Chern–Weil version of the Cartan homotopy formula [11,12]. Notice that $\omega_{2n-1}(A_2, A_1)$ is invariant under simultaneous gauge transformations of A_1 and A_2. For $p = 1$, $A_t = t_1A_1 + t_2A_2 + (1 - t_1 - t_2)A_3$ and T_2 the corresponding simplex, we get

$$\begin{aligned} -\int_{\partial T_2} \ell_t P(F^n_t) &= \omega_{2n-1}(A_1, A_2) + \omega_{2n-1}(A_2, A_3) + \omega_{2n-1}(A_3, A_1) \\ &= \frac{n(n-1)}{2} d\int_{T_2} SP(d_tA_t, d_tA_t, F^{n-2}_t) \\ &= \frac{n(n-1)}{2} d\int_0^1 \mathrm{d}t_1 \int_0^{1-t_1} \mathrm{d}t_2 SP(A_2 - A_3, A_1 - A_3, F^{n-2}_t) \\ &\equiv \mathrm{d}\chi(A_1, A_2, A_3), \end{aligned} \tag{25}$$

where $\mathrm{d}t_1$ and $\mathrm{d}t_2$ are ordinary differentials, and SP is the symmetrized form of the polynomial P (see [9,11]). Equation (25) will be used very often in the rest of this paper, and we shall refer to it as the 'triangle formula'. This formula had been used previously in [11,24] with a different derivation.

3. Chiral anomalies as elements of H^1 (Lie $\mathscr{G}$, Γ_{loc})

In the known field theory models involving a gauge field A, and possibly a fixed background gauge field A_0 (in the sequel we shall *not* transform A_0, i.e., $\mathscr{S}A_0 = 0$) whenever $P(M, G)$ is not trivial, anomalies appear as the right-hand side of an anomalous Ward identity [2,5]

$$\mathscr{S}\Gamma(\cdot, A, A_0) = \int_M \mathscr{A}(v; A, A_0)\,, \tag{26}$$

where $\Gamma(\cdot, A, A_0)$ is the vertex functional of the theory under consideration in which the dot collectively denotes all other fields, which transform linearly under $\mathscr{G}$. $\mathscr{A}(v; A, A_0)$

is linear in v and depends locally on A and A_0. Thus, from the algebraic property $\mathscr{S}^2 = 0$ we get the consistency condition

$$\int \mathscr{S}\mathscr{A}(v; A, A_0) = 0, \tag{27}$$

which characterizes $\mathscr{A}(v; A, A_0)$ as a representative of an element of $H^1(\text{Lie}\,\mathscr{G}, \Gamma_{\text{loc}})$ since $\Gamma(\cdot, A, A_0)$ is ambiguous up to local counterterms consistent with power counting and other symmetry laws as implied by renormalization theory. This mere fact has to be stressed since it implies that, in general, there is no standard formula for $\mathscr{A}(v; A, A_0)$. $\mathscr{A}(v; A, A_0)+\mathscr{S}\Gamma_{\text{loc}}+\text{d}\chi$ is just as good a candidate if Γ_{loc} is an admissible counterterm, χ a local form. We shall see in the sequel several examples in which this ambiguity helps the anomaly to assume quite different disguises, not to speak of the case of gravitational anomalies [8,13,20,21] which will not be covered here.

Although there is only one case in which $H^1(\text{Lie}\,\mathscr{G}, \Gamma_{\text{loc}})$ has been computed [5], namely the case of perturbatively renormalizable theories in four dimensions, a large class of solutions of the consistency conditions is known, which it is fair to call the Adler–Bardeen [25] class, and may very well exhaust the set of all solutions (as this paper was being completed, M Dubois-Violette, M Talon, C M Viallet kindly informed us that they had computed $H^*(\text{Lie}\,\mathscr{G}, \mathscr{A})$ where $\mathscr{A}$ is the space of local funcfionals of A, $F(A)$, confirming the general belief if G involves at most one $U(1)$ factor). It is obtained as follows: Consider symmetric polynomials of degree n on Lie G, invariant under the adjoint action of G (these are tabulated for all compact simple groups and can therefore be obtained for all reductive groups). Then, a simultaneous application of the Russian formula (17) and the Cartan homotopy formula (24) yields

$$\begin{aligned} P(F^n(A)) - P(F^n(A_0)) &= n\,(d + \mathscr{S}) \int_0^1 \text{d}t\, P(A + v - A_0, \mathscr{F}(A_t)) \\ &= (d + \mathscr{S})\omega_{2n-1}(A + v, A_0)\,, \end{aligned} \tag{28}$$

where $A_t = t(A + v) + (1 - t)A_0$ and

$$\mathscr{F}(A_t) = (d + \mathscr{S})A_t + \frac{1}{2}\,[A_t, A_t]\,. \tag{29}$$

Expanding ω_{2n-1} in powers of v,

$$\omega_{2n-1}(A + v, A_0) = \sum_{p=0}^{2n-1} \omega^p_{2n-1-p}(v; A, A_0), \tag{30}$$

where the lower index denotes the form degree and the upper index denotes the power of v (the degree in $\{\lambda\}$ space) that is involved, we get

$$\begin{aligned} &P(F^n\,(A)) - P(F^n\,(A_0)) = \text{d}\omega^0_{2n-1}, \\ &\mathscr{S}\omega^p_{2n-1-p} = -\text{d}\omega^{p+1}_{2n-2-p}, \quad p = 0, 1, \ldots, 2n-2, \\ &\mathscr{S}\omega^{2n-1}_0 = 0. \end{aligned} \tag{31}$$

This is the set of 'descent equations' considered for instance, in [11], generalized to the case in which there is a background field. This shows in particular that

$$\mathscr{A}(v; A, A_0) = \omega^1_{2n-2}(v; A, A_0) \tag{32}$$

solves the consistency condition (27).

Remark that all formulae so far written are global on $P(M, G)$ and that only for a trivial bundle one can choose $A_0 = 0$ and recover the usual local formulae [11,12]. Also for two different background fields A_0^1, A_0^2, the anomalies differ by a coboundary. Combining the Russian formula with the 'triangle formula', we have

$$\begin{aligned}&\omega_{2n-1}(A_0^1, A+v) + \omega_{2n-1}(A+v, A_0^2) + \omega_{2n-1}(A_0^2, A_0^1)\\&\quad = (d+\mathscr{S})\chi(A_0^2, A_0^1, A+v).\end{aligned} \tag{33}$$

The term linear in v gives the difference of the anomalies in background fields A_0^1, A_0^2 as an allowed ambiguity

$$\begin{aligned}&\omega_{2n-2}^1(v; A, A_0^2) - \omega_{2n-2}^1(v; A, A_0^1)\\&\quad = \mathscr{S}\chi\big(A_0^2, A_0^1, A\big) + \mathrm{d}\chi_{2n-3}^1(A_0^2, A_0^1, A+v).\end{aligned} \tag{34}$$

In a recent paper [32] a new 'simpler' expression for $\omega_{2n-1-p}^p(v;\, A, 0)$ has been proposed. This new expression has a reduced dependence in the field A, attained through the inclusion of powers of $\mathrm{d}v$ which contribute to the form degree. This can be generalized to the presence of a background field by expanding instead of $\omega_{2n-1}(A+v, A_0)$ the following form:

$$\begin{aligned}\hat{\omega}_{2n-1}(v; A, A_0) &\equiv \omega_{2n-1}(A+v, A_0+v) + \omega_{2n-1}(A_0+v, A_0)\\&= n\int_0^1 \mathrm{d}t\, P(A-A_0, \mathscr{F}^{n-1}(A_t))\\&\quad + n\int_0^1 \mathrm{d}\mu\, P(v, \mathscr{F}^{n-1}(A_\mu)),\end{aligned} \tag{35}$$

where $A_t = tA + (1-t)A_0 + v$, $A_\mu = \mu v + A_0$ and $\mathscr{F}$ are given by eq. (29). By the 'triangle formula' the relation between $\hat{\omega}_{2n-1}$ and ω_{2n-1} is

$$\begin{aligned}\hat{\omega}_{2n-1}(v; A, A_0) &= \omega_{2n-1}(A+v, A_0)\\&\quad + (d+\mathscr{S})\chi(A+v, A_0+v, A_0).\end{aligned} \tag{36}$$

This means that the new $\hat{\omega}_{2n-1-p}^p$ will differ from the old ones by allowed ambiguities [12,22].

Now, it is a remarkable fact [2] that, although $\omega_{2n-2}^1(v; A, A_0)$ represents a nontrivial class in $H^1(\text{Lie}\,\mathscr{G}, \Gamma_{\text{loc}})$, this class can be 'killed' by enlarging $\Gamma_{\text{loc}}(A, A_0)$ into $\Gamma_{\text{loc}}(g; A, A_0)$, where g belongs to $\mathscr{G}_0$. We define the action of $\mathscr{L}$ on g by

$$\mathscr{S}g = -vg. \tag{37}$$

We lift the whole situation to $P(M\times[0,1], G) = P(M, G)\times[0,1]$ by considering a family of connections A_t on $P(M, G)$ such that

$$A_t = A_0 \text{ for } t = 0, \quad A_1 = A, \tag{38}$$

and a family g_t of gauge transformations satisfying

$$g_0 = id\mathscr{G}, \quad g_1 = g. \tag{39}$$

We continue v, $\mathscr{S}$ into V, $\mathbf{S}$ such that

$$\begin{aligned}
\mathbf{S}V &= -\frac{1}{2}\,[V, V]\,, \\
\mathbf{S}A_t &= -d_{\text{tot}}V - [A_t, V]\,, \\
\mathbf{S}g_t &= -V g_t\,,
\end{aligned} \tag{40}$$

and define

$$\Gamma_{wzw}\,(g_t; A_t, A_0) = \int_{Mx[0,1]} (\omega^0_{2n-1}\,(A_t, A_0) - \omega^0_{2n-1}(A_{tg_t}, A_0)), \tag{41}$$

where $d_{\text{tot}} = d + d_t$ is involved everywhere.

First we have

$$\mathbf{S}\Gamma_{wzw}\,(g_t; A_t, A_0) = \int_M \mathscr{A}(v; A_t, A_0)\,. \tag{42}$$

This is because, by construction, the second term in the right-hand side of eq. (41) is invariant under $\mathbf{S}$, and

$$\mathbf{S}\omega^0_{2n-1}(A_t, A_0) = d_{\text{tot}}\mathscr{A}(V; A_t, A_0)\,. \tag{43}$$

The result follows by the application of Stokes theorem and the vanishing of the integrand for $t = 0$ (notice that the rule given by (18) has to be used, with $\mathbf{S}$ instead of d_t).

Secondly, we shall show that, for a suitable choice of g_t, Γ_{wzw} can be expressed as an integral over M of a functional which is manifestly local in the gauge potential A, but not obviously local in g. To this end, we can write

$$\begin{aligned}
\Gamma_{wzw} &= \int_{Mx[0,1]} (\omega^0_{2n-1}\,(A_t, A_0) - \omega^0_{2n-1}(A_{tg_t}, A_0)) \\
&= -\int_0^1 \mathrm{d}s \frac{\partial}{\partial s} \int_{Mx[0,1]} \omega^0_{2n-1}(A_{tg_t(s)}, A_0) \\
&= -\int_0^1 \mathbf{S} \int_{Mx[0,1]} \omega^0_{2n-1}(A_{tg_t(s)}, A_0),
\end{aligned} \tag{44}$$

where $g_t(s)$ is a family such that $g_t(0) = id\mathscr{G}$, $g_t(1) = g_t$.

The latter expression reads [2],

$$\begin{aligned}
\Gamma_{wzw} &= -\int_0^1 \mathrm{d}s \int_{Mx[0,1]} d_{\text{tot}}\mathscr{A}\left(g_t^{-1}(s)\frac{\partial g_t(s)}{\partial s}; A_{tg_t(s)}, A_0\right) \\
&= -\int_0^1 \mathrm{d}s \int_M \mathscr{A}\left(g_1^{-1}(s)\frac{\partial g_1(s)}{\partial s}; A_{g_1(s)}, A_0\right),
\end{aligned} \tag{45}$$

where, after commuting $\mathbf{S}$ with the integral over $M \times [0, 1]$ (see eq. (18)), we have contracted

$$\mathbf{S}\omega^0_{2n-1} = -d_{\text{tot}}\omega^1_{2n-2} = -d_{\text{tot}}\mathscr{A}$$

with $\partial/\partial s$, using

$$V \llcorner \frac{\partial}{\partial s} = g_t^{-1}(s)\frac{\partial g_t(s)}{\partial s}. \tag{46}$$

Taking advantage of the exponentiability of $\mathscr{G}_0$, we may, for instance, choose

$$g_t(s) = \mathrm{e}^{s\xi\phi(t)}, \quad \varphi(0) = 0, \quad \varphi(1) = 1 \tag{47}$$

with $\xi \in \mathrm{Lie}\,\mathscr{G}$.

Now, Γ_{wzw} as given by eq. (44) can be split into two parts [7,22–26]: a purely mesonic part

$$\begin{aligned}\tilde{\Gamma}_w &= -\int_{Mx[0,1]} \omega^0_{2n-1}(A_{0g_t}, A_0)\\ &= -\int_0^1 \mathrm{d}s \int_M \mathscr{A}\left(g_1^{-1}(s)\frac{\partial g_1(s)}{\partial s}, A_{0g_1(s)}, A_0\right)\end{aligned} \tag{48}$$

and a gauge field part

$$\begin{aligned}\Gamma_A &= -\int_{Mx[0,1]} \omega^0_{2n-1}(A_{tg_t}, A_0) + \omega^0_{2n-1}(A_0, A_{0g_t})\\ &\quad + \omega^0_{2n-1}(A_{0g_t}, A_{tg_t}),\end{aligned} \tag{49}$$

where we have used the gauge invariance of ω^0_{2n-1}. After using the 'triangle formula' and Stokes theorem we get

$$\Gamma_A = \int_M \chi(A_g, A_{0g}, A_0). \tag{50}$$

To sum up

$$\Gamma_{wzw} = -\int_0^1 \mathrm{d}s \int_M \mathscr{A}(\xi, A_{0g(s)}, A_0) + \int_M \chi(A_g, A_{0g}, A_0), \tag{51}$$

where χ is given by eq. (25), and

$$g\,(s) \equiv g_1\,(s) = \mathrm{e}^{s\xi}, \quad g = \mathrm{e}^{\xi}. \tag{52}$$

It is easy to see that the change of Γ_{wzw} under a change of the background field from A_0^1 to A_0^2 is given by

$$\Gamma_{wzw}(g; A, A_0^2) - \Gamma_{wzw}(g; A, A_0^1) = \int_M (\chi(A, A_0^2, A_0^1) - \chi(A_g, A_0^2, A_0^1)). \tag{53}$$

Finally, let us point out that in this section we have been very careful to distinguish between A, g, $\mathscr{S}$ and v and their corresponding continuations into $M \times [0, 1]$ which we wrote A_t, g_t, $\mathbf{S}$ and V. In the next section, to keep the notation from becoming too heavy, we shall always write A, g, $\mathscr{S}$ and v, considering the t-dependence implicit when necessary.

4. The covariant Bardeen vertex anomaly

(This is not to be confused with the covariant current anomalies [13], which can be derived from the present formulae (see also [19])). Assume [7,12] there is a subgroup K of G with the property that the invariant symmetric polynomial P vanishes when its arguments are restricted to Lie K, and that $P(M, G)$ is reducible to K (e.g., its transition functions may be chosen to lie in K), so that A_0 may be chosen to belong to Lie K. Then, we may decompose A and v along Lie K and an invariantly defined orthogonal complement (Lie $K)_\perp$

$$\begin{aligned} A &= A_K + A_\perp, \quad A_K, v_K \in \text{Lie } K, \\ v &= v_K + v_\perp, \quad A_\perp, v_\perp \in (\text{Lie } K)_\perp\,. \end{aligned} \tag{54}$$

The anomaly $\mathscr{A}(v; A, A_0)$ as it stands does not vanish along K; there, it reduces to $\mathscr{A}(v_K; A, A_0)$, where A does not belong to Lie K.

Define the Bardeen (local) counterterm

$$\begin{aligned} \Gamma_B(A, A_0) &= -\int_{Mx[0,1]} (\omega^0_{2n-1}(A, A_0)+\omega^0_{2n-1}(A_0, A_K)+\omega^0_{2n-1}(A_K, A)) \\ &= \int_M \chi(A, A_K, A_0). \end{aligned} \tag{55}$$

The first term cancels the canonical anomaly. The second term is identically zero, since its arguments are both in Lie K. The covariant anomaly $\mathscr{A}_{\text{cov}}(v; A_K, A_\perp, A_0)$ is thus given by

$$\mathscr{S}\omega^0_{2n-1}(A, A_K) = -\mathrm{d}\mathscr{A}_{\text{cov}}(v; A_K, A_\perp)\,, \tag{56}$$

and clearly does not depend anymore on A_0.

We use the family

$$A_t = t(A + v) + (1 - t)\,(A_K + v_K) = A_K + v_K + t(A_\perp + v_\perp)\,, \tag{57}$$

so that the homotopy formula yields

$$\begin{aligned} &P(\mathscr{F}^n(A + v)) - P(\mathscr{F}^n(A_K + v_K)) \\ &\quad = P(\mathscr{F}^n(A)) \\ &\quad = n(d + \mathscr{S}) \int_0^1 \mathrm{d}t\, P(A_\perp + v_\perp, \mathscr{F}^{n-1}(A_t)) \\ &\quad = (d + \mathscr{S})\omega_{2n-1}(A + v, A_K + v_K) \end{aligned} \tag{58}$$

with

$$\begin{aligned} \mathscr{F}(A_t) &= (d + \mathscr{S})A_t + \frac{1}{2}\,[A_t, A_t] \\ &= F(A_K + tA_\perp) + t^2\,[A_\perp, v_\perp] \\ &\quad - t\,[A_\perp, v_\perp]_\perp - [A_\perp, v_\perp]_K + O\left(v^2\right). \end{aligned} \tag{59}$$

This expression for $\mathscr{F}(A_t)$ is obtained by using

$$\begin{aligned} \mathscr{S}A_K &= -\mathrm{d}v_K - [A_K, v_K] - [A_\perp, v_\perp]_K\,, \\ \mathscr{S}A_\perp &= -\mathrm{d}v_\perp - [A_K, v_\perp] - [A_\perp, v_K] - [A_\perp, v_\perp]_\perp\,, \\ \mathscr{S}v_K &= -\frac{1}{2}\,[v_K, v_K] - \frac{1}{2}\,[v_\perp, v_\perp]_K\,, \\ \mathscr{S}v_\perp &= -\,[v_K, v_\perp] - \frac{1}{2}\,[v_\perp, v_\perp]_\perp\,. \end{aligned} \tag{60}$$

Collecting the linear terms in eq. (58) yields

$$\begin{aligned}
&\mathscr{A}_{\rm cov}(v; A_K, A_\perp)\\
&\quad = n \int_0^1 {\rm d}t\, P(v_\perp, F^{n-1}(A_K + tA_\perp))\\
&\qquad + n(n-1) \int_0^1 {\rm d}t\, P(A_\perp, t^2[A_\perp, v_\perp] - t[A_\perp, v_\perp]_\perp\\
&\qquad\qquad - [A_\perp, v_\perp]_K, F^{n-2}(A_K + tA_\perp)),
\end{aligned}\tag{61}$$

which for $A_\perp = 0$ reduces to the well-known result

$$\mathscr{A}_{\rm cov}(v; A_K, 0) = nP(v_\perp, F^{n-1}(A_K)).\tag{62}$$

Remark, as a check, that the v_K dependence has disappeared as it should, i.e.,

$$\mathscr{A}_{\rm cov}(v_K; A_K, A_\perp) = 0.\tag{63}$$

Although the anomaly does not depend on the background field, Γ_B does, and it is clear that its variation is given by

$$\Gamma_B(A, A_0^2) - \Gamma_B(A, A_0^1) = -\int_M \chi(A, A_0^2, A_0^1).\tag{64}$$

It is also possible to obtain a new expression $\bar{\Gamma}_{wzw}$ for the WZW action which gives the covariant form of the anomaly and is expressible in terms of $\hat{g} \in \mathscr{G}/\mathscr{K}$. By eq. (56), a functional which gives the anomaly directly in the covariant form is:

$$\bar{\Gamma}_{wzw}\,(g; A) = \int_{Mx[0,1]} (\omega^0_{2n-1}\,(A, A_K) - \omega^0_{2n-1}(A_g, A_{gK})).\tag{65}$$

We consider $g \in \mathscr{G}$ decomposed in the following way:

$$g = \hat{g}k, \quad \hat{g} \in \mathscr{G}/\mathscr{K}, \quad k \in \mathscr{K}.\tag{66}$$

Since by (56) and (63) $\omega^0_{2n-1}(A, A_K)$ is invariant under gauge transformations $k \in \mathscr{K}$ (or at least under $k \in \mathscr{K}_0$), we have the following identity:

$$\omega^0_{2n-1}(A_g, A_{gK}) = \omega^0_{2n-1}(A_{\hat{g}}, A_{\hat{g}K}),\tag{67}$$

and therefore

$$\bar{\Gamma}_{wzw}(g; A) = \bar{\Gamma}_{wzw}(\hat{g}; A),\tag{68}$$

i.e., $\bar{\Gamma}_{wzw}$ depends on $\hat{g}$ alone. By using the 'triangle formula' it is easy to get the following expression for $\bar{\Gamma}_{wzw}$ in terms of the more conventional Γ_{wzw}:

$$\bar{\Gamma}_{wzw}(\hat{g}; A) = \Gamma_{wzw}(\hat{g}; A, A_0) + \Gamma_B(A, A_0) - \Gamma_B(A_{\hat{g}}, A_0),\tag{69}$$

where $\Gamma_{wzw}(\hat{g}; A, A_0)$ is given by eq. (51). Notice that under a change in the background field, the variation of Γ_{wzw} is cancelled by the variation of the counterterms (eqs (53) and (64)) as it should be, since by definition $\bar{\Gamma}_{wzw}(g; A)$ is intrinsically independent of A_0. It is only to get an expression local in A and globally defined on a non-trivial $P(M, G)$ that we are forced to introduce the background field in the right-hand side of eq. (69).

It should be noticed that the variation of $\hat{g}$ under a gauge transformation (i.e., $\mathscr{L}\,\hat{g}$) depends on the particular way the decomposition of eq. (66) is defined.

The most popular application [2,7,22–26] of this formalism is when $G = G_R \times G_L$. In this case the following notation is used:

$$
\begin{aligned}
&(g_R, g_L) \in \mathscr{G} = \mathscr{G}_R \times \mathscr{G}_L, \quad (g, g) \in \mathscr{K} = \operatorname{diag}\mathscr{G},\\
&A = (A_R, A_L)\,, \quad A_R = \mathbf{V} + \mathbf{A}, \quad A_L = \mathbf{V} - \mathbf{A},\\
&A_K = (\mathbf{V}, \mathbf{V})\,, \quad A_\perp = (\mathbf{A}, -\mathbf{A})\,,\\
&v_R = v_V + v_A, \quad v_L = v_V - v_A,\\
&v_K = (v_V, v_V)\,, \quad v_\perp = (v_A, -v_A)\,,\\
&P(F^n(A)) \equiv P(F^n(A_R)) - P(F^n(A_L)).
\end{aligned}
\tag{70}
$$

In this case the covariant vertex anomaly is given by specializing eq. (61), with the result

$$
\begin{aligned}
&\mathscr{A}^{\text{chiral}}_{\text{cov}}\,(v_A; \mathbf{V}, \mathbf{A})\\
&\quad = n \int_0^1 \mathrm{d}t\, P(v_A, F^{n-1}(\mathbf{V} + t\mathbf{A}))\\
&\qquad + n\,(n-1) \int_0^1 \mathrm{d}t\, P(\mathbf{A}, (t^2 - 1)\,[\mathbf{A}, v_A]\,, F^{n-2}(\mathbf{V} + t\mathbf{A}))\\
&\qquad\qquad - (v_A \to -v_A, \mathbf{A} \to -\mathbf{A})\,,
\end{aligned}
\tag{71}
$$

which for $\mathbf{A} = 0$ reduces to

$$
\mathscr{A}^{\text{chiral}}_{\text{cov}}\,(v_A; \mathbf{V}, 0) = 2n\, P(v_A, F^{n-1}\,(\mathbf{V})).
\tag{72}
$$

However, one may get a shorter formula for the full anomaly by choosing, instead of

$$
\begin{aligned}
\Gamma_B = -\int_{Mx[0,1]} &(\omega^0_{2n-1}(A_R, A_0) - \omega^0_{2n-1}(A_L, A_0) + \omega^0_{2n-1}(\mathbf{V}, A_R)\\
&- \omega^0_{2n-1}(\mathbf{V}, A_L)),
\end{aligned}
\tag{73}
$$

the following expression:

$$
\begin{aligned}
\Gamma'_B &= -\int_{Mx[0,1]} (\omega^0_{2n-1}(A_R, A_0) + \omega^0_{2n-1}(A_0, A_L) + \omega^0_{2n-1}(A_L, A_R))\\
&= \int_M \chi(A_L, A_R, A_0)\,,
\end{aligned}
\tag{74}
$$

which differs from Γ_B by

$$
\begin{aligned}
\Gamma_B - \Gamma'_B &= -\int_{Mx[0,1]} (\omega^0_{2n-1}(\mathbf{V}, A_R) + \omega^0_{2n-1}(A_R, A_L) + \omega^0_{2n-1}(A_L, \mathbf{V})\\
&= \int_M \chi(A_L, A_R, \mathbf{V})\,.
\end{aligned}
\tag{75}
$$

Then, the new covariant anomaly satisfies

$$\mathscr{S}\omega^0_{2n-1}(A_R, A_L) = -\mathrm{d}\mathscr{A}'^{\mathrm{chiral}}_{\mathrm{cov}}(v_A; \mathbf{V}, \mathbf{A})\,, \tag{76}$$

and is obtained from the identity

$$\begin{aligned}
&P(F(A_R)) - P(F(A_L)) \\
&\quad = n(d+\mathscr{S})\int_0^1 \mathrm{d}t\, P(A_R + v_R - A_L - v_L, \mathscr{F}^{n-1}(A_t)) \\
&\quad = (d+\mathscr{S})\,\omega^0_{2n-1}(A_R + v_R, A_L + v_L)\,,
\end{aligned} \tag{77}$$

where now:

$$\begin{aligned}
A_t &= t\,(A_R + v_R) + (1-t)\,(A_L + v_L)\,, \\
\mathscr{F}_t &= (d+\mathscr{S})A_t + \frac{1}{2}\,[A_t, A_t] \\
&= F(tA_R + (1-t)A_L) + 4t(t-1)\,[\mathbf{A}, v_A] + O(v^2) \\
&= F(\mathbf{V} - (1-2t)\mathbf{A}) + 4t(t-1)\,[\mathbf{A}, v_A] + O(v^2).
\end{aligned} \tag{78}$$

Picking the term linear in v in the right-hand side of eq. (77) yields the wanted form of the anomaly

$$\begin{aligned}
&\mathscr{A}'^{\mathrm{chiral}}_{\mathrm{cov}}(v_A; \mathbf{V}, \mathbf{A}) \\
&\quad = 2n\int_0^1 \mathrm{d}t\, P(v_A, F_t^{n-1}) \\
&\qquad + 8n(n-1)\int_0^1 \mathrm{d}t\, P(\mathbf{A}, t\,(t-1)\,[\mathbf{A}, v_A]\,, F_t^{n-2}),
\end{aligned} \tag{79}$$

where

$$F_t \equiv F(\mathbf{V} - (1-2t)\,\mathbf{A})\,.$$

As before

$$\mathscr{A}'^{\mathrm{chiral}}_{\mathrm{cov}}(v_A; \mathbf{V}, 0) = 2nP(v_A, F^{n-1}(\mathbf{V})). \tag{80}$$

Equation (69) with either Γ_B or Γ'_B can be used to construct a WZW action which depends only on $\hat{g} \in \mathscr{G}_R \times \mathscr{G}_L/\mathrm{diag}\,\mathscr{G}$. A possible decomposition of g is [24]

$$\begin{aligned}
&(g_R, g_L) = (e, g_L g_R^{-1})\,(g_R, g_R)\,, \\
&\hat{g} \equiv (e, g_L g_R^{-1}) \in \mathscr{G}_R \times \mathscr{G}_L/\mathrm{diag}\,\mathscr{G}.
\end{aligned} \tag{81}$$

From (35), we know how g_R and g_L transform:

$$\mathscr{S}g_R = -v_R g_R, \quad \mathscr{S}g_L = -v_L g_L, \tag{82}$$

and if we define $U = g_L g_R^{-1}$, we obtain immediately the action of $\mathscr{S}$ on $\hat{g}$

$$\begin{aligned}
\mathscr{S}U &= \mathscr{S}\left(g_L g_R^{-1}\right) = \mathscr{S}g_L g_R^{-1} + g_L \mathscr{L} g_R^{-1} \\
&= -v_L g_L g_R^{-1} + g_L g_R^{-1} v_R = -v_L U + U v_R.
\end{aligned} \tag{83}$$

Appendix

Here we consider the problem of defining the action of ℓ_t on the algebra of polynomials generated by a particular set of forms $A_t, F_t, \ldots$. This is applied to polynomials in $\{A(x,\lambda), F(x,\lambda), v, \mathrm{d}v\}$, where v is the geometric Faddeev–Popov ghost and

$$\begin{aligned} A(x,\lambda) &= g^{-1}A(x)\,g + g^{-1}\mathrm{d}g, \quad g = g(x,\lambda)\,, \\ A(x,\lambda) &= \mathrm{d}A(x,\lambda) + \frac{1}{2}\left[A(x,\lambda)\,, A(x,\lambda)\right] = g^{-1}F(x)g, \end{aligned} \tag{A.1}$$

and the meaning of the resulting extended Cartan homotopy formula is exhibited.

In general, given a family of connections A_t with curvatures F_t, we want to extend the algebra of polynomials $P(A_t, F_t)$ with values, e.g., in the enveloping algebra ε(Lie G) of the relevant Lie algebra in such a way that it becomes stable by applications of d (exterior derivative with respect to base space), d_t (exterior derivative with respect to parameter space) and ℓ_t, a homotopy derivation which increases the degree in $\mathrm{d}t$ by one and decreases the degree in $\mathrm{d}x$ ($x \in$ base space) by one, such that

$$\begin{aligned} &\ell_t d - d\ell_t = d_t, \\ &\ell_t d_t - d_t \ell_t = 0, \\ &d d_t + d_t d = 0. \end{aligned} \tag{A.2}$$

So, we need in general the generators in table 1.

Notice that the generator of degrees (3, 0) in ($\mathrm{d}x$, $\mathrm{d}t$) is not independent, since by the Bianchi identity we have $\mathrm{d}F_t = F_t A_t - A_t F_t$. Similarly, the generator of degrees (0, 3) can be written as

$$\frac{1}{2}\ell_t^2 d_t F_t = (\ell_t d_t A_t)(\ell_t A_t) - (\ell_t A_t)(\ell_t d_t A_t)\,. \tag{A.3}$$

Also, the following generators are identically zero (they would have a negative degree in $\mathrm{d}t$)

$$\ell_t^2 A_t = \ell_t^2 d_t A_t = \ell_t^3 F_t = 0. \tag{A.4}$$

Table 1.

	d-deg		
d_t-deg	0	1	2
0		A_t	F_t
1	$\ell_t A_t$	$d_t A_t$ $\ell_t F_t$	$d_t F_t$
2	$\ell_t d_t A_t$ $\ell_t^2 F_t$	$\ell_t d_t F_t$	

Now we may subject the free algebra generated by elements in the table to relations consistent with eq. (A.2). We consider two examples:

(1) Impose the relations

$$\ell_t F_t = d_t A_t, \qquad \ell_t A_t = \ell_t d_t A_t = \ell_t^2 F_t = \ell_t d_t F_t = 0. \tag{A.5}$$

Only $A_t, F_t, d_t A_t$ and $d_t F_t$ remain as independent generators. This is the case considered in eq. (12).

(2) Introduce a new form v_t of degrees $(0, 1)$ in $(\mathrm{d}x, \mathrm{d}t)$ and impose the following relations:

$$\ell_t A_t = v_t, \qquad \ell_t F_t = \ell_t v_t = 0. \tag{A.6}$$

From $\ell_t F_t = 0$ we get

$$\begin{aligned} 0 = \ell_t F_t &= \ell_t \mathrm{d} A_t + \ell_t A_t^2 \\ &= (\mathrm{d}\ell_t + d_t) A_t + (\ell_t A_t) A_t + A_t(\ell_t A_t) \\ &= \mathrm{d}v_t + d_t A_t + [v_t, A_t]\,. \end{aligned}$$

This defines $d_t A_t$ as

$$d_t A_t = -\mathrm{d}v_t - [v_t, A_t]\,. \tag{A.7}$$

From $\ell_t^2 F_t = 0$ we get in a similar way

$$0 = \ell_t^2 F_t = 2 d_t v_t + [v_t, v_t]\,,$$

which defines $d_t v_t$ as

$$d_t v_t = -\frac{1}{2}\,[v_t, v_t]\,. \tag{A.8}$$

In this case the remaining independent generators are $\{A_t, F_t, v_t, \mathrm{d}v_t\}$. From (A.7) and (A.8) it is clear that we can identify v_t with the geometric Faddeev–Popov ghost v, d_t with $\mathscr{S}$ and A_t with $A(x, \lambda)$ as given by (A.1). Therefore, we define the action of ℓ_λ on polynomials in $\{A(x, \lambda), F(x, \lambda), v, \mathrm{d}v\}$ by

$$\begin{aligned} &\ell_\lambda A = v, \\ &\ell_\lambda F = \ell_\lambda v = 0, \\ &\ell_\lambda \mathrm{d}v = (\mathrm{d}\ell_\lambda + \mathscr{S})\, v = \mathscr{S} v = -\frac{1}{2}\,[v, v]\,. \end{aligned} \tag{A.9}$$

We shall now write the extended Cartan homotopy formula for $\mathscr{Q}$ given by

$$\mathscr{Q} \equiv \omega_{2n-1}^0(A, A_0)\,, \quad \mathrm{d}\omega_{2n-1}^0 = P(F^n(A)) - P(F^n(A_0)), \tag{A.10}$$

where P is an invariant symmetric polynomial, i.e.,

$$\mathrm{d}P = \mathscr{S} P = 0. \tag{A.11}$$

In this case eq. (16) becomes

$$\mathscr{S}\frac{\ell_\lambda^p}{p!}\omega_{2n-1}^0 = -d\left(\frac{\ell_\lambda^{p+1}}{(p+1)!}\omega_{2n-1}^0\right), \tag{A.12}$$

since by eq. (A.9) $\ell_\lambda P = 0$.

This coincides with the ordinary descent eq. (31) if we identify

$$\omega_{2n-1-p}^p \equiv \frac{\ell_\lambda^p}{p!}\omega_{2n-1}^0. \tag{A.13}$$

To evaluate this expression we need, in addition to eq. (A.9), the action of ℓ_λ on A_0 and F_0. We set

$$\ell_\lambda A_0 = \ell_\lambda F_0 = 0, \tag{A.14}$$

which is consistent with $\mathscr{S}A_0 = \mathscr{S}F_0 = 0$ and eq. (A.2). (Strictly speaking, we are considering the algebra of polynomials in $\{A, F, A_0, F_0, v, \mathrm{d}v\}$ with the constraints $\ell_\lambda F = \ell_\lambda v = \ell_\lambda A_0 = \ell_\lambda F_0 = \mathscr{S}A_0 = \mathscr{S}F_0 = 0$ and $\ell_\lambda A = v$.)

It is clear that the formula for $\hat{\omega}_{2n-1-p}^p$ given in eq. (A.13) coincides with the one obtained by expanding $\omega_{2n-1}(A+v, A_0)$ in powers of v (eq. (30)), since ℓ_λ carries A into $A+v$ into itself.

The integral form (eq. (20)) of the extended Cartan homotopy formula, with (A.13) is

$$\int_{\partial T_{p+1}} \omega_{2n-1-p}^p = (-)^p\, d\int_{T_{p+1}} \omega_{2n-2-p}^{p+1}. \tag{A.15}$$

If T_p is a p-simplex which has as vertices the gauge group elements $\{g_0, g_1, ..., g_p\}$ and we define

$$\alpha_p(A, A_0; g_0, g_1, \ldots, g_p) = \int_X \beta_p(A(x), A_0(x); g_0(x), \ldots, g_p(x)), \tag{A.16}$$

where β_p is the following density in x-space:

$$\begin{aligned}&\beta_p(A(x), A_0(x); g_0(x), g_1(x), \ldots, g_p(x))\\ &\quad = \int_T \omega_{2n-1-p}^p(v; A(x), A_0(x)),\end{aligned} \tag{A.17}$$

then α_p is a p-cocycle in the (simplicial) gauge group cohomotogy with coboundary operator given by

$$\begin{aligned}&(\Delta\alpha_p)(A, A_0; g_0, g_1, \ldots, g_p)\\ &\quad = \sum_{i=0}^{p+1}(-)^i\,\alpha_p(A, A_0; g_0, g_1, \ldots, \hat{g}_i, \ldots, g_{p+1}).\end{aligned} \tag{A.18}$$

This can be easily checked for α_1:

$$\begin{aligned}&(\Delta\alpha_1)(A, A_0; g_0, g_1, g_2)\\ &\quad = \alpha_1(A, A_0; g_1, g_2) - \alpha_1(A, A_0; g_0, g_2) + \alpha_1(A, A_0; g_0, g_1)\\ &\quad = \int_X\int_{\partial T_2}\omega_{2n-2}^1 = -\int_X d\int_{T_2}\omega_{2n-3}^2 = 0,\end{aligned} \tag{A.19}$$

where eq. (A.15) has been used in the last step. In general, eq. (A.15) can be written as

$$\Delta\beta_p = \int_{\partial T_{p+1}} \omega^p_{2n-1-p} = (-)^p \, \mathrm{d}\beta_{p+1}, \tag{A.16}$$

which vanishes upon integration over x-space. Notice that iteration of eq. (A.20) gives all the higher cocycles once β_1 is defined. The cohomology of the gauge group and an explicit construction of its cocycles have been considered recently in [30–32,34], whose results are recovered quite directly here.

By considering different parametrizations and different algebras of polynomials, the formalism presented at the beginning of this Appendix can be used to obtain new sets of 'descent equations' from the extended Cartan homotopy formulae (eqs (16) and (20)).

Acknowledgements

This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the US Department of Energy under contract DE-AC03-76SF00098 and in part by the National Science Foundation under research grant PHY81-18547.

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