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Remarks on quantum field theory on de Sitter and anti-de Sitter space-times

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Abstract. This is a short review of work done in common with Jacques Bros, Michel Gaudin, Ugo Moschella, and Vincent Pasquier. Among results are explicit Källén–Lehmann representations for products of two free-field two-point functions in the de Sitter and the anti-de Sitter spaces and applications to particle decay.

Keywords. Quantum field theory; de Sitter; anti-de Sitter.

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1. Introduction

The literature devoted to quantum field theory (QFT) on the de Sitter and anti-de Sitter space-times is enormous. This short review is limited to work done in recent years with the authors quoted in the abstract. The most significant results described here are the two explicit Källén–Lehmann representations appearing in §3.5 and 4.2, and the discussion of 'tachyons' in de Sitter space-time in §3.7.

2. Real and complex Minkowski space-time in $d+1$ dimensions

$$M_{d+1} = \mathbf{R}^{d+1}, \quad M_{d+1}^{(c)} = \mathbf{C}^{d+1}. \tag{2.1}$$

Scalar product:

$$x \cdot y = x^0 y^0 - x^1 y^1 - \cdots - x^d y^d = x^0 y^0 - \vec{x} \cdot \vec{y}. \tag{2.2}$$

$$x^2 \stackrel{\text{def}}{=} x \cdot x. \tag{2.3}$$

Future light-cone:

$$V_+ = -V_- = \{x \in \mathbf{R}^{d+1}: \ x^{(0)} > 0, \ \ x \cdot x > 0\}. \tag{2.4}$$

Future tube:

$$T_+ = \{x + iy \in \mathbf{C}^{d+1}:\quad y \in V_+\} = -T_-. \tag{2.5}$$

n-point future tube:

$$T_{n+} = \{(z_1, \ldots, z_n) \in \mathbf{C}^{n(d+1)}:\quad \mathrm{Im}(z_{j+1} - z_j) \in V_+,\ 1 \le j \le n-1\}$$
$$= -T_{n-}. \tag{2.6}$$

$L_+^\uparrow(d+1) = \mathrm{SO}_0(1, d; \mathbf{R})$ and $L_+(\mathbf{C}, d+1)$ are the connected real and complex Lorentz groups.

3. *d*-dimensional de Sitter space-time

The real and complex d-dimensional de Sitter (dS) space-times with radius $R > 0$, respectively denoted by X_d and $X_d^{(c)}$, are identified with the hyperboloids in real and complex $(d+1)$-dimensional Minkowski space-times defined by

$$X_d = \{x \in M_{d+1}:\quad x \cdot x = -R^2\}, \tag{3.1}$$

$$X_d^{(c)} = \{x \in M_{d+1}^{(c)}:\quad x \cdot x = -R^2\}. \tag{3.2}$$

The forward and backward tuboids are defined as

$$\mathcal{T}_\pm = \{x + iy \in X_d^{(c)}:\quad y \in \pm V_+\} = X_d^{(c)} \cap T_\pm. \tag{3.3}$$

We denote S_d the 'Euclidian' version of the de Sitter space-time, i.e. the sphere:

$$\{z \in X_d^{(c)}:\quad \mathrm{Re}\, z^0 = 0, \mathrm{Im}\, \vec{z} = 0\}. \tag{3.4}$$

3.1 *Free fields in Minkowski and de Sitter space-times*

The free scalar neutral fields used for the de Sitter space-time are the most similar to those of the Minkowski QFT. In both cases they are labelled by a mass $m \ge 0$ and are fully characterized by their two-point function

$$(\Omega, \phi(x)\phi(y)\Omega) = \mathcal{W}_{m+}(x, y) = \mathcal{W}_{m-}(y, x). \tag{3.5}$$

This is a tempered distribution with the following properties:

Hermiticity

$$\overline{\mathcal{W}_{m\pm}(x, y)} = \mathcal{W}_{m\pm}(y, x). \tag{3.6}$$

Solution of Klein–Gordon equation

$$(\Box_x + m^2)\mathcal{W}_{m+}(x, y) = (\Box_y + m^2)\mathcal{W}_{m+}(x, y) = 0. \tag{3.7}$$

Invariance and analyticity. There exist a function $W_m(z_1, z_2)$ and a function w_m of one complex variable, holomorphic in $\mathbf{C}\backslash\mathbf{R}_+$ such that

$$W_m(z_1, z_2) = w_m((z_1 - z_2)^2), \tag{3.8}$$

$$\mathcal{W}_{m+}(x_1, x_2) = \lim_{\substack{z_1 \in T_-,\, z_2 \in T_+ \\ z_1 \to x_1,\, z_2 \to x_2}} W_m(z_1, z_2). \tag{3.9}$$

Positivity

$$\int_{X_d \times X_d} \overline{f(x)}\, \mathcal{W}_+(x, y)\, f(y)\, \mathrm{d}y \geq 0 \tag{3.10}$$

for every test-function f. This will be somewhat relaxed in the case of tachyons.

Fock space and Wick powers can be uniquely constructed starting from such a two-point function in the same way for de Sitter as for Minkowski space-time, and the same is true for generalized free fields which only differ from the free fields by not having to satisfy the Klein–Gordon equation.

3.2 *Special facts for the de Sitter free fields*

In the case of free Klein–Gordon fields on the de Sitter space-time, the mass can be related to a dimensionless parameter ν:

$$m^2 R^2 = \left(\frac{d-1}{2}\right)^2 + \nu^2. \tag{3.11}$$

$$W_\nu(z_1, z_2) = W_{-\nu}(z_1, z_2) = \frac{\Gamma\left(((d-1)/2) + i\nu\right)\Gamma\left(((d-1)/2) - i\nu\right)}{(4\pi)^{d/2} R^{d-2}\Gamma(d/2)} \times F\left(\frac{d-1}{2} + i\nu, \frac{d-1}{2} - i\nu;\ \frac{d}{2};\ \frac{1-\zeta}{2}\right), \tag{3.12}$$

$$\zeta = R^{-2} z_1 \cdot z_2\,. \tag{3.13}$$

The right-hand side of (3.12) is a meromorphic function of ν but positivity occurs only for special values of ν. In these cases, the '1-particle subspace' of Fock space carries an irreducible unitary representation of the de Sitter (i.e. Lorentz) group:

$\nu \in \mathbf{R}$: $m \geq \frac{d-1}{2}$: principal series

$-\frac{d-1}{2} \leq i\nu \leq \frac{d-1}{2}$: $0 \leq m \leq \frac{d-1}{2}$: complementary series

$-\frac{d-1}{2} - i\nu$ integer ≥ 0: $m^2 \leq 0$: tachyons

In the last case, positivity is only satisfied in the physical subspace (see §3.7).

3.3 *Axioms for QFT on de Sitter space-time*

All Wightman axioms can be straightforwardly adapted to dS, except for the spectral property. One possibility (see [1]) is to suppose that the n-point Wightman function $\mathcal{W}_n(x_1, \dots, x_n)$ is Lorentz invariant and is the boundary value of a function $W_n(z_1, \dots, z_n)$, holomorphic in the tuboid

$$\mathcal{T}_{n+} = \{(z_1, \dots, z_n) \in X_d^{(c)n}\colon\ \mathrm{Im}(z_{j+1} - z_j) \in V_+ \quad \forall j = 1, \dots, n-1\} \tag{3.14}$$

which is the intersection of $X_d^{(c)n}$ with the tube in ambient space where a $(d+1)$-Minkowski n-point Wightman function would be analytic, i.e. (in the sense of distributions)

$$\begin{aligned}\mathcal{W}_n(x_1, \dots, x_n) &\overset{\text{def}}{=} (\Omega, \phi(x_1)\dots\phi(x_n)\Omega)\\ &= \lim_{\substack{\mathrm{Im}(z_{j+1} - z_j) \in V_+\\ z_j \to x_j}} W(z_1, \dots, z_n).\end{aligned} \tag{3.15}$$

Note that the right-hand side is expressed in terms of the differences $(z_{j+1} - z_j)$ of the variables z_k (considered as points of $M_{d+1}^{(c)}$) while there is no translational invariance in dS. This nevertheless makes sense since the invariant function W is a function of the invariants $z_k^2 = -R^2$ and $(z_j - z_k)^2$. If π is a permutation of $(1, \dots, n)$ the permuted tuboid $\mathcal{T}_{n\pi}$ is defined as

$$\mathcal{T}_{n\pi} = \{(z_1, \dots, z_n) \in X_d^{(c)n}\colon\ (z_{\pi 1}, \dots, z_{\pi n}) \in \mathcal{T}_{n+}\} \tag{3.16}$$

and the extended permuted tuboid $\mathcal{T}'_{n\pi}$ is

$$\mathcal{T}'_{n\pi} = \bigcup_{\Lambda \in L_+(\mathbf{C})} \Lambda \mathcal{T}_{n\pi}. \tag{3.17}$$

From locality and invariance it follows, as in the Minkowskian case [2–4], that W_n extends to a function holomorphic in the union of the permuted extended tuboids. Actually the locality, invariance and analyticity assumptions formulated above can all be replaced by the assumption that W_n is holomorphic in the union of the permuted extended tuboids, and is Lorentz invariant. (This is described in detail in [1].)

These axioms are satisfied by Wick powers of free or generalized free fields, for which $W(z_1, \dots, z_n)$ extends to a function holomorphic in the 'Complement of the Cuts':

$$\{(z_1, \dots, z_n) \in X_d^{(c)n}\colon\ (z_j - z_k)^2 \in \mathbf{C}\backslash\mathbf{R}_+ \text{ for all } j \neq k\}. \tag{3.18}$$

This domain contains the union of the extended permuted tuboids. Several well-known consequences of the Minkowskian axioms (see [2,3]) extend to the dS case (see [1]):

- CPT Theorem
- Bisognano–Wichmann analyticity
- Reeh–Schlieder Theorem
- Bisognano–Wichmann Theorem

(The proof of the Reeh–Schlieder Theorem requires some more work than in the Minkowskian case and relies on a theorem of Glaser [5]). Note also that the union of the permuted extended tuboids contains all the non-coinciding points of the 'Euclidean' de Sitter world, i.e.

$$\{(z_1, \ldots, z_n) \in S_d^n\colon\ z_j \neq z_k \quad \forall j \neq k\}. \tag{3.19}$$

3.4 *Perturbation theory*

Perturbation theory can be set up in de Sitter space-time, as it is a special case of the theory initiated in [6] and pursued in several subsequent works. The group invariance of de Sitter space-time provides some notable simplifications. A remarkable recent result, obtained independently by Hollands [7] and by Marolf and Morrison [8], is that, at each order of perturbation theory, W_n is holomorphic in the 'Complement of the Cuts' (3.18). Thus, perturbation theory satisfies the axioms in the sense of formal power series.

3.5 *Källén–Lehmann representations*

In Minkowski space-time, for every two-point function W satisfying the axioms, there is a tempered ρ with support in $\mathbf{R}_+$ such that

$$W(z_1, z_2) = \int_0^\infty \rho(m^2) W_m(z_1, z_2)\, \mathrm{d}m^2, \tag{3.20}$$

and ρ is a positive measure iff W satisfies the positivity condition (see e.g. p. 360 of [9]). A similar general theorem holds in de Sitter space-time [10] provided W decreases sufficiently at infinity:

$$W(z_1, z_2) = \int_{\mathbf{R}} \kappa\rho(\kappa)\, W_\kappa(z_1, z_2)\, \mathrm{d}\kappa. \tag{3.21}$$

However in the dS case, ρ need not have any support property. In Minkowski space-time, if W_{m_1} and W_{m_2} are the two-point functions of free fields $\phi_1(x)$ and $\phi_2(x)$ with masses m_1 and m_2, their product is the two-point function of $\phi_1(x)\phi_2(x)$ and thus

$$W_{m_1}(z_1, z_2) W_{m_2}(z_1, z_2) = \int_{(m_1+m_2)^2}^\infty \rho_{\text{Mink}}(m_0^2; m_1, m_2)\, W_{m_0}(z_1, z_2)\, \mathrm{d}m_0^2. \tag{3.22}$$

$\rho_{\text{Mink}}(m_0^2; m_1, m_2)$ is trivially explicitly computable:

$$\rho_{\text{Mink}}(m_0; m_1, m_2) = \frac{1}{2^{2d-3}\pi^{(d-1)/2}\Gamma((d-1)/2)m_0^{d-2}} \times \theta(m_0 - m_1 - m_2) \prod_{\epsilon_1,\ \epsilon_2 = \pm 1} (m_0 - \epsilon_1 m_1 - \epsilon_2 m_2)^{(d-3)/2}. \tag{3.23}$$

A similar explicit expression can be given in the de Sitter case:

$$W_\nu(z_1, z_2)\, W_\lambda(z_1, z_2) = R^{2-d} \int_{\mathbf{R}} \kappa\rho(\kappa; \nu, \lambda) W_\kappa(z_1, z_2)\, \mathrm{d}\kappa, \tag{3.24}$$

$$\kappa\,\rho(\kappa;\,\nu,\ \lambda)$$
$$= \frac{\kappa\ \mathrm{sh}\,\pi\kappa}{2^5\pi^{(d+5)/2}\Gamma((d-1)/2)\Gamma\left(((d-1)/2)+i\kappa\right)\Gamma\left(((d-1)/2)-i\kappa\right)}$$
$$\times \prod_{\epsilon,\ \epsilon',\ \epsilon''=\pm 1} \Gamma\left(\frac{d-1}{4}+\frac{i\epsilon\kappa+i\epsilon'\nu+i\epsilon''\lambda}{2}\right). \tag{3.25}$$

Obtaining this explicit expression is very far from trivial (see [11–13]). It is valid if ν and λ belong to the principal series ($\nu,\ \lambda > 0$. Recall that $W_\nu = W_{-\nu}$). The general case can be obtained by analytic continuation in ν and λ [11,12]. Discrete contributions to the right-hand side of (3.24) then appear as a consequence of poles crossing the contour of integration. A remarkable feature of the above formulae is that $\kappa \mapsto \rho(\kappa;\,\nu,\lambda)$ is analytic on $\mathbf{R}$ so that its support is the whole of $\mathbf{R}$. An application of this formula will be described in the next subsection.

3.6 *Particle decay in first order*

Although the concept of particle is not very clear in de Sitter space-time, it is possible to compute, in the first order of perturbation theory, the probability of decay of a 'one-particle state' into a 'two-particle state', in the same way as in Minkowski space-time. We start from e.g. three commuting free fields ϕ_0, ϕ_1 and ϕ_2 with masses m_0, m_1 and m_2 (parameters ν_0, ν_1, ν_2 in the de Sitter case) and switch on an interaction

$$\int \gamma\, g(x)\, \phi_0(x)\phi_1(x)\phi_2(x)\,\mathrm{d}x. \tag{3.26}$$

g is a switching-off factor which, in the end, must be made to tend to 1 (adiabatic limit). Denoting $\Psi_0 = \int f(x)\,\phi_0(x)\Omega\,\mathrm{d}x$ (f a test function), the total probability of transition from Ψ_0 to any state $\int h(x_1,x_2)\phi_1(x_1)\phi_2(x_2)\Omega\,\mathrm{d}x_1\,\mathrm{d}x_2$ is

$$\frac{\gamma^2}{\int \overline{f_0(x)}\,\mathcal{W}_{m_0}(x,y)\,f_0(y)\,\mathrm{d}x\,\mathrm{d}y}\int \overline{f_0(x)}\,f_0(y)\,g(u)\,g(v)$$
$$\times\ \mathcal{W}_{m_0}(x,u)\,\left\{\mathcal{W}_{m_1}(u,v)\,\mathcal{W}_{m_2}(u,v)\right\}\mathcal{W}_{m_0}(v,y)\,\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y. \tag{3.27}$$

The product $\{\mathcal{W}_{m_1}(u,v)\,\mathcal{W}_{m_2}(u,v)\}$ can be expressed through its Källén–Lehmann representation, i.e. (3.22) or (3.24). However, the adiabatic limit does not exist in either de Sitter or Minkowski space-time. In the latter case the remedy (see e.g. [14]) is to take g as the indicator function of a time-slice of width T, divide the above expression by T and then take the limit as T tends to 0. In this way one finds, in the Minkowski case, that the probability of transition per unit time (whose reciprocal is the lifetime of the 0-particle) is

$$\Gamma_{f_0} = \frac{(2\pi)\gamma^2\int(2p^0)^{-1}\,|\tilde{f}_0(p)|^2\delta(p^2-m_0^2)\theta(p^0)\,\mathrm{d}p}{\int|\tilde{f}_0(p)|^2\,\delta(p^2-m_0^2)\theta(p^0)\,\mathrm{d}p}\,\rho_{\mathrm{Mink}}(m_0^2;\,m_1,m_2). \tag{3.28}$$

$\tilde{f}_0$ is the Fourier transform of f_0. Letting $|\tilde{f}_0(p)|^2$ tend to $\delta(\vec{p})$, we obtain the inverse lifetime of the 0-particle at rest:

$$\frac{\pi\gamma^2}{m_0}\,\rho_{\mathrm{Mink}}(m_0^2;\,m_1,m_2). \tag{3.29}$$

The same procedure can be used in the de Sitter case, at the cost of some non-trivial calculations [11–13]. One finds, in the case when all the masses involved belong to the principal series,

$$\Gamma_{f_0} = \frac{\gamma^2 \pi \coth(\pi \nu_0)^2 R}{|\nu_0|} \rho(\nu_0; \nu_1, \nu_2) \tag{3.30}$$

with ρ as given in (3.25). This formula has two striking features. First the dependence on f_0 has disappeared in the course of taking the (averaged) adiabatic limit. This is an effect of the well-known immensity of de Sitter space-time at large distances. Second, because the support of $\nu_0 \mapsto \rho(\nu_0; \nu_1, \nu_2)$ is the whole of $\mathbf{R}$ (while $\rho_{\text{Mink}}(m_0^2; m_1, m_2)$ vanishes if $m_0 < m_1 + m_2$), the decay occurs even if the mass of the 0-particle is smaller than the sum of the masses of the decay products. This phenomenon was discovered in 1968 by Nachtmann [15].

3.7 *Tachyons in de Sitter space-time*

'Tachyons' in de Sitter space-time have negative square mass, but are not as exotic as tachyons in Minkowski space-time (if they existed there). To study them, it is convenient to adopt the parameter λ, related to ν and m by

$$\begin{aligned} \lambda &= -\frac{d-1}{2} - i\nu, \\ m^2 R^2 &= -\lambda(\lambda + d - 1). \end{aligned} \tag{3.31}$$

In the complex λ-plane the various irreducible unitary representations are located as shown in figure 1 (only half the picture appears, the whole picture being symmetric across the line $\text{Re}\,\lambda = -(d-1)/2$).

With the new parametrization,

$$\begin{aligned} W_\nu(z_1, z_2) &= W_{(\lambda)}(z_1, z_2) \\ &= \frac{\Gamma(-\lambda)\Gamma(\lambda + d - 1)}{(4\pi)^{d/2}\Gamma(d/2)\, R^{d-2}} F\left(-\lambda,\ \lambda + d - 1;\ \frac{d}{2};\ \frac{1-\zeta}{2}\right) \\ &= \Gamma(-\lambda) G_\lambda(\zeta), \quad \zeta = \frac{z_1 \cdot z_2}{R^2}. \end{aligned} \tag{3.32}$$

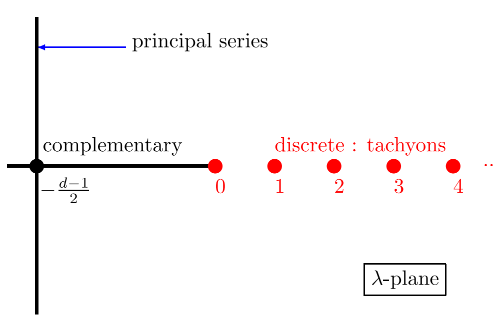

Figure 1. Irreducible unitary representations.

This is a meromorphic function of λ, with poles at all non-negative integers (as well as their symmetric images across $\mathrm{Re}\,\lambda = -(d-1)/2$). If n is a non-negative integer (and $R = 1$),

$$G_n(\zeta) = \frac{\Gamma(n+1)\Gamma(d+1)}{(4\pi)^{d/2}\Gamma(d/2)\,R^{d-2}} C_n^{(d-1)/2}(\zeta). \tag{3.33}$$

$C_n^{(d-1)/2}$ is a Gegenbauer polynomial of degree n (see [16], p. 175). If λ is close to n, $\Gamma(-\lambda) \sim [(-1)^{n+1} n!(\lambda - n)]^{-1}$. We can try to obtain a two-point function at $\lambda = n$ by defining

$$\begin{aligned}\widehat{W}_n(z_1, z_2) = \widehat{w}_n(\zeta) &= \lim_{\lambda\to n} \Gamma(-\lambda)[G_\lambda(\zeta) - G_n(\zeta)] \\ &= \frac{(-1)^{n+1}}{n!} \frac{\partial}{\partial\lambda} G_\lambda(\zeta)\bigg|_{\lambda=n}.\end{aligned} \tag{3.34}$$

We denote $\widehat{\mathcal{W}}_{n+}$ and $\widehat{\mathcal{W}}_{n-}$ the boundary values of $\widehat{W}_n$ from the forward and backward tuboids. Since G_n has no discontinuity, the commutator function

$$\begin{aligned}c_n(x_1, x_2) &= \widehat{\mathcal{W}}_{n+}(x_1, x_2) - \widehat{\mathcal{W}}_{n-}(x_1, x_2) = \lim_{\lambda\to n} c_\lambda(x_1, x_2), \\ c_\lambda(x_1, x_2) &= \mathcal{W}_{(\lambda)+}(x_1, x_2) - \mathcal{W}_{(\lambda)-}(x_1, x_2).\end{aligned} \tag{3.35}$$

Thus, c_λ has the limit c_n as $\lambda \to n$ without the need for any subtraction, and c_n therefore satisfies the Klein–Gordon equation with square mass $-n(n+d-1)R^{-2}$ in x_1 and in x_2. But this is not the case for $\widehat{W}_n$. Since G_λ does satisfy the Klein–Gordon equation, it easily follows from (3.34) and (3.33) that

$$\begin{aligned}&[\Box_{z_1} - n(n+d-1)R^{-2}]\widehat{W}_n(z_1, z_2) \\ &\quad= \frac{(-1)^{n+1}(2n+d-1)\Gamma((d-1)/2)}{4\pi^{(d+1)/2}\,R^d} C_n^{(d-1)/2}(\zeta)\,.\end{aligned} \tag{3.36}$$

A (non-Hilbertian) Fock space and a local field ϕ can be built up by using $\widehat{W}_n(z_1, z_2)$ as a two-point function. The field ϕ satisfies an inhomogeneous Klein–Gordon equation

$$[\Box - n(n+d-1)R^{-2}]\phi = Q_n. \tag{3.37}$$

One can then define a 'physical subspace' as the set of states Ψ such that

$$Q_n^- \Psi = 0. \tag{3.38}$$

In the one-particle subspace, the 'physical subspace' is the subspace $\mathcal{E}_n$ of test functions defined by

$$\mathcal{E}_n = \left\{ \psi \in \mathcal{S}(X_d)\colon\ \int_{X_d} G_n(x_1\cdot x_2)\,\psi(x_2)\,\mathrm{d}x_2 = 0 \quad \forall x_1 \in X_d \right\}. \tag{3.39}$$

It can be shown that on this subspace the sesquilinear form defined by $\widehat{\mathcal{W}}_{n+}$ is positive definite, i.e.

$$\int_{X_d\times X_d} \overline{f(x_1)}\,\widehat{\mathcal{W}}_{n+}(x_1, x_2)\, f(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \geq 0 \quad \forall f \in \mathcal{E}_n. \tag{3.40}$$

For a sketch of the proof, see [17]. It follows that the scalar product of the Fock space is positive definite on the physical subspace defined by (3.38). On this subspace, which is invariant under the de Sitter group, ϕ satisfies the (homogeneous) Klein–Gordon equation. The situation is analogous to what happens with the free electromagnetic field in the Gupta–Bleuler formalism. The field ϕ and its Wick powers satisfy all the axioms described in §3.3 except positivity which still exists in the slightly weakened form described above. We also note that $\mathcal{E}_n$ has a finite co-dimension which, however, tends to infinity when n tends to infinity.

4. Anti-de Sitter space-time ($d \geq 2$)

We identify the d-dimensional anti-de Sitter space-time (AdS) with a quadric embedded in the $(d+1)$-dimensional ambient space $E_{d+1} = \mathbf{R}^{d+1}$ or its complexified $E_{d+1}^{(c)} = \mathbf{C}^{d+1}$, both equipped with the scalar product

$$x \cdot y = x^0 y^0 + x^d y^d - x^1 y^1 - \cdots - x^{(d-1)} y^{(d-1)}. \tag{4.1}$$

The notation $x^2 \stackrel{\text{def}}{=} x \cdot x$ is used if no ambiguity arises. We denote $G_0 = SO_0(2, d-1, \mathbf{R})$ (resp. $G_0^{(c)} = SO_0(2, d-1, \mathbf{C})$) the connected group of real (resp. complex) linear transformations of E_{d+1} (resp. $E_{d+1}^{(c)}$) which preserve the above scalar product. The real and complex AdS space-times are respectively

$$X_d = \{x \in E_{d+1}\colon\ x \cdot x = R^2\} \tag{4.2}$$

and

$$X_d^{(c)} = \{x \in E_{d+1}^{(c)}\colon\ x \cdot x = R^2\}. \tag{4.3}$$

The universal covering $\tilde{X}_d$ of X_d is also frequently studied.

The forward and backward tuboids in $X_d^{(c)}$ are given by

$$\mathcal{T}_{1+} = (\mathcal{T}_{1-})^* = \{x + iy \in X_d^{(c)}\colon\ y \cdot y > 0,\quad y^0 x^d - y^d x^0 > 0\}. \tag{4.4}$$

4.1 *Free fields in anti-de Sitter space-time*

A real scalar free field in AdS is again characterized by its two-point function, which is labelled by an integer n (we take $R = 1$ for simplicity).

$$\begin{aligned}
&(\Omega, \phi(x_1)\phi(x_2)\,\Omega) = \mathcal{W}_{n+\frac{d-1}{2}}(x_1, x_2),\\
&\mathcal{W}_{n+((d-1)/2)}(x_1, x_2) = \lim_{\substack{z_1 \in \mathcal{T}_{1-},\ z_2 \in \mathcal{T}_{1+}\\ z_1 \to x_1,\ z_2 \to x_2}} W_{n+((d-1)/2)}(z_1, z_2),\\
&W_{n+((d-1)/2)}(z_1, z_2) = w_{n+((d-1)/2)}(z_1 \cdot z_2).
\end{aligned} \tag{4.5}$$

The function $w_{n+((d-1)/2)}$ is holomorphic in the domain $\Delta_1 = \mathbf{C}\backslash[-1, 1]$, and is given by

$$w_{n+((d-1)/2)}(z) = \frac{\Gamma((d-1)/2)}{2\pi^{(d+1)/2}} D_n^{(d-1)/2}(z), \tag{4.6}$$

$$D_n^\lambda(z) = \frac{\pi\Gamma(n+2\lambda)}{\Gamma(\lambda)\Gamma(n+\lambda+1)}(2z)^{-n-2\lambda} \times F\left(\frac{n+2\lambda}{2}, \frac{n+2\lambda+1}{2}; n+\lambda+1; \frac{1}{z^2}\right) \quad (4.7)$$

$$= \frac{\pi\Gamma(n+2\lambda)}{\Gamma(\lambda)\Gamma(n+\lambda+1)}(\zeta)^{-n-2\lambda} F\left(n+2\lambda, \lambda; n+\lambda+1; \frac{1}{\zeta^2}\right), \quad (4.8)$$

where the variables z and ζ are related as follows:

$$\zeta = z + (z^2-1)^{1/2}, \quad \zeta^{-1} = z - (z^2-1)^{1/2}, \quad z = \frac{\zeta+\zeta^{-1}}{2}. \quad (4.9)$$

The field ϕ satisfies the Klein–Gordon equation with squared mass $m^2 = n(n+d-1)$. $W_{n+((d-1)/2)}$ is positive definite if $2n+d+1 > 0$.

The same formulae apply to the covering $\tilde{X}_d$ of the AdS space-time, but in this case the parameter n need not be an integer, the function $w_{n+((d-1)/2)}$ is now holomorphic on the universal covering $\tilde{\Delta}_1$ of the cut-plane Δ_1, and the variable z appearing in (4.6) and (4.7) must be regarded as a point of $\tilde{\Delta}_1$. We shall not deal in detail with $\tilde{X}_d$.

Axioms can be formulated for quantum fields on the anti-de Sitter space-time, and in particular a positive energy condition can be defined without ambiguity. As a consequence of these axioms, the two-point function must have the same analyticity properties as the free-field two-point functions mentioned above (see e.g. [18,19]).

4.2 *A Källén–Lehmann formula in AdS*

As in the Minkowskian and de Sitterian cases, any function with the same general linear properties as the two-point function of a local field satisfying the axioms has a Källén–Lehmann representation, i.e. can be expressed as a linear combination of free-field two-point functions: the general theorem appears in [20]. We can also ask for the explicit Källén–Lehmann representation of a product of two free-field two-point functions, as we did in the Minkowskian and de Sitterian cases. The answer is given by

$$W_{m+((d-1)/2)}(z_1, z_2)\, W_{n+((d-1)/2)}(z_1, z_2) = \sum_{\substack{l=m+n+d-1+2k \\ k\in\mathbf{Z},\ 0\le k}} \rho(l; m, n)\, W_{l+((d-1)/2)}(z_1, z_2), \quad (4.10)$$

$$\rho(l; m, n) = \frac{\Gamma(\lambda)}{2\pi^\lambda} \frac{\alpha_\lambda\left(\frac{l+m-n}{2}\right)\alpha_\lambda\left(\frac{l-m+n}{2}\right)\alpha_\lambda\left(\frac{l+m+n}{2}+\lambda\right)\alpha_\lambda\left(\frac{l-m-n}{2}-\lambda\right)}{\alpha_\lambda(l)\alpha_\lambda(l+\lambda)},$$

$$\alpha_\lambda(t) \stackrel{\text{def}}{=} \frac{\Gamma(t+\lambda)}{\Gamma(\lambda)\Gamma(t+1)}, \quad \lambda = \frac{d-1}{2}. \quad (4.11)$$

Here $d \geq 2$ is an integer and the conditions

$$m+d-1>0, \quad n+d-1>0, \quad m+\frac{d+1}{2}>0, \quad n+\frac{d+1}{2}>0 \tag{4.12}$$

must be satisfied. This again results from non-trivial calculations. The details are given in [21]. In fact the above formula continues to hold when m and n are not integers, provided the conditions (4.12) hold. Then $W_{m+((d-1)/2)}$ and $W_{n+((d-1)/2)}$ are the two-point functions of free fields on the universal covering $\tilde{X}_d$ of X_d. It is worth noting that the formula (4.11) reflects some non-obvious identities among hypergeometric functions, for example

$$\begin{aligned} &F(x, 1-\eta; x+\eta; v)F(y,\ 1-\eta; y+\eta; v) \\ &\quad = \sum_{k=0}^{\infty} \frac{(x)_k(y)_k(x+y+2\eta-1+k)_k(1-\eta)_k}{(x+\eta)_k(y+\eta)_k(x+y+\eta-1+k)_k\, k!} \\ &\qquad \times v^k F(x+y+2k, 1-\eta; x+y+\eta+2k; v). \end{aligned} \tag{4.13}$$

(Here $(t)_k \stackrel{\text{def}}{=} \Gamma(t+k)/\Gamma(t)$ for any integer $k \geq 0$.) This holds (and the right-hand side converges) provided $|v| < 1$ and

$$\begin{aligned} &x>0, \quad y>0, \quad x+y+2\eta-1 \geq 0, \quad 1-\eta \geq 0, \\ &x+\eta>0, \quad y+\eta>0, \quad x+y+\eta>0. \end{aligned} \tag{4.14}$$

As an application of the formula (4.11) we may again consider, as in §3.6, three commuting free fields ϕ_0, ϕ_1 and ϕ_2 with (integer) parameters n_0, n_1 and n_2, and switch on the interaction (3.26). Denoting again $\Psi_0 = \int f(x)\,\phi_0(x)\Omega\,\mathrm{d}x$, the total probability of transition from Ψ_0 to any state of the form $\int h(x_1, x_2)\phi_1(x_1)\phi_2(x_2)\Omega\,\mathrm{d}x_1\,\mathrm{d}x_2$ is given, in the lowest order, by the formula (3.27). In the present case, the integral over u and v is uniformly convergent even when $g \to 1$ and, by general arguments, must be proportional to the denominator, i.e., as in the de Sitter case but for different reasons, the dependence on f_0 disappears. The result can be calculated exactly under the conditions

$$\begin{aligned} &n_1+d-1>0, \quad n_2+d-1>0, \\ &n_1+\frac{d-1}{2}+1>0, \quad n_2+\frac{d-1}{2}+1>0, \end{aligned} \tag{4.15}$$

$$n_0+d-1>0, \quad n_0+n_1+n_2+2(d-1)>0\,, \tag{4.16}$$

and is given by 0 unless $n_0 - n_1 - n_2 - d + 1$ is an even non-negative integer, and otherwise by

$$\text{Prob.}\ (\psi_0 \to n_1, n_2) = \left(\frac{2\pi\gamma}{2n_0+d-1}\right)^2 \rho(n_0; n_1, n_2). \tag{4.17}$$

In order to compare this with the Minkowskian and de Sitterian results of §3.6, we should divide the above total probability by a plausible 'total time' K_0 ($K_0 R$ when $R \neq 1$). One might be tempted to take $K_0 = 2\pi$, the total length of a time-like geodesic. However, $K_0 = 4\pi$ is the choice which makes the lifetime tend to the Minkowskian lifetime when R tends to infinity (see [21]).

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