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# Cosmological model of interacting phantom and Yang–Mills fields

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**Abstract.** In this paper, we consider a model of interacting phantom and Yang–Mills (YM) fields by assuming dilaton-type coupling. Using the specific solution for YM equation previously found by the author, we obtain simple exact solutions for the accelerated expansion of the Friedmann–Robertson–Walker (FRW) cosmological model. Besides, we derive induced potentials of phantom field corresponding to some given regimes of expansion. The effective equations of state (EoS) have been reconstructed for all types of models considered here.

**Keywords.** Cosmological model; interaction; phantom; Yang–Mills field; accelerated expansion.

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## 1. Introduction

The present accelerated expansion of the Universe is well proved in many papers [1–8]. In order to explain this unexpected behaviour of our Universe, one can modify the gravitational theory [9–14], or construct various field models of the so-called dark energy (DE) by which EoS satisfies $w = p/\rho < -1/3$. The simplest candidate of DE is the cosmological constant with fixed EoS ($w = -1$). If it is quintessence then $-1 < w < -1/3$ and if it is phantom then $w < -1$. The constant EoS, i.e., $w = -1$ is called the phantom divide. There are some dark energies which can cross the phantom divide from both sides [15]. So far, a large class of scalar-field DE models has been studied, including tachyon [16], ghost condensate [17], quintom [18,19], and so forth. Other proposals on DE include interacting DE models [20], braneworld models [21], holographic DE models [22], etc.

Other class of DE models is based on the conjecture that a vector field can be the origin of DE [23,24]. The YM field can be a candidate for such a vector field [25–27]. At the same time, it is well known that a pure YM field (with its EoS $w = 1/3$) cannot provide accelerated expansion of the Universe, for which $w < -1/3$ is required. This is a direct consequence of conformal symmetry of the Lagrangian for a massless YM field.

Any violations of conformal symmetry (e.g., as a result of quantum corrections [28] or of non-minimal coupling to gravity [29]) give a good chance for involving YM fields in the reconstruction of DE. The alternative chance for YM fields to be involved in DE problem is the consideration of some interaction of YM field with different sources of gravity. In this aspect, the idea of induced nonlinearity is fairly attractive for the realization of phantom field in cosmology. In this paper, we turn our attention to the issue of YM fields interacting with a phantom field in FRW cosmology. Using the specific solution of YM equation previously considered in FRW cosmology [30–34], we generalize the model investigated in [30] in the case of interacting phantom and YM fields. This allows us to obtain some exact solutions for the accelerated expansion of FRW cosmological model. Besides, we derive the induced potentials of phantom field corresponding to some given regimes of expansion. The effective EoS have been reconstructed for all types of models considered below.

## 2. Basic equations

The main equations of the model follow from the Lagrangian density [30]:

$$\mathcal{L} = \frac{R}{2\kappa} + \epsilon\frac{1}{2}\varphi_{,\alpha}\varphi^{,\alpha} - \frac{1}{16\pi}F^a_{\alpha\beta}F^{a\alpha\beta}\Psi(\varphi), \tag{1}$$

where $R$ is the Ricci curvature scalar, $\varphi$ is a scalar field, $F^a_{\alpha\beta}$ is the YM strength tensor, $\Psi(\varphi)$ is the coupling analytical function, $\epsilon = +1$ represents quintessence while $\epsilon = -1$ represents phantom field. Variation of (1) with respect to the metric $g^{ik}$ and fields yields the Einstein equation $G^v_\mu = \kappa T^v_\mu$ with the following energy–momentum tensor for the system of fields:

$$T^v_\mu = \epsilon\varphi_{,\mu}\varphi^{,v} - \frac{1}{4\pi}F^a_{\mu\beta}F^{av\beta}\Psi(\varphi) \\ -\delta^v_\mu\left[\frac{\epsilon}{2}\varphi_{,\alpha}\varphi^{,\alpha} - \frac{1}{16\pi}F^a_{\alpha\beta}F^{a\alpha\beta}\Psi(\varphi)\right], \tag{2}$$

and the modified YM equation: $D_v\left(\sqrt{-g}F^{av\mu}\Psi(\varphi)\right) = 0$, where $D_v$ denotes the covariant derivative. The scalar field equation is as follows:

$$\frac{\epsilon}{\sqrt{-g}}\frac{\partial}{\partial x^v}\left(\sqrt{-g}g^{v\mu}\frac{\partial\varphi}{\partial x^\mu}\right) + \frac{1}{16\pi}I\,\frac{\mathrm{d}\Psi(\varphi)}{\mathrm{d}\varphi} = 0, \tag{3}$$

where the first invariant of Yang–Mills field $I = F^a_{\alpha\beta}F^{a\alpha\beta}$. We assume that the Universe is described by a FRW geometry with a scale factor $a(t)$ and the sign of space curvature $k = +1, 0, -1$.

The nontrivial solution for the YM equation obtained in [30] has such a valuable feature that the YM invariant built on this solution depends only on time: $I = I(t) = 6/g^2a^4(t)$. In view of the latter, for the phantom field ($\epsilon = -1$), the rest of the set of Einstein and scalar field equations is as follows:

$$\dot\varphi^2 = 6\kappa^{-1}a^{-2}(\dot a^2 + a\ddot a + k), \quad \Psi(\varphi) = 8\pi g^2\kappa^{-1}a^2(2\dot a^2 + a\ddot a + 2k), \tag{4}$$

$$\frac{1}{a^3}\frac{\partial}{\partial t}\left(a^3\dot\varphi\right) - \frac{3}{8\pi g^2}\frac{1}{a^4}\frac{\mathrm{d}\Psi(\varphi)}{\mathrm{d}\varphi} = 0. \tag{5}$$

Besides, comparing eq. (5) in this case with the usual one for the phantom field in FRW cosmology, we can derive the relationship between an effective potential of phantom field $V_{\rm eff}$ and the coupling function $\Psi(\varphi)$:

$$\frac{{\rm d}V_{\rm eff}(\varphi)}{{\rm d}\varphi} = \frac{3}{8\pi g^2}\frac{1}{a^4}\frac{{\rm d}\Psi(\varphi)}{{\rm d}\varphi}. \tag{6}$$

From the first equation in (4) it follows that the real solution for phantom field exists when $\dot{a}^2 + a\ddot{a} + k \geq 0$. Besides, the simple study of effective energy density and pressure followed from (2), i.e.,

$$\rho(t) = -\frac{\dot{\varphi}^2}{2} + 3\frac{1}{8\pi g^2 a^4}\Psi(\varphi), \quad p(t) = -\frac{\dot{\varphi}^2}{2} + \frac{1}{8\pi g^2 a^4}\Psi(\varphi), \tag{7}$$

shows that the effective EoS in this model is restricted by $p \leq -\rho/3$. Therefore, the accelerated expansion, for which $p < -\rho/3$ is required, can be realized in our model.

## 3. Simple examples of exact solution

To demonstrate some interesting features of this model, we are going to consider two illustrative examples of exact solutions further to our general study. We shall consider the accelerated expansion, that is $\ddot{a} > 0$, when the asymptotic value of the Hubble parameter $H = H_0 =$ const. is achieved in the course of time.

### 3.1 *The case* $a(t) = H_0^{-1}\sinh(H_0 t)$

Note that this dependence of $a(t)$ on time satisfies inequality $\dot{a}^2 + a\ddot{a} + k \geq 0$ for all signs of the curvature, and eqs (4) become

$$\dot{\varphi}^2 = H_0^2\frac{(2\sinh^2(H_0 t) + 1 + k)}{2\lambda^2\sinh^2(H_0 t)},$$

$$\Psi(t) = \frac{\Psi_0\sinh^2(H_0 t)}{3}(3\sinh^2(H_0 t) + 2 + 2k). \tag{8}$$

Consider the cases of negative or positive signs of curvature separately. Here and further, we use the following notations: $\lambda = \sqrt{\kappa/12}$, $\Psi_0 = 24\pi g^2/\kappa H_0^2$, $V_0 = 3H_0^2/\lambda^2$.

(i) Open model: $k = -1$. From eqs (8), one can find that $\dot{\varphi} = \lambda^{-1}H_0$, $\Psi(t) = \Psi_0\sinh^4(H_0 t)$. Integrating the first equation and taking into account the second one together with the explicit expression of $a(t)$, we obtain

$$\varphi(t) = \lambda^{-1}H_0 t + \varphi_0, \qquad \Psi(\varphi) = \Psi_0\sinh^4[\lambda(\varphi - \varphi_0)]. \tag{9}$$

With the help of eq. (6), we can derive the following effective potential which is induced by the coupling to the YM field:

$$V_{\rm eff}(\varphi) = V_0\ln(\sinh[\lambda(\varphi - \varphi_0)]) + U_0, \tag{10}$$

where $U_0$ is a constant.

(ii) Closed model: $k = +1$. As it follows from eqs (4), in this case, we have

$$\dot{\varphi} = \lambda^{-1} H_0 \coth(H_0 t), \qquad \Psi(t) = \Psi_0 \sinh^2(H_0 t)\Big(\sinh^2(H_0 t) + \frac{4}{3}\Big). \quad (11)$$

Then from eq. (11), we can find that

$$\varphi(t) = \lambda^{-1} \ln\Big[\sinh(H_0 t)\Big] + \varphi_0, \quad \Psi(\varphi) = \Psi_0 e^{2\lambda(\varphi-\varphi_0)}\Big[e^{2\lambda(\varphi-\varphi_0)} + \frac{4}{3}\Big]. \quad (12)$$

The induced phantom potential, which corresponds to the coupling function (12), can be obtained in the following form:

$$V_{\text{eff}}(\varphi) = V_0\Big[\lambda(\varphi - \varphi_0) - \frac{1}{3}e^{-2\lambda(\varphi-\varphi_0)}\Big] + U_0. \quad (13)$$

### 3.2 *The case* $a(t) = H_0^{-1}\cosh(H_0 t)$

In this case, eqs (4) take the following form:

$$\dot{\varphi}^2 = H_0^2 \frac{(2\cosh^2(H_0 t) - 1 + k)}{2\lambda^2 \cosh^2(H_0 t)},$$
$$\Psi(t) = \frac{\Psi_0 \cosh^2(H_0 t)}{3}(3\cosh^2(H_0 t) - 2 + 2k). \quad (14)$$

(iii) Open model: $k = -1$. From eqs (14), we have

$$\dot{\varphi} = \lambda^{-1} H_0 \tanh(H_0 t), \quad \Psi(t) = \Psi_0 \cosh^2(H_0 t)\Big(\cosh^2(H_0 t) - \frac{4}{3}\Big). \quad (15)$$

In view of the second equation in (15) and explicit expression for $a(t)$, we can solve the first equation in (15), and then obtain

$$\varphi(t) = \lambda^{-1} \ln\big[\cosh(H_0 t)\big] + \varphi_0, \quad \Psi(\varphi) = \Psi_0 e^{2\lambda(\varphi-\varphi_0)}\Big[e^{2\lambda(\varphi-\varphi_0)} - \frac{4}{3}\Big]. \quad (16)$$

Due to eq. (6), the effective phantom potential becomes

$$V_{\text{eff}}(\varphi) = V_0\Big[\lambda(\varphi - \varphi_0) + \frac{1}{3}e^{-2\lambda(\varphi-\varphi_0)}\Big] + U_0. \quad (17)$$

(iv) Closed model: $k = +1$. For this sign of curvature, it follows from eqs (14), that $\dot{\varphi} = \lambda^{-1} H_0$, $\Psi(t) = \Psi_0 \cosh^4(H_0 t)$. Therefore, we have the following solution in this case:

$$\varphi(t) = \lambda^{-1} H_0 t + \varphi_0, \quad \Psi(\varphi) = \Psi_0 \cosh^4\left[\lambda(\varphi - \varphi_0)\right]. \quad (18)$$

In view of eq. (6), it can be easily obtained that now the effective phantom potential

$$V_{\text{eff}}(\varphi) = V_0 \ln(\cosh\left[\lambda(\varphi - \varphi_0)\right]) + U_0. \quad (19)$$

With the help of expressions (7), all solutions obtained above can be arranged in three groups according to their EoS.

In Case (1): $a(t) = H_0^{-1}\cosh(H_0 t)$ $(k = -1)$, and in Case (2): $a(t) = H_0^{-1}\sinh(H_0 t)$ $(k = +1)$, one can find consequently

$$w_1 = -\frac{3\sinh^2(H_0 t) + 1}{3\sinh^2(H_0 t) - 3}, \qquad w_2 = -\frac{3\sinh^2(H_0 t) + 2}{3\sinh^2(H_0 t) + 6}. \tag{20}$$

In Case (3): $a(t) = H_0^{-1}\sinh(H_0 t)$ with $k = -1$, or $a(t) = H_0^{-1}\cosh(H_0 t)$ with $k = +1$, we have $w_3 = -1$. The plots of EoS parameters for all three cases considered above can be viewed on figure 1. As one can see, the curve of EoS when $a = H_0^{-1}\sinh(H_0 t)$ with $k = -1$ displays a rather unusual behaviour around the value $x = H_0 t$ determined by the solution of $\sinh(H_0 t) = 1$. To understand this, let us find the energy densities from eq. (7) for the cases defined above. So we can find for Cases (1) and (2) that

$$\rho_1 = \frac{H_0^2}{4\lambda^2\cosh^2(H_0 t)}\left(\sinh^2(H_0 t) - 1\right),$$
$$\rho_2 = \frac{H_0^2}{4\lambda^2\sinh^2(H_0 t)}\left(\sinh^2(H_0 t) + 2\right). \tag{21}$$

For Case (3), we have $\rho_3 = H_0^2/4\lambda^2$, that is $\rho_3 = \lim_{t\to\infty}\rho_1(t) = \lim_{t\to\infty}\rho_2(t)$. It could be supposed that the constant density $\rho_3$ represents some energy density of an effective

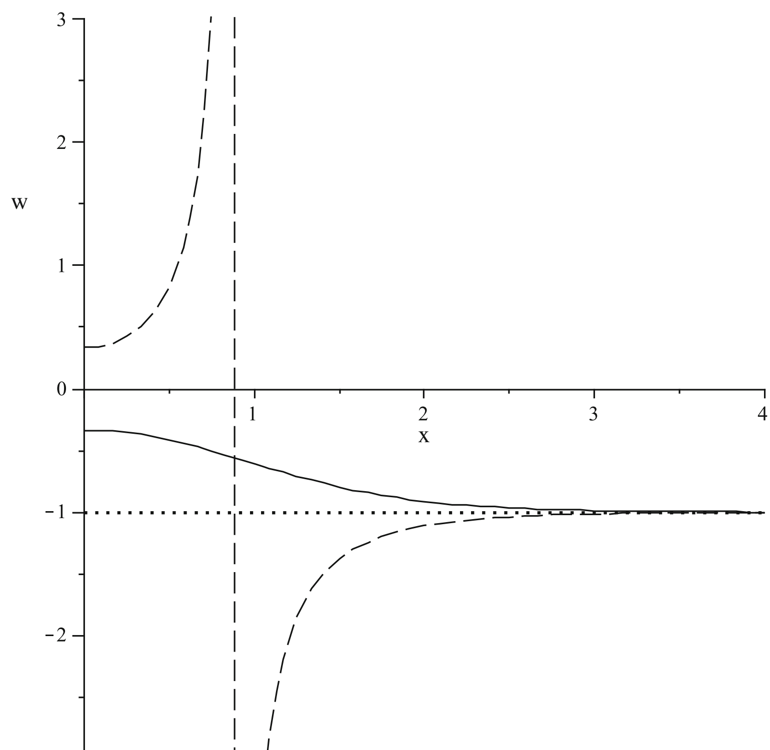


**Figure 1.** Effective EoS parameter $w$ vs. $x = H_0 t$ in Cases (i) and (iv) (dotted line), in Case (ii) (solid line) and in Case (iii) (dashed line).

cosmological constant $\Lambda$, that is $\Lambda/\kappa = H_0^2/4\lambda^2$. This implies the well-known result for a de Sitter model: $H_0 = \sqrt{\Lambda/3}$. At the same time, one can conclude from (21) that while $\sinh^2(H_0 t) < 1$ the energy density $\rho_1$ is negative, i.e. the weak energy condition is violated. But as soon as $\sinh^2(H_0 t) > 1$, $\rho_1$ becomes positive, and the energy condition takes effect. From eq. (21), it is obvious that there is no problem of this sort in Case (2), as well as in Case (3).

## 4. Conclusion

In summary, the model of interacting phantom and YM fields in FRW non-flat cosmology are briefly studied in this paper. First of all, we have derived the set of main equations which determines the model dynamics. Using the specific solution of YM equation previously considered in FRW scalar field cosmology, we generalized the model investigated in [30] in the case of interacting phantom and YM fields. This allowed us to obtain some exact solutions for the accelerated expansion of FRW cosmological model. Besides, we derive the induced potentials of phantom field corresponding to cases in §3.1 and 3.2 in which the Hubble parameter changes as $H(t) = H_0 \coth(H_0 t)$ or $H(t) = H_0 \tanh(H_0 t)$ respectively. At that, it follows from (9), (12), (16) and (18) that all cases considered above are related by the conditions $\dot{\varphi} \sim H_0$ or $\dot{\varphi} \sim H(t)$. In other words, the rate of phantom field change is proportional to either asymptotical value of the Hubble parameter ($H_0 = \lim_{t\to\infty} H(t)$) or to its contemporary value. The effective EoS have been reconstructed for all types of models considered above. Somewhat unexpected are the results of Case (1). Nevertheless, this model considered from the moment $t = H_0^{-1} \sinh^{-1}(1)$ does not demonstrate any oddity. As can be seen from our examples, all EoS parameters considered do not cross the phantom divide $-1$ at late time. This is not a characteristic property of the model but only the consequence of the given simplest expansion regimes.

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