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A rotating charged black hole solution in $f(R)$ gravity

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Abstract. In the context of $f(R)$ theories of gravity, we address the problem of finding a rotating charged black hole solution in the case of constant curvature. A new metric is obtained by solving the field equations and we show that its behaviour is typical of a rotating charged source. In addition, we analyse the thermodynamics of the new black hole. The results ensure that the thermodynamical properties in $f(R)$ gravities are qualitatively similar to those of standard General Relativity.

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1. Introduction

Increasing attention has been paid recently to modified theories of gravity in order to understand several open cosmological questions such as the accelerated expansion of the Universe and the nature of dark matter. A common option is to modify General Relativity by adding higher powers of the scalar curvature R, the Riemann and Ricci tensors, or their derivatives [1] in the Lagrangian formulation. Lovelock and $f(R)$ theories are some examples of these attempts. Therefore, it is quite natural to ask about the existence of black hole and its features in those gravitational theories. One can expect that some signatures of black holes in these theories will be in disagreement with the expected physical results of Einstein's gravity. For these purposes, research on the thermodynamical quantities of black holes is of particular interest.

We shall consider the $f(R)$ gravity theories in metric formalism in Jordan's frame. The gravitational Lagrangian is given by $R + f(R)$ where $f(R)$ is an arbitrary function of the curvature scalar R. Einstein's equations are usually fourth order in the metric [2,3] and when working with constant curvature, solutions are very similar to those of General Relativity with a cosmological constant. An example of this can be seen in the $f(R)$-Maxwell static black hole obtained in [4–6], where it has been shown that all of its thermodynamic quantities are similar to those of the Reissner–Nordström–AdS black hole when making appropriate replacements.

In this paper, we find the rotating charged static black hole of $f(R)$ gravity in the case of constant curvature scalar and study some of the thermodynamic aspects of the new solution.

2. $f(R)$ Theories with constant curvature scalar

Consider the action for $f(R)$ gravity with Maxwell term in four dimensions

$$S = S_{\rm g} + S_{\rm M}, \tag{1}$$

where $S_{\rm g}$ is the four-dimensional gravitational action given by

$$S_{\rm g} = \frac{1}{16\pi}\int {\rm d}^D x\sqrt{|g|}(R + f(R)) \tag{2}$$

and $S_{\rm M}$ is the electromagnetic action given by

$$S_{\rm M} = -\frac{1}{16\pi}\int {\rm d}^4 x\sqrt{-g}\left[F_{\mu\nu}F^{\mu\nu}\right], \tag{3}$$

where g is the determinant of the metric, R is the scalar curvature and $R + f(R)$ is the function defining the theory under consideration. From the above action, the Maxwell equation takes the form

$$\nabla_\mu F^{\mu\nu} = 0 \tag{4}$$

while the field equations in the metric formalism take the form

$$R_{\mu\nu}\left(1+f'(R)\right)-\frac{1}{2}\left(R+f(R)\right)g_{\mu\nu}+\left(g_{\mu\nu}\nabla^2-\nabla_\mu\nabla_\nu\right)f'(R) = 2T_{\mu\nu}, \tag{5}$$

where $R_{\mu\nu}$ is the Ricci tensor, ∇ is the usual covariant derivative and the stress-energy tensor of the electromagnetic field is given by

$$T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{g_{\mu\nu}}{4}F_{\rho\sigma}F^{\rho\sigma} \tag{6}$$

with

$$T^\mu_{\ \mu} = 0. \tag{7}$$

Considering the constant curvature scalar $R = R_0$, the trace of (5) leads to

$$R_0\left(1 + f'(R_0)\right) - 2\left(R_0 + f(R_0)\right) = 0 \tag{8}$$

which determines the constant curvature scalar,

$$R_0 = \frac{2f(R_0)}{f'(R_0) - 1} \tag{9}$$

as long as $f'(R_0) - 1 \neq 0$. This shows that the selection of $f(R_0)$ fixes the value of the Ricci scalar. Therefore, some theories of gravity can give multiple real values of R_0 while other theories may not have real solution for the constant Ricci scalar. Some examples of the different models of $f(R)$ theories and the respective values of R_0 are reported in [7–9].

If the gravity theory (i.e. the selection of $f(R_0)$) gives a real value for R_0, we can use eq. (9) in eq. (5) to obtain

$$R_{\mu\nu} = \frac{1}{2}\left(\frac{f(R_0)}{f'(R_0)-1}\right) g_{\mu\nu} + \frac{2}{1+f'(R_0)} T_{\mu\nu} \tag{10}$$

which shows similarity between this theory and General Relativity with an effective cosmological constant $\Lambda_{\text{eff}} = \frac{1}{2}\left(f(R_0)/(f'(R_0)-1)\right)$ and an effective Newtonian constant $G_{\text{eff}} = G/(1+f'(R_0))$. To ensure a positive effective gravitational constant, we shall impose $1+f'(R_0) > 0$. It is also known [9] that $f''(R_0) > 0$ (when requiring the existence of a stable high-curvature regime such as the matter-dominated Universe) and $f'(R_0) < 0$ (if we want to recover General Relativity at early times). These conditions restrict $f'(R_0)$ to be a negative, monotic function between the values $-1 < f'(R_0) < 0$. Therefore, $(f(R_0)/f'(R_0)-1) < 0$ as long as $f(R_0) > 0$, and it implies that we shall consider a negative value of the Ricci scalar (i.e. the effective cosmological constant) from now on.

Some of the most important models of $f(R)$ gravity theories are

(1) $f(R_0) = \alpha\,|R_0|^{\beta}$. This model accounts for a Universe with accelerated expansion. Equation (9) gives a curvature scalar given by

$$R_0 = \pm\left[\frac{\pm 1}{\alpha\,(\beta-2)}\right]^{1/(\beta-1)} = 4\Lambda_{\text{eff}}. \tag{11}$$

Since we are considering only negative values of R_0, the parameters of this kind of solution are restricted to $\{\alpha < 0, \beta > 2\}$ or $\{\alpha > 0, \beta < 1\}$.

(2) $f(R_0) = \pm\,|R_0|^{\beta}\,\mathrm{e}^{\alpha/R} - R_0$. This time, eq. (9) gives the curvature scalar as

$$R_0 = \frac{\alpha}{\beta-2} = 4\Lambda_{\text{eff}}. \tag{12}$$

The negative values of R_0 are obtained for parameters in the ranges $\{\alpha < 2, \beta > 2\}$ or $\{\alpha > 0, \beta < 2\}$.

(3) $f(R_0) = R_0\left[\log(\alpha R_0)\right]^{\beta} - R_0$. The curvature scalar given by eq. (9) is

$$R_0 = \frac{\mathrm{e}^{\beta}}{\alpha} = 4\Lambda_{\text{eff}}. \tag{13}$$

The negative values of R_0 are obtained for $\alpha < 0$ and the condition $1+f'(R_0) > 0$ restrict $\beta > 0$.

3. Rotating charged black hole in $f(R)$ theory

Inspired by the Kerr–Newman–AdS black hole solution, we introduce the axisymmetric ansatz in Boyer–Lindquist-type coordinates (t, r, θ, φ),

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{B(r)}{\rho^2}\left[\mathrm{d}t - \frac{a\sin^2\theta}{C(r)}\mathrm{d}\varphi\right]^2 + \frac{\rho^2}{B(r)}\mathrm{d}r^2 \\ &+ \frac{\rho^2}{D(\theta)}\mathrm{d}\theta^2 + \frac{D(\theta)\sin^2\theta}{\rho^2}\left[a\,\mathrm{d}t - \frac{r^2+a^2}{C(r)}\mathrm{d}\varphi\right]^2 \end{aligned} \tag{14}$$

with

$$\rho^2 = r^2 + a^2 \cos^2\theta \tag{15}$$

and $B(r)$, $C(r)$ and $D(\theta)$ are functions to be determined by the field equations. However, the field equations alone are insufficient to determine all the unknown functions uniquely. Since we are interested in solutions possesing a regular horizon at $r = r_+$ we shall impose the condition $B\,(r_+) = 0$. Additionally, we shall consider that in the asymptotic region, the metric will be flat if $R_0 = 0$ and $T_{\mu\nu} \to 0$.

Solving the field equations (10) together with the condition of constant curvature scalar, we obtain the solution

$$B = \Delta_r = \left(r^2 + a^2\right)\left(1 + \frac{R_0}{12}r^2\right) - 2Mr + \frac{Q^2}{(1 + f'(R_0))} \tag{16}$$

$$C = \Xi = 1 - \frac{R_0}{12}a^2 \tag{17}$$

$$D = \Delta_\theta = 1 - \frac{R_0}{12}a^2\cos^2\theta \tag{18}$$

and the line element can be written in the convenient form as

$$\begin{aligned} \mathrm{d}s^2 = &- \frac{\Delta_r}{\rho^2}\left[\mathrm{d}t - \frac{a\sin^2\theta}{\Xi}\mathrm{d}\varphi\right]^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 \\ &+ \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left[a\,\mathrm{d}t - \frac{r^2 + a^2}{\Xi}\mathrm{d}\varphi\right]^2. \end{aligned} \tag{19}$$

Note that when $a = 0$ the solution reproduces the charged rotating black hole reported in [4–6] and it is a generalization of the black hole for a limited case of $f(R)$ theories studied in [7]. The gauge field considered has the potential

$$A_t(r) = -\frac{Qr}{\sqrt{\rho^2\Delta_r}}. \tag{20}$$

4. Thermodynamics

To complete the analysis of the rotating charged solution, we shall calculate some thermodynamical quantities. The radius of the horizon r_+ is defined by the condition $\Delta_r = 0$, i.e.

$$\left(r_+^2 + a^2\right)\left(1 + \frac{R_0}{12}r_+^2\right) - 2Mr_+ + \frac{Q^2}{(1 + f'(R_0))} = 0, \tag{21}$$

and it gives the horizon area

$$A = \frac{4\pi\left(r_+^2 + a^2\right)}{\Xi} = \frac{4\pi\left(r_+^2 + a^2\right)}{1 - (R_0/12)a^2}. \tag{22}$$

The Hawking temperature is defined as

$$T = \frac{\kappa}{2\pi} \tag{23}$$

with the surface gravity κ given by

$$\kappa^2 = -\frac{1}{2}\nabla^{\mu}\chi^{\nu}\nabla_{\mu}\chi_{\nu}, \tag{24}$$

where χ^{ν} are null Killing vectors. The metric (19) has the Killing vectors $\xi^{\nu} = \partial_t$ and $\zeta^{\nu} = \partial_{\varphi}$ that are associated with the time translation and rotational invariance respectively. Thus, we define

$$\chi^{\nu} = \xi^{\nu} + \Omega\zeta^{\nu} \tag{25}$$

and we shall find Ω imposing χ^{ν} to be a null vector. This gives

$$\chi^{\nu}\chi_{\nu} = g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi} = 0, \tag{26}$$

from which

$$\Omega = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} \pm \sqrt{\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \tag{27}$$

and therefore at the event horizon $\Delta\,(r_+) = 0$, it reduces to

$$\Omega_+ = \frac{a\,\Xi}{r_+^2 + a^2}. \tag{28}$$

This gives the surface gravity

$$\kappa = \frac{1}{2\left(r_+^2 + a^2\right)}\left.\frac{\mathrm{d}\Delta_r}{\mathrm{d}r}\right|_{r=r_+} \tag{29}$$

$$\kappa = \frac{r_+^2 - a^2 - \frac{Q^2}{(1+f'(R_0))} + \frac{R_0}{12}a^2 r_+^2 + \frac{R_0}{4}r_+^4}{2r_+\left(r_+^2 + a^2\right)} \tag{30}$$

and the corresponding Hawking temperature

$$T = \frac{r_+^2 - a^2 - \frac{Q^2}{(1+f'(R_0))} + \frac{R_0}{12}a^2 r_+^2 + \frac{R_0}{4}r_+^4}{4\pi r_+\left(r_+^2 + a^2\right)}. \tag{31}$$

The same result is obtained by making an analytical continuation of the Lorentzian metric by $t \to i\tau$ and $a \to ia$, which gives the Euclidean section. Here, the regularity at $r = r_+$ requires the identification $\tau \sim \tau + \beta$ and $\varphi \sim \varphi + i\beta\Omega_+$ where

$$\beta = \frac{4\pi r_+\left(r_+^2 + a^2\right)}{r_+^2 - a^2 - \frac{Q^2}{(1+f'(R_0))} + \frac{R_0}{12}a^2 r_+^2 + \frac{R_0}{4}r_+^4} = \frac{1}{T}. \tag{32}$$

4.1 *Generalized Smarr formula*

To obtain a generalized Smarr formula we find the total energy (mass) E and angular momentum J of the black hole by means of Komar integrals. To do it, we use the Killing vectors $\frac{1}{\Xi}\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial\varphi}$ to obtain

$$E = \frac{M}{\Xi^2} \tag{33}$$

$$J = \frac{aM}{\Xi^2}. \tag{34}$$

Note that the time Killing vector has been normalized in order to generate an $so\,(3, 2)$ algebra from the corresponding conserved quantities. Using these quantities and the horizon area (22) in eq. (21), we obtain

$$E^2 = \frac{A}{16\pi} + \frac{\pi}{A}\left[4J^2 + \frac{Q^4}{(1+f'(R_0))^2}\right] + \frac{Q^2}{2\,(1+f'(R_0))} - \frac{R_0}{12}J^2 - \frac{R_0 A}{96\pi}\left[\frac{Q^2}{(1+f'(R_0))} + \frac{A}{4\pi} - \frac{R_0 A^2}{384\pi^2}\right]. \quad (35)$$

By identifying the entropy of the black hole as

$$S = \frac{A}{4} = \frac{\pi\left(r_+^2 + a^2\right)}{1 - \frac{R_0}{12}a^2} \quad (36)$$

we obtain the generalized Smarr formula

$$E^2 = \frac{S}{4\pi} + \frac{\pi}{4S}\left[4J^2 + \frac{Q^4}{(1+f'(R_0))^2}\right] + \frac{Q^2}{2\,(1+f'(R_0))} - \frac{R_0}{12}J^2 - \frac{R_0 S}{24\pi}\left[\frac{Q^2}{(1+f'(R_0))} + \frac{S}{\pi} - \frac{R_0 S^2}{24\pi^2}\right]. \quad (37)$$

As is well known, this relation contains all the thermodynamical information of the black hole. Therefore, one can define the quantities conjugate to S, J and Q as the temperature, angular velocity and electric potential respectively,

$$T = \left(\frac{\partial E}{\partial S}\right)_{J,Q} = \frac{1}{8\pi E}\left[1 - \frac{\pi^2}{S^2}\left(4J^2 + \frac{Q^4}{(1+f'(R_0))^2}\right) - \frac{R_0}{6}\left(\frac{Q^2}{(1+f'(R_0))} + \frac{2S}{\pi} - \frac{R_0 S^2}{8\pi^2}\right)\right] \quad (38)$$

$$\Omega = \left(\frac{\partial E}{\partial J}\right)_{S,Q} = \frac{J}{E}\left(\pi S - \frac{R_0}{12}\right) \quad (39)$$

$$\Phi = \left(\frac{\partial E}{\partial Q}\right)_{S,J} = \frac{Q}{2E\,(1+f'(R_0))}\left[\frac{\pi}{S}\frac{Q^2}{(1+f'(R_0))} + 1 - \frac{R_0 S}{12\pi}\right]. \quad (40)$$

It is easy to verify that the relation (38) for temperature coincides with eq. (31). However, it is not true that the angular momentum (39) coincides with eq. (28). In order to clarify this point, let us write the metric (19) in the form

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{\vartheta^2\sin^2\theta}{\rho^2\Xi^2}\left(\mathrm{d}\varphi - N^\varphi\mathrm{d}t\right)^2, \quad (41)$$

where

$$\vartheta^2 = \left(r^2 + a^2\right)\Delta_\theta - a^2\Delta_r\sin^2\theta \quad (42)$$

$$N = \frac{\rho^2\Delta_r\Delta_\theta}{\vartheta^2} \quad (43)$$

and

$$N^{\varphi} = \frac{a\,\Xi}{\vartheta^2}\left[\left(r^2+a^2\right)\Delta_\theta - \Delta_r\right]. \tag{44}$$

At the horizon, the function N^{φ} coincides with Ω_+ but asymptotically ($r \to \infty$) it becomes

$$N^{\varphi} \to \frac{R_0}{12}a \tag{45}$$

that can be interpreted as the angular velocity at infinity. Therefore, the angular velocity defined in eq. (39) actually corresponds to the difference

$$\Omega = \Omega_+ - \frac{R_0}{12}a \tag{46}$$

as can be easily checked.

Finally, the thermal capacity C at constant angular momentum and charge is another thermodynamical quantity of interest. It is given by

$$C = T\left(\frac{\partial S}{\partial T}\right)_{J,Q} = \frac{4\pi ETS}{1-4\pi T\,(2M+TS) - \frac{R_0}{6}\left(Q^2+\frac{3S}{\pi}-\frac{R_0 S^2}{4\pi^2}\right)}. \tag{47}$$

5. Conclusion

In this work we have obtained a rotating charged solution in $f(R)$ theory of gravity with constant curvature representing a black hole. A new metric is obtained by solving the field equations and we have also calculated some thermodynamical quantities. The behaviour of the new solution is typical of a rotating charged source and the analysis shows that the thermodynamical properties in $f(R)$ gravities, and specially in the constant curvature case, are qualitatively similar to those of standard General Relativity.

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