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Light squeezing in optical parametric amplification beyond the slowly-varying amplitude approximation

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Abstract. Optical parametric amplification (OPA) described usually by the coupled-wave equations with the first-order derivatives of the signal and idler waves, is solved under the slowly-varying amplitude approximation (SVA). In this article, by keeping the second-order derivatives in the coupled-wave equations, we obtained an analytical solution for the output signal and idler waves up to the first order of $(\kappa/k)^1$; the ratio of coupling constant to the wave number. Furthermore, here the signal and the idler waves are distinguished only by their polarizations with the same frequency. Light squeezing is observed in normally ordered variances of the two quadrature operators of the output combined mode when plotted against κL, where κ is the coupling constant and L the interaction length. The variances have different signs for a range of values of κL and their variations are in opposite directions. We also show that this property is strongly dependent on the relative refractive index of the medium (n). It is worth mentioning that the relative index dependency is not an explicit feature in squeezing of OPA under SVA approximation. Furthermore, the squeezing vanishes when $n \to 1$ and $\kappa/k \to 0$.

Keywords. Quantum nonlinear optics; squeezed light; optical parametric amplification.

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1. Introduction

Squeezed states of light are the minimum uncertainty states in which the fluctuations in one quadrature can be reduced, below the vacuum fluctuations, by increasing the fluctuations of other quadratures [1–4]. It has long been known that optical parametric amplification (OPA) can produce these nonclassical states of the electromagnetic fields [5–9]. The destruction of a photon of frequency ω_p from a single narrow-band incident pump beam, in a nonlinear medium with second-order nonlinearity, creates simultaneously two photons with frequencies ω and $\omega - \omega_p$ which form signal and idler beams, respectively. Nowadays, the squeezing of light is an established technique and has many applications, for example, in reducing noises in optical communication systems [10], and in the possible detection of gravitational waves by optical interferometry [2]. OPA has some applications in generating a single or two-mode squeezed states of light [11–13], as a source of the entangled light beams and images [14,15] and in improving optical resolution [16].

OPA is usually described by the coupled-mode equations with the first-order derivatives of the signal and idler waves, under the slowly-varying amplitude approximation (SVA). Here, by keeping the second-order derivatives in the coupled-wave equations we obtained an analytical solution for the output signal and idler waves up to the first order of $(\kappa/k)^1$. In this study we also consider the signal and the idler waves distinguished only by their polarization states with the same frequency. The classical aspects and physical interpretation of this OPA solution have been investigated in our previous work [17]. Here we introduce the quantum aspects of this solution to study the squeezed state properties of the output modes. Light squeezing is observed in normally-ordered variances of the quadratures of the operator which is produced by the combination of the output signal and idler modes. The results show that in certain regions of the normalized coupling constant κL, light squeezing is strongly dependent on the relative refractive index of the medium.

Similar approaches have been already utilized by the degenerate four-wave mixing (DFWM) phenomenon without the SVA approximation [18]. In that case, we have shown that the squeezing property of the proper output combination mode has an oscillatory behaviour in certain regions of κL. We saw that by increasing the relative refractive index

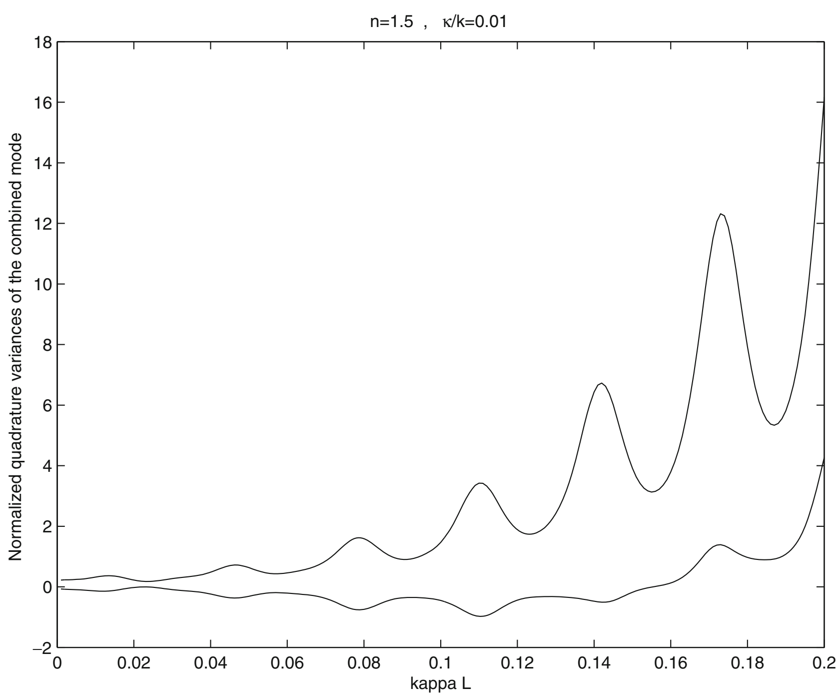

Figure 1. Normally ordered variances of quadrature operators of the output combined mode for $\kappa/k = 0.01$ and $n = 1.5$.

of the medium the squeezing effect is increased, the same property we have obtained in our present study.

This article proceeds by presenting a brief review of the quantum behaviour of the OPA solution with the SVA approximation under the same situation that we have used for the solution of OPA without SVA, in §2. The solutions of the output modes beyond the SVA approximation in OPA are included in §3. Later, the quantum fluctuations of the output generated modes are discussed in §4, and finally the article will be concluded in §5.

2. Quantum aspects of the OPA solution within the SVA approximation

Optical parametric amplification is simply the transfer of energy from photons of the pump at $\omega_3(\omega_p)$ to the signal and idler photons at frequencies $\omega_1(\omega_s)$ and $\omega_2(\omega_i)$ respectively. Let us consider two optical waves of frequencies ω_3 and ω_1 with the corresponding quantized field amplitudes a_3 and a_1 interacting with each other in a lossless nonlinear medium, via the second-order susceptibility, $\chi^{(2)}$, to produce an output wave at frequency $\omega_2 = \omega_3 - \omega_1$ with quantized amplitude a_2. It is customary to consider the pump beam to be very strong and to treat it with $\chi^{(2)}$ of the medium classically, i.e. it is not necessary to treat A_3 as an operator. In the SVA approximation and perfect phase-matching

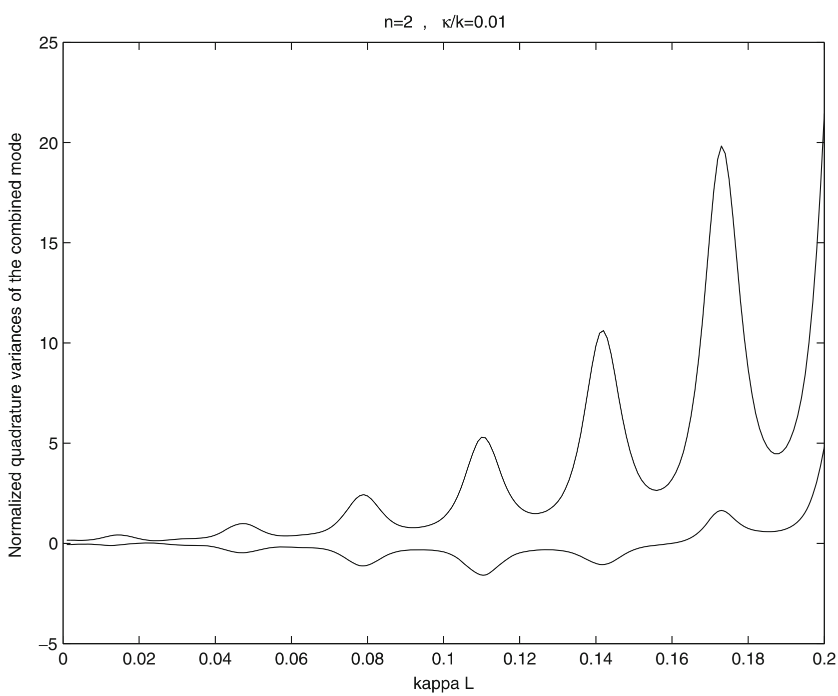

Figure 2. Normally ordered variances of quadrature operators of the output combined mode for $\kappa/k = 0.01$ and $n = 2$.

condition, the quantized field amplitudes of the amplified signal mode a_1 and the generated idler mode a_2 satisfy the following coupled mode equations. Here we consider the case $\omega_1 = \omega_2 = \omega$ $(k_1 = k_2 = k)$, where these modes are distinguished only by their polarizations

$$\frac{\mathrm{d}a_1}{\mathrm{d}z} = \kappa a_2^{\dagger}, \tag{1}$$

$$\frac{\mathrm{d}a_2}{\mathrm{d}z} = \kappa a_1^{\dagger}, \tag{2}$$

where $\kappa = 4\pi k\chi^{(2)}|A_3|$, is the coupling constant. The solutions of these coupled mode equations for the output signal and idler modes with respect to the input modes are

$$a_1(L) = (\cosh\kappa L)a_1(0) + (\sinh\kappa L)a_2^{\dagger}(0) \tag{3}$$

$$a_2(L) = (\sinh\kappa L)a_1^{\dagger}(0) + (\cosh\kappa L)a_2(0), \tag{4}$$

where L is the interaction length and here κ is real. In contrast with the classical case [2,9], the introduction of the idler mode from the rare side of the medium, $a_2(0)$, is necessary to describe the quantum characteristics of the output modes, $a_1(L)$ and $a_2(L)$. By assuming vacuum-state fields at the input of the medium, where $z = 0$, the expectation values of

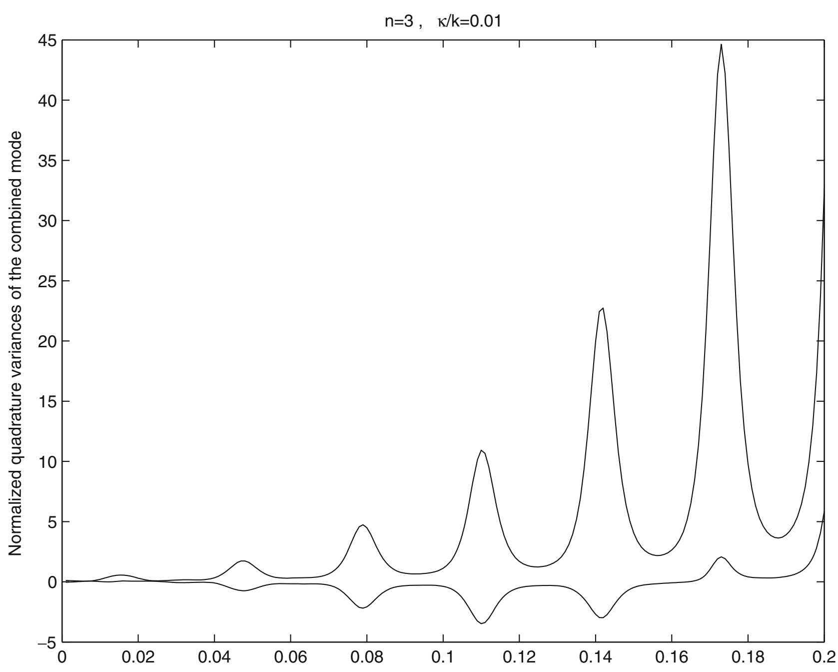

Figure 3. Normally ordered variances of quadrature operators of the output combined mode for $\kappa/k = 0.01$ and $n = 3$.

the output signal and idler modes vanish, i.e., $\langle a_1(L)\rangle = \langle a_2(L)\rangle = 0$. The second-order correlation functions within the signal and idler modes, at $z = L$, are not equal to zero and are given by

$$\langle a_1^\dagger(L)a_1(L)\rangle = \langle a_2^\dagger(L)a_2(L)\rangle = \sinh^2(\kappa L). \tag{5}$$

Equation (5) shows that the mean photon number of the outgoing signal and idler modes are not zero in the absence of input signals.

Here we employed the situation in which the signal and the idler modes are distinguishable only by their polarization states. For example, they may have orthogonal linear polarizations. Thus, it is suitable here to consider one output mode which is constructed from the outgoing modes as follows:

$$a_{\text{out}} = \frac{2^{1/2}}{2}(a_1(L) + a_2(L)). \tag{6}$$

The quadrature components of the combination output mode are given by

$$a_{\text{out}}^{(1)} = \frac{a_{\text{out}} + a_{\text{out}}^\dagger}{2}, \tag{7}$$

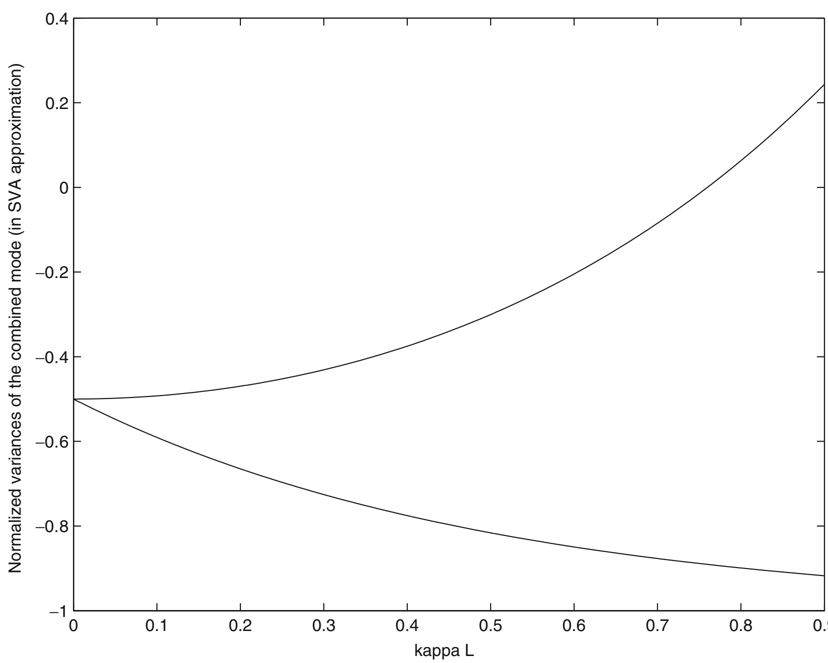

Figure 4. Normally ordered variances of the quadrature operators for output combined mode in OPA with SVA approximation. These quadratures can be calculated using eq. (27) with the simple transformation $\hat{A}_{\text{out}} \rightarrow a_{\text{out}}$. The squeezing effect, i.e. variances with different signs, occurred after $\kappa L \simeq 0.76$ and is not like squeezing in OPA without SVA approximation which occurred in 0.001–0.15 accompanied by some peaks (see figures 1–3).

and

$$a_{\rm out}^{(2)} = \frac{a_{\rm out} - a_{\rm out}^{\dagger}}{2\imath}. \tag{8}$$

Quantum variances of these quadratures for coherent incoming modes with quadrature fluctuations $\langle(\Delta a_1^{(1)}(0))^2\rangle = \langle(\Delta a_2^{(1)}(0))^2\rangle = \langle(\Delta a_1^{(2)}(0))^2\rangle = \langle(\Delta a_1^{(2)}(0))^2\rangle = 1/2$ are

$$\langle(\Delta a_{\rm out}^{(1)})^2\rangle = \frac{1}{2}(\cosh\kappa L + \sinh\kappa L)^2, \tag{9}$$

and

$$\langle(\Delta a_{\rm out}^{(2)})^2\rangle = \frac{1}{2}(\cosh\kappa L - \sinh\kappa L)^2. \tag{10}$$

According to eqs (9) and (10) we have

$$\langle(\Delta a_{\rm out}^{(1)})^2\rangle\langle(\Delta a_{\rm out}^{(2)})^2\rangle = \frac{1}{4}. \tag{11}$$

This shows that the combination mode is in the minimum uncertainty state. However, in contrast with the coherent state, the combination mode has unequal uncertainty

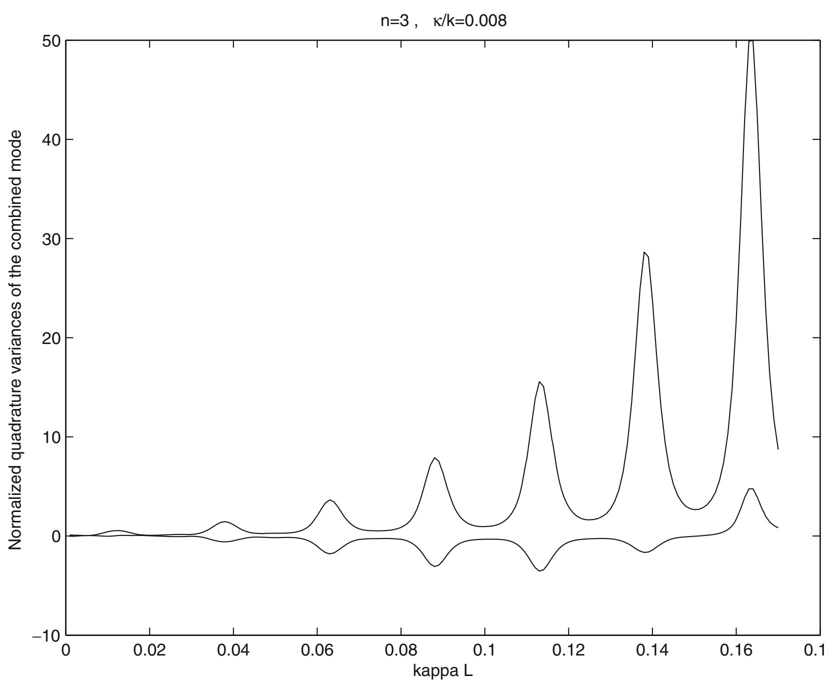

Figure 5. Normally ordered variances of quadrature operators of the output combined mode for $\kappa/k = 0.008$ and $n = 3$.

in the two field quadratures for $\kappa L \neq 0$. This feature is the footprint of the squeezed property of light in OPA process under SVA approximation. In the next section, we shall introduce the solution for the output modes in OPA without SVA approximation to study the squeezing properties of these modes, under the same situation as considered above.

3. Output modes in OPA beyond the SVA approximation

In this section, we extend our classical work which was previously done on OPA without SVA approximation [17] to phenomenological and semiclassical quantum treatment. For this purpose, initially, we use the corresponding classical coupled-wave equations with second-order derivatives [17]

$$\frac{\mathrm{d}^2 A_1}{\mathrm{d}z^2} - 2\imath k_1 \frac{\mathrm{d}A_1}{\mathrm{d}z} = \frac{4\pi \omega_1^2 \chi^{(2)}}{c^2} A_3 A_2^*, \tag{12}$$

$$\frac{\mathrm{d}^2 A_2}{\mathrm{d}z^2} - 2\imath k_2 \frac{\mathrm{d}A_2}{\mathrm{d}z} = \frac{4\pi \omega_2^2 \chi^{(2)}}{c^2} A_3 A_1^*, \tag{13}$$

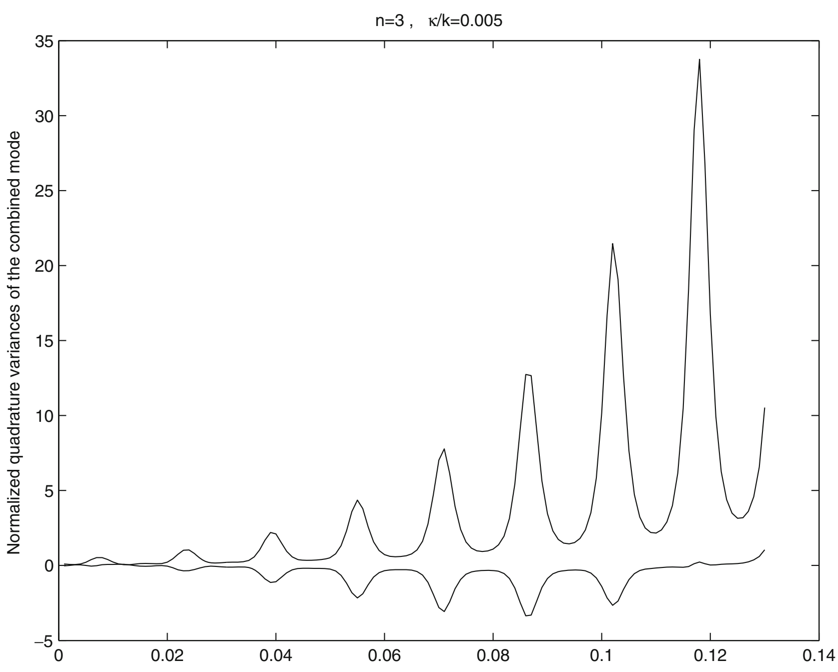

Figure 6. Normally ordered variances of quadrature operators of the output combined mode for $\kappa/k = 0.005$ and $n = 3$.

where A_1, A_2 and A_3 are the amplitudes of the signal, idler and pump fields, respectively. If simultaneous solutions of eqs (12) and (13) are considered for the case $\omega_1 = \omega_2$, one easily finds the following equation for the signal:

$$\frac{\mathrm{d}^4 A_1}{\mathrm{d}z^4} + 4k^2 \frac{\mathrm{d}^2 A_1}{\mathrm{d}z^2} - 4\kappa^2 k^2 A_1 = 0. \tag{14}$$

The general solution of eq. (14) up to the first order of κ/k can be written as

$$A_1(z) = C_1 \exp(\kappa z) + C_2 \exp(-\kappa z) + C_3 \exp(2\imath kz) + C_4 \exp(-2\imath kz). \tag{15}$$

The first two terms of eq. (15) are similar to the SVA solution but the last two terms are new and imply an oscillation with spatial frequency $2k$. To determine the constants in eq. (15), C_1, ..., C_4, suitable boundary conditions should be applied. The continuity of the tangential components of the electric and magnetic fields of $A_1(z)$ at $z = L$ accompanied with $A_2(0) = 0$ and an arbitrary value for $A_1(0)$, yield spatial variations of $A_1(z)$. The same procedure is followed for $A_2(z)$ by substituting $A_1(z)$ into eq. (12). These classical solutions were found and analysed in our previous work [17]. Here, in order

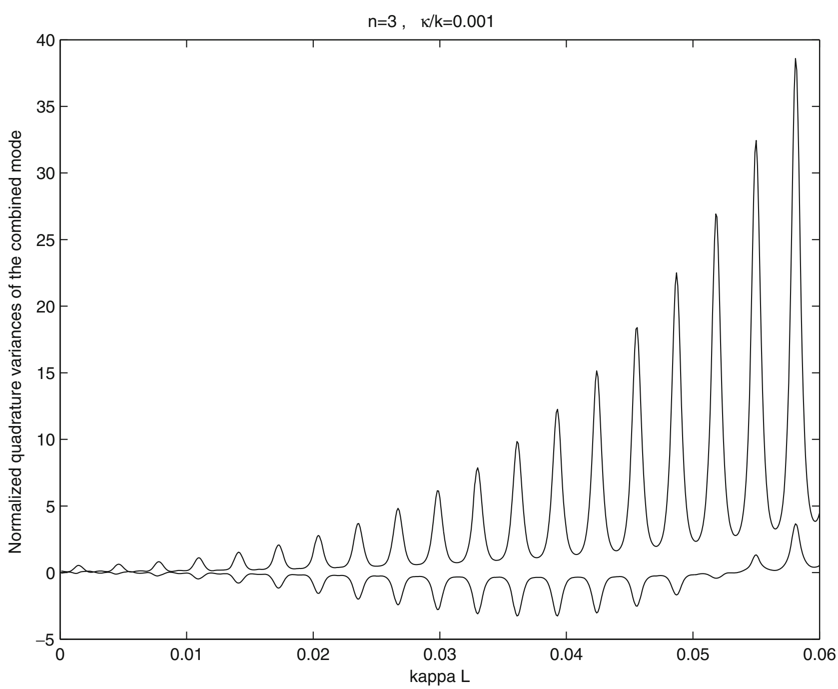

Figure 7. Normally ordered variances of quadrature operators of the output combined mode for $\kappa/k = 0.001$ and $n = 3$.

to consider the vacuum fluctuations in our quantum investigation, we keep the incoming idler amplitude $A_2(0) \neq 0$, with the above boundary conditions, for finding $A_1(z)$ and $A_2(z)$. Thus, after some lengthy calculations, the output fields at $z = L$ relate to the input fields at $z = 0$, by

$$A_1(L) = \eta_1 A_1(0) + \eta_2 A_2(0)^\star, \tag{16}$$

and

$$A_2(L) = \eta_3 A_1(0)^\star + \eta_4 A_2(0). \tag{17}$$

The quantities η_1, η_2, η_3 and η_4 are complex constants and have some large expressions (see appendix A), and $\star$ denotes the complex conjugate. The quantum mechanical versions of relations (16) and (17) are obtained when the input and output amplitudes, A_1 and A_2, are replaced by their corresponding operators, $\hat{A}_1$ and $\hat{A}_2$. Then

$$\hat{A}_1(L) = \eta_1 \hat{A}_1(0) + \eta_2 \hat{A}_2(0)^\dagger, \tag{18}$$

and

$$\hat{A}_2(L) = \eta_3 \hat{A}_1(0)^\dagger + \eta_4 \hat{A}_2(0). \tag{19}$$

As shown in eqs (18) and (19), the expectation values of the output signal and the idler operators vanish when the incoming fields are in vacuum states. But the second-order

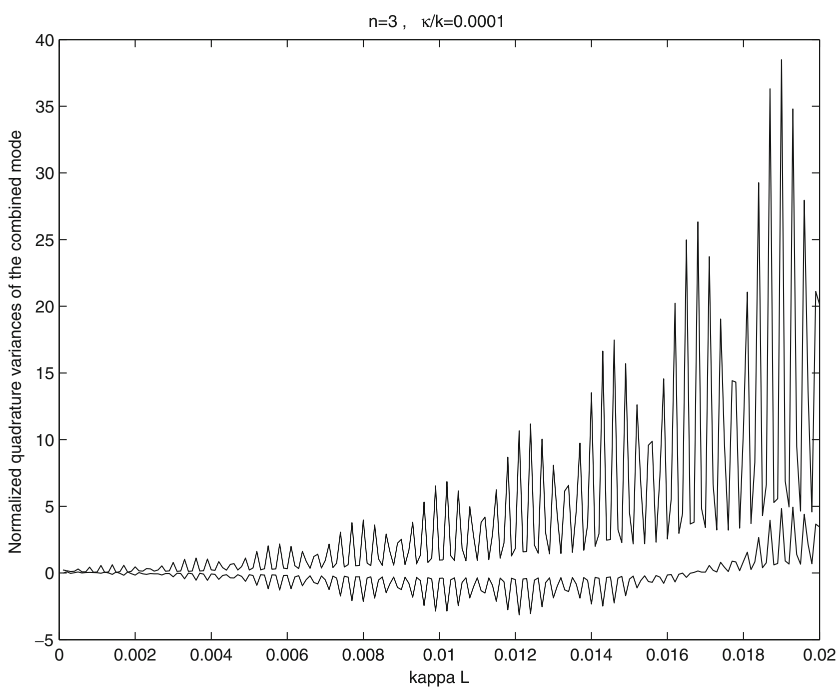

Figure 8. Normally ordered variances of quadrature operators of the output combined mode for $\kappa/k = 0.0001$ and $n = 3$. This figure has the same value of n as in figures 5–7 with smaller values of κ/k.

correlation functions or the mean photon number in the outgoing signal and idler modes are not zero and are given, respectively, by

$$\langle \hat{A}_1(L)^\dagger \hat{A}_1(L)\rangle = |\eta_2|^2, \tag{20}$$

and

$$\langle \hat{A}_2(L)^\dagger \hat{A}_2(L)\rangle = |\eta_4|^2. \tag{21}$$

4. Quantum fluctuations of the output modes

To investigate the squeezed state properties of the output modes, eqs (18) and (19), when these modes are distinguished only by their polarizations, it is better to consider only one output mode produced from the output signal and idler modes, the same as eq. (6), by

$$\hat{A}_{\text{out}} = 1/(2)^{1/2}[\hat{A}_1(L) + \hat{A}_2(L)]. \tag{22}$$

The quadrature operators belonging to the output combined mode are

$$\hat{A}^{(1)}_{\text{out}} = \frac{\hat{A}_{\text{out}} + \hat{A}^\dagger_{\text{out}}}{2}, \tag{23}$$

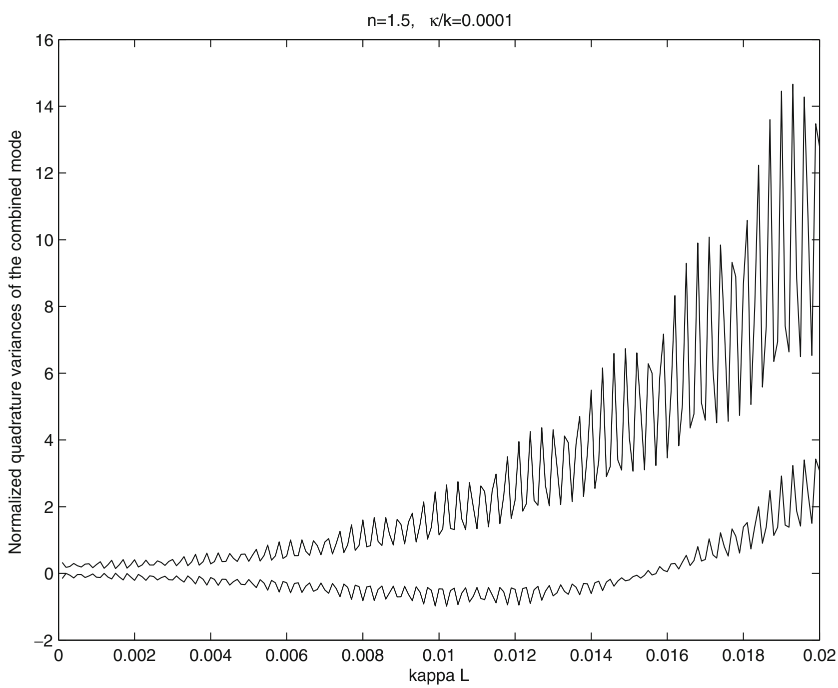

Figure 9. Normally ordered variances of quadrature operators of the output combined mode for $\kappa/k = 0.0001$ and $n = 1.5$. The value of κ/k is the same as that in figure 8 with a smaller value of relative refractive index.

and

$$\hat{A}_{\text{out}}^{(2)} = \frac{\hat{A}_{\text{out}} - \hat{A}_{\text{out}}^{\dagger}}{2\imath}. \tag{24}$$

By using the quadrature variances of the coherent incoming modes, $\langle(\Delta A_1^{(1)}(0))^2\rangle = \langle(\Delta A_2^{(1)}(0))^2\rangle = \langle(\Delta A_1^{(2)}(0))^2\rangle = \langle(\Delta A_1^{(2)}(0))^2\rangle = 1/2$, one can evaluate the fluctuations of these output quadratures as follows:

$$\langle(\Delta \hat{A}_{\text{out}}^{(1)})^2\rangle = \frac{1}{4}\left[(\eta_{1\text{r}} + \eta_{3\text{r}})^2 + (\eta_{1\text{i}} - \eta_{3\text{i}})^2 + (\eta_{2\text{r}} + \eta_{4\text{r}})^2 + (\eta_{2\text{i}} - \eta_{4\text{i}})^2\right], \tag{25}$$

and

$$\langle(\Delta \hat{A}_{\text{out}}^{(2)})^2\rangle = \frac{1}{4}\left[(\eta_{1\text{r}} - \eta_{3\text{r}})^2 + (\eta_{1\text{i}} + \eta_{3\text{i}})^2 + (\eta_{2\text{r}} - \eta_{4\text{r}})^2 + (\eta_{2\text{i}} + \eta_{4\text{i}})^2\right], \tag{26}$$

where $\eta_{j\text{r}}$ and $\eta_{j\text{i}}$ for $j = 1, 2, 3, 4$ are the real and imaginary parts of η_j, respectively. It can be found from eqs (25) and (26), that the variations of these fluctuations can take place

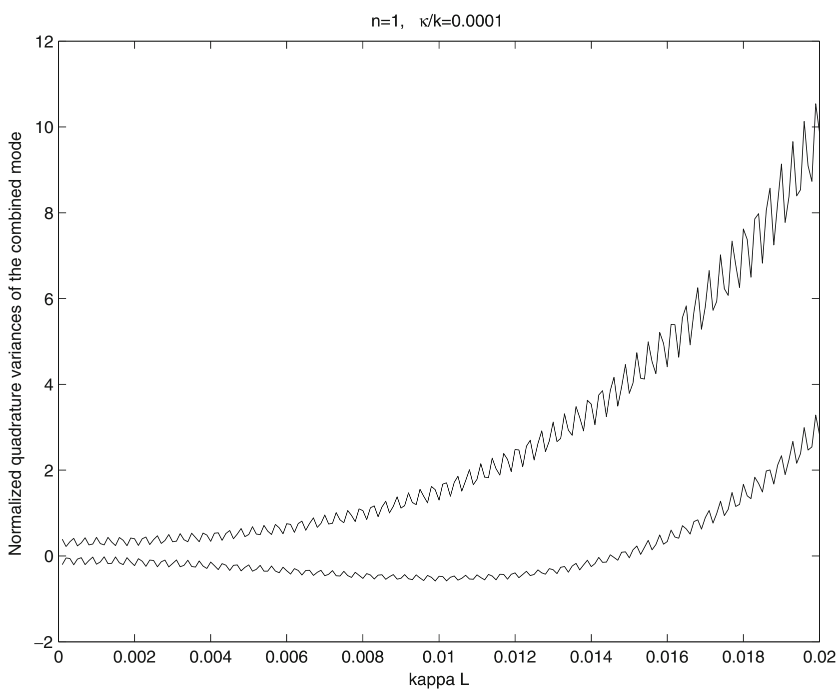

Figure 10. Normally ordered variances of quadrature operators of the output combined mode for $\kappa/k = 0.0001$ and $n = 1$. Compare this figure to figures 8 and 9. As n approaches 1, the squeezing effect takes place in smaller values of κl.

in opposite directions in certain regions of κL. To analyse the squeezing properties of the output combined mode we plotted the normally ordered quadrature variances defined by

$$\langle : (\Delta \hat{A}_{\text{out}}^{(1,2)})^2 : \rangle = \langle (\Delta \hat{A}_{\text{out}}^{(1,2)})^2 \rangle - [\hat{A}_{\text{out}}, \hat{A}_{\text{out}}^{\dagger}], \tag{27}$$

where

$$[\hat{A}_{\text{out}}, \hat{A}_{\text{out}}^{\dagger}] = \frac{1}{2}\left(|\eta_1|^2 - |\eta_2|^2 - |\eta_3|^2 + |\eta_4|^2\right) \tag{28}$$

is a function of κL (normalized coupling constant). In this expression, the values of the quadrature variances may be positive or negative where their signs are important in squeezing effects. As shown in figure 1, for $n = 1.5$ and $\kappa/k = 0.01$, in certain region of κL (0.001–0.15) the values of $\langle :(\Delta \hat{A}_{\text{out}}^{(1)})^2 : \rangle$ are positive and the values of $\langle :(\Delta \hat{A}_{\text{out}}^{(2)})^2 : \rangle$ are negative and their variations are in opposite directions, and in this region the squeezing effect occurs. We see some clear simultaneous occurrences of the positive maxima and negative minima in quadrature variances near certain values of κL like 0.05, 0.08, 0.11, 0.14 (see figures 1–3). Furthermore, when the relative refractive index, n, is

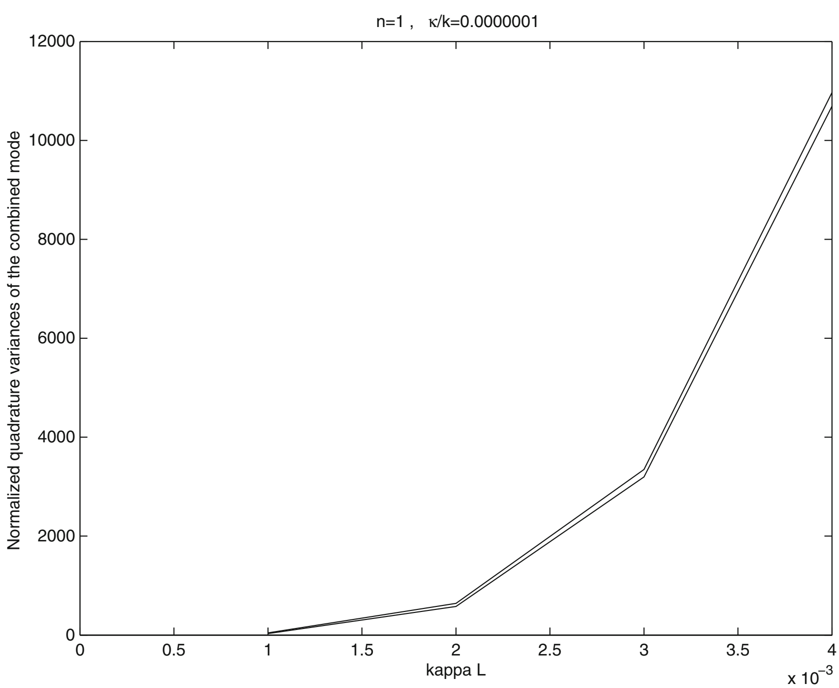

Figure 11. Normally ordered variances of quadrature operators of the output combined mode for $\kappa/k = 10^{-9}$ and $n = 1$. Comparing this figure with figures 5–10, we see that the reduction of n ($n \to 1$) is not sufficient for the disappearance of the squeezing effect, however, the reduction of κ/k ($\kappa/k \to 0$) is also necessary.

increased, as shown in figures 2 and 3, the amplitudes of the quadratures are also increased and consequently the squeezing properties are enhanced . On the other hand, we do not observe any squeezing effects in the same region of κL, for the corresponding quadrature variances under SVA approximation (see figure 4). Thus, according to figures 1–3 the squeezing effect is strongly dependent on the relative refractive index of the medium. Such an important behaviour is not an explicit feature in the squeezing properties of OPA under SVA approximation.

It should be noted that, the squeezing effect is very sensitive to the values of κ/k. From figures 5 and 6 one can see that reducing the values of κ/k implies the reduction of squeezing range of the quadratures. Thus, the position of the peaks, simultaneous occurrence of positive maxima and negative minima in quadrature variances, shift to smaller values of κL (compare figures 5–7). Decreasing κ/k continuously, not only shifts the peaks to lower values of κL, but also increases the number of peaks (see figures 7 and 8). Keeping κ/k constant, any decrease in the value of the relative refractive index will also decrease the amplitude of the corresponding peaks, indicating the reduction of squeezing effects. This can be seen by comparing figures 9 and 10, where both have the same values of κ/k as in figure 8. Finally, as shown in figure 11, by considering smaller values of κ/k when $n \to 1$ the squeezing property disappears and both quadrature variances approach nearly to the same positive values.

5. Conclusion

The coupled-mode equations of the non-degenerate optical parametric amplification are solved analytically beyond the SVA approximation up to the first order of κ/k, where the signal and the idler waves have the same frequencies but different polarizations. The quantum aspects of the resultant output signal and the idler modes in terms of the input signal and idler modes are considered. We plotted the normally ordered quadrature variances of the combined mode vs. κL to investigate the squeezing effects. In a certain region of κL, the variance of one quadrature is positive while the variance of the other is negative and their variations are in opposite directions. In such a region, squeezing occurs. The results show that the quadrature variances of the combined mode and the consequent squeezing properties are strongly dependent on the relative refractive index of the medium. It is worth mentioning that the relative refractive index dependency is not an explicit feature in the squeezing effect of output modes in ordinary OPA under SVA approximation. The reason is that in applying the corresponding approximation the boundary conditions of the end face of the medium and its reflection effects are not considered.

Similar to the classical approach [17], the squeezing effect is more appreciable for the media of short lengths (in the range of medium coherent length) and large relative refractive indices. Besides, one must apply high pump intensity in a medium with a large value of second-order susceptibility in order to fairly adjust the value of κ/k. Its value has an important role for observing the light squeezing in normally ordered quadrature variances of the output combined mode. Furthermore, the squeezing properties vanish when κ/k becomes very small and the relative refractive index approaches 1.

Appendix A: The explicit form of parameters η_1, η_2, η_3 and η_4

The complex quantities η_1, η_2, η_3 and η_4 which are appeared in eqs (18), (19), (25), (26) and (28) have the following expressions for their real and imaginary parts:

$$\begin{aligned}
\eta_{1\mathrm{r}} = {} & (c_{11}c_{13} + c_{12}c_{11})\exp(\kappa L)/\left(c_{13}^2 + c_{14}^2\right) \\
& + (c_{21}c_{23} + c_{22}c_{24})\exp(-\kappa L)/\left(c_{23}^2 + c_{24}^2\right) \\
& + ((c_{31}c_{13} + c_{32}c_{14})\cos(2kL) - (c_{32}c_{13} - c_{31}c_{14})\sin(2kL))/\left(c_{13}^2 + c_{14}^2\right) \\
& + (\kappa/k)((c_{41}c_{43} + c_{42}c_{44})\cos(2kL) + (c_{42}c_{43} - c_{41}c_{44})\sin(2kL))/\left(c_{43}^2 + c_{44}^2\right),
\end{aligned}$$

$$\begin{aligned}
\eta_{1\mathrm{i}} = {} & (c_{12}c_{13} - c_{11}c_{14})\exp(\kappa L)/\left(c_{13}^2 + c_{14}^2\right) \\
& + (c_{22}c_{23} - c_{21}c_{24})\exp(-\kappa L)/\left(c_{23}^2 + c_{24}^2\right) \\
& + ((c_{31}c_{13} + c_{32}c_{14})\sin(2kL) + (c_{32}c_{13} - c_{31}c_{14})\cos(2kL))/\left(c_{13}^2 + c_{14}^2\right) \\
& + (\kappa/k)(-(c_{41}c_{43} + c_{42}c_{44})\sin(2kL) \\
& + (c_{42}c_{43} - c_{41}c_{44})\cos(2kL))/\left(c_{43}^2 + c_{44}^2\right),
\end{aligned}$$

$$\begin{aligned}
\eta_{2\mathrm{r}} = {} & (c_{15}c_{13} + c_{16}c_{14})\exp(\kappa L)/\left(c_{13}^2 + c_{14}^2\right) \\
& + (c_{25}c_{23} + c_{26}c_{24})\exp(-\kappa L)/\left(c_{23}^2 + c_{24}^2\right) \\
& + ((c_{35}c_{13} + c_{36}c_{14})\cos(2kL) - (c_{36}c_{13} - c_{35}c_{14})\sin(2kL))/\left(c_{13}^2 + c_{14}^2\right) \\
& + (\kappa/k)((c_{45}c_{43} + c_{46}c_{44})\cos(2kL) \\
& + (c_{46}c_{43} - c_{45}c_{44})\sin(2kL))/\left(c_{43}^2 + c_{44}^2\right),
\end{aligned}$$

$$\begin{aligned}
\eta_{2\mathrm{i}} = {} & (c_{16}c_{13} - c_{15}c_{14})\exp(\kappa L)/\left(c_{13}^2 + c_{14}^2\right) \\
& + (c_{26}c_{23} - c_{25}c_{24})\exp(-\kappa L)/\left(c_{23}^2 + c_{24}^2\right) \\
& + ((c_{35}c_{13} + c_{36}c_{14})\sin(2kL) + (c_{36}c_{13} - c_{35}c_{14})\cos(2kL))/\left(c_{13}^2 + c_{14}^2\right) \\
& + (\kappa/k)(-(c_{45}c_{43} + c_{46}c_{44})\sin(2kL) \\
& + (c_{46}c_{43} - c_{45}c_{44})\cos(2kL))/\left(c_{43}^2 + c_{44}^2\right),
\end{aligned}$$

$$\begin{aligned}
\eta_{3\mathrm{r}} = {} & ((c_{11}c_{13} + c_{12}c_{14})\exp(\kappa L)(\kappa/2k))/\left(c_{13}^2 + c_{14}^2\right) \\
& - ((c_{11}c_{14} - c_{12}c_{13})\exp(\kappa L))/\left(c_{13}^2 + c_{14}^2\right) \\
& + ((c_{21}c_{23} + c_{22}c_{24})\exp(-\kappa)(\kappa/2k))/\left(c_{23}^2 + c_{24}^2\right) \\
& + ((c_{21}c_{24} - c_{22}c_{23})\exp(-\kappa L))/\left(c_{23}^2 + c_{24}^2\right) \\
& - ((c_{41}c_{43} + c_{42}c_{44})2\cos(2kL))/\left(c_{43}^2 + c_{44}^2\right) \\
& - ((c_{41}c_{44} - c_{42}c_{43})2\sin(2kL))/\left(c_{43}^2 + c_{44}^2\right),
\end{aligned}$$

$$\begin{aligned}
\eta_{3\mathrm{i}} = {} & ((c_{11}c_{13} + c_{12}c_{14})\exp(\kappa L))/\left(c_{13}^2 + c_{14}^2\right) \\
& + ((c_{11}c_{14} - c_{12}c_{13})\exp(\kappa L)(\kappa/2k))/\left(c_{13}^2 + c_{14}^2\right) \\
& - ((c_{21}c_{23} + c_{22}c_{24})\exp(-\kappa L))/\left(c_{23}^2 + c_{24}^2\right) \\
& + ((c_{21}c_{24} - c_{22}c_{23})\exp(-\kappa L)(\kappa/2k))/\left(c_{23}^2 + c_{24}^2\right) \\
& + ((c_{41}c_{43} + c_{42}c_{44})2\sin(2kL))/\left(c_{43}^2 + c_{44}^2\right) \\
& - ((c_{41}c_{44} - c_{42}c_{43})2\cos(2kL))/\left(c_{43}^2 + c_{44}^2\right),
\end{aligned}$$

$$\begin{aligned}
\eta_{4\mathrm{r}} =\ & ((c_{15}c_{13} + c_{16}c_{14})\exp(\kappa L)(\kappa/2k))/(c_{13}^2 + c_{14}^2) \\
& - ((c_{15}c_{14} - c_{16}c_{13})\exp(\kappa L))/(c_{13}^2 + c_{14}^2) \\
& + ((c_{25}c_{23} + c_{26}c_{24})\exp(-\kappa L)(\kappa/2k))/(c_{23}^2 + c_{24}^2) \\
& + ((c_{25}c_{24} - c_{26}c_{23})\exp(-\kappa L))/(c_{23}^2 + c_{24}^2) \\
& - ((c_{45}c_{43} + c_{46}c_{44})2\cos(2kL))/(c_{43}^2 + c_{44}^2) \\
& - ((c_{45}c_{44} - c_{46}c_{43})2\sin(2kL))/(c_{43}^2 + c_{44}^2),
\end{aligned}$$

$$\begin{aligned}
\eta_{4\mathrm{i}} =\ & ((c_{15}c_{13} + c_{16}c_{14})\exp(\kappa L))/(c_{13}^2 + c_{14}^2) \\
& + ((c_{15}c_{14} - c_{16}c_{13})\exp(\kappa L)(\kappa/2k))/(c_{13}^2 + c_{14}^2) \\
& - ((c_{25}c_{23} + c_{26}c_{24})\exp(-\kappa L))/(c_{23}^2 + c_{24}^2) \\
& + ((c_{25}c_{24} - c_{26}c_{23})\exp(-\kappa L)(\kappa/2k))/(c_{23}^2 + c_{24}^2) \\
& + ((c_{45}c_{43} + c_{46}c_{44})2\sin(2kL))/(c_{43}^2 + c_{44}^2) \\
& - ((c_{45}c_{44} - c_{46}c_{43})2\cos(2kL))/(c_{43}^2 + c_{44}^2).
\end{aligned}$$

The parameters c_{ij}, $i = 1, 2, 4$ and $j = 1, ..., 6$ with c_{3l}, $l = 1, 2, 5, 6$ which are appeared in the above equations are given by

$$\begin{aligned}
c_{11} =\ & -((64 + 128n + 64n^2)(2\exp(-\kappa L) + \exp(\kappa L)) \\
& + 32\kappa/k(1 - n^2)(-\sin(2kL) + 2\cos(2kL) + 2\cos(2kL)\exp(-\kappa L) \\
& - 3\sin(2kL)\exp(-\kappa L))), \\
c_{12} =\ & -(32(\kappa/k)(1 - n^2)\cos(2kL) + 2\sin(2kL) \\
& + 2\exp(-\kappa L)\sin(2kL) + 3\cos(2kL)\exp(-\kappa L)), \\
c_{13} =\ & -4(\kappa/k)\exp(-\kappa L)(16n^2\exp(-\kappa L)2\sin(2kL) \\
& - 32\exp(-\kappa L)\sin(2kL)) \\
& - 4(\exp(\kappa L) + \exp(-\kappa L))(32 + 32n^2 + 64n \\
& + \exp(\kappa L)\cos(2kL)(16n^2 - 16) \\
& + \exp(\kappa L)\sin(2kL)(4(\kappa/k) + 28(\kappa/k)n^2) \\
& + \exp(-\kappa L)\cos(2kL)(16n^2 - 16) \\
& + \exp(-\kappa L)\sin(2kL)(-12(\kappa/k) - 44(\kappa/k)n^2) \\
& + \exp(-\kappa L)\cos(2kL)(-16n^2 + 16) \\
& + \exp(-\kappa L)\sin(2kL)(-4(\kappa/k) + 4(\kappa/k)n^2) \\
& + \exp(\kappa L)\cos(2kL)(16n^2 - 16) \\
& + \exp(\kappa L)\sin(2kL)(-8(\kappa/k) + 8(\kappa/k)n^2)), \\
c_{14} =\ & -4(\kappa/k)\exp(-\kappa L)(2\exp(\kappa L)2\cos(2kL) \\
& - 16\exp(\kappa L)\cos(2kL) + 32n^2 + 64n + 32) \\
& - 4(\exp(\kappa L) + \exp(-\kappa L))(\exp(\kappa L)\cos(2kL)(4(\kappa/k) + 28(\kappa/k)n^2) \\
& - \exp(\kappa L)\sin(2kL)(16n^2 - 16) \\
& + \exp(-\kappa L)\cos(2kL)(-12(\kappa/k) - 44(\kappa/k)n^2) \\
& + \exp(-\kappa L)\sin(2kL)(-16n^2 + 16) \\
& + \exp(-\kappa L)\cos(2kL)(4(\kappa/k) - 4(\kappa/k)n^2) \\
& + \exp(-\kappa L)\sin(2kL)(16n^2 - 16) \\
& + \exp(\kappa L)\cos(2kL)(-8(\kappa/k) + 8(\kappa/k)n^2) \\
& + \exp(\kappa L)\sin(2kL)(16n^2 - 16) \\
& + 64n(\kappa/k) + 16(\kappa/k) - 16n^2(\kappa/k)),
\end{aligned}$$

$$
\begin{aligned}
c_{15} = &-(\exp(\kappa L)(-32(\kappa/k) - 32(\kappa/k)n^2 - 64n(\kappa/k) + 128n)\\
&+ \exp(-\kappa L)(-69(\kappa/k) + 64 - 192(\kappa/k)n - 96(\kappa/k)n^2)\\
&+ \cos(2kL)(16(\kappa/k) - 16(\kappa/k)n^2\\
&+ 128(\kappa/k)n) + \sin(2kL)(64n^2 - 64)\\
&+ (-16n^2(\kappa/k) + 16(\kappa/k))(\cos(2kL) + \cos(2kL)\exp(-\kappa L))\\
&+ \cos(2kL)\exp(-\kappa L)(128n(\kappa/k) - 80n^2(\kappa/k) + 80(\kappa/k))\\
&+ \sin(2kL)\exp(-\kappa L)(64n^2 - 64)),\\
c_{16} = &-(\exp(\kappa L)(128n + 64 + 64n^2) + \exp(-\kappa L)(64n^2 + 128n)\\
&+ \cos(2kL)(64n^2 - 64) - \sin(2kL)(16(\kappa/k)\\
&- 16(\kappa/k)n^2 + 128(\kappa/k)n)\\
&+ (-16n^2(\kappa/k) + 16(\kappa/k))(\sin(2kL) + \sin(2kL)\exp(-\kappa L))\\
&+ \cos(2kL)(64n^2 - 64)\\
&- \sin(2kL)\exp(-\kappa L)(128n(\kappa/k) - 80n^2(\kappa/k) + 80(\kappa/k))),\\
c_{21} = &\,2(\exp(\kappa L)(-8(\kappa/k)n^2 - 8(\kappa/k) - 16n(\kappa/k))\\
&+ \exp(-\kappa L)(-24n^2(\kappa/k) - 48n(\kappa/k) - 24(\kappa/k))\\
&+ \cos(2kL)(16(\kappa/k) - 16n^2(\kappa/k))\\
&+ \sin(2kL)(16(\kappa/k) - 16n^2(\kappa/k))\\
&+ \exp(2\kappa L)(\sin(2kL)(-16n^2(\kappa/k) + 16))),\\
c_{22} = &-2(\exp(\kappa L)(-16n^2 - 16 - 32n) + \exp(-\kappa L)(-32n - 16 - 16n^2)\\
&+ \cos(2Lk)(16(\kappa/k) - 16n^2(\kappa/k))\\
&- \sin(2kL)(16(\kappa/k) - 16n^2(\kappa/k))\\
&+ \exp(2\kappa L)\cos(2kL)(-16n^2(\kappa/k) + 16)),\\
c_{23} = &\,(\exp(\kappa L) - \exp(-\kappa L))(\kappa/k)(\exp(\kappa L)2\cos(2kL)\\
&- 32\exp(\kappa L)\cos(2kL) + 32n^2 + 64n + 32)\\
&- 2(\exp(\kappa L) + \exp(-\kappa L))(\exp(\kappa L)\cos(2kL)(4(\kappa/k) + 28(\kappa/k)n^2)\\
&- \exp(\kappa L)\sin(2kL)(16n^2 - 16)\\
&+ \exp(-\kappa L)\cos(2kL)(-12(\kappa/k) - 44(\kappa/k)n^2)\\
&+ \exp(-\kappa L)\sin(2kL)(-16n^2 + 16)\\
&+ \exp(-\kappa L)\cos(2kL)(4(\kappa/k) - 4(\kappa/k)n^2)\\
&+ \exp(-\kappa L)\sin(2kL)(16n^2 - 16)\\
&+ \exp(\kappa L)\cos(2kL)(-8(\kappa/k) + 8(\kappa/k)n^2)\\
&+ \exp(\kappa)\sin(2kL)(16n^2 - 16)\\
&+ 64n(\kappa/k) + 16(\kappa/k) - 16n^2(\kappa/k)),\\
c_{24} = &\,(\exp(\kappa L) - \exp(-\kappa L))(\kappa/k)(-32n^2\exp(-\kappa L)\sin(2kL)\\
&+ 32\exp(-\kappa L)\sin(2kL))\\
&+ 2(\exp(\kappa L) + \exp(-\kappa L))(32 + 32n^2 + 64n\\
&+ \exp(\kappa L)\cos(2kL)(16n^2 - 16)\\
&+ \exp(\kappa L)\sin(2kL)(4(\kappa/k) + 28(\kappa/k)n^2)\\
&+ \exp(-\kappa L)\cos(2kL)(16n^2 - 16)\\
&+ \exp(-\kappa L)\sin(2kL)(-12(\kappa/k) - 44(\kappa/k)n^2)\\
&+ \exp(-\kappa L)\cos(2kL)(-16n^2 + 16)\\
&+ \exp(-\kappa L)\sin(2kL)(-4(\kappa/k) + 4(\kappa/k)n^2)\\
&+ \exp(\kappa L)\cos(2kL)(16n^2 - 16)\\
&+ \exp(\kappa L)\sin(2kL)(-8(\kappa/k) + 8(\kappa/k)n^2)),
\end{aligned}
$$

$$
\begin{aligned}
c_{25} = {} & 2(\exp(\kappa L)(+16n^2 + 16 + 32n) + \exp(-\kappa L)(32n + 16 + 16n^2) \\
& + \sin(2kL)(4(\kappa/k) - 4n^2(\kappa/k)) + \cos(2kL)(-16 + 16n^2) \\
& - \sin(2kL)(12(\kappa/k) - 44n^2(\kappa/k)) \\
& + \exp(2\kappa L)\cos(2kL)(-16 + 16n.^2) \\
& - \exp(2\kappa L)\sin(2kL)(-4(\kappa/k) - 28n^2(\kappa/k)) \\
& + \exp(2\kappa L)\sin(2kL)(4(\kappa/k) - 4n^2(\kappa/k))), \\
c_{26} = {} & -2(\exp(-\kappa L)(-16(\kappa/k)n^2 - 16(\kappa/k) - 32n(\kappa/k)) \\
& + \cos(2kL)(4(\kappa/k) - 4n^2(\kappa/k)) \\
& + \cos(2kL)(12(\kappa/k) - 44n^2(\kappa/k)) \\
& + \sin(2kL)(-16 + 16n^2) \\
& + \exp(2\kappa L)\cos(2kL)(-4(\kappa/k) - 28n^2(\kappa/k)) \\
& + \exp(2\kappa L)\sin(2kL)(-16 + 16n^2) \\
& + \exp(2\kappa L)\cos(2kL)(4(\kappa/k) - 4n^2(\kappa/k))), \\
c_{31} = {} & \cos(2kL)(96(\kappa/k) - 128n^2 + 128) \\
& + \sin(2kL)(128n(\kappa/k) - 224(\kappa/k)n^2) \\
& + \exp(-2\kappa L)\cos(2kL)(64 - 64n^2) \\
& + \exp(-2\kappa L)\sin(2kL)(128(\kappa/k)n - 112n^2(\kappa/k) + 112) \\
& + \exp(2\kappa)\cos(2kL)(64 - 64n^2) \\
& + \exp(2\kappa L)\sin(2kL)(-112n^2(\kappa/k) - 16), \\
c_{32} = {} & \exp(\kappa L)(128n(\kappa/k) + 128n^2(\kappa/k)) \\
& + \exp(-\kappa L)(-128(\kappa/k) + 128n^2(\kappa/k)) \\
& + \cos(2kL)(128n(\kappa/k) - 224n^2(\kappa/k)) \\
& - \sin(2kL)(96(\kappa/k) - 128n^2 + 128) \\
& + \exp(2\kappa L)\cos(2kL)(128n(\kappa/k) - 112n^2(\kappa/k) + 112) \\
& - \exp(2\kappa L)\sin(2kL)(64 - 64n^2) \\
& + \exp(2\kappa L)\cos(2kL)(-16(\kappa/k) - 112n^2(\kappa/k)) \\
& - \exp(2\kappa L)\sin(2kL)(64 - 64n^2), \\
c_{35} = {} & \cos(2kL)(128n(\kappa/k) + 192n^2(\kappa/k) \\
& - 64(\kappa/k))\exp(2\kappa L)\cos(2kL)(144n^2(\kappa/k) - 16(\kappa/k)) \\
& + \exp(-2\kappa L)\cos(2kL)(128n(\kappa/k) - 80n^2(\kappa/k) + 80(\kappa/k)) \\
& + \exp(-2\kappa L)\sin(2kL)(64n^2 - 64) \\
& + \exp(2\kappa L)\sin(2kL)(-64n^2 + 64), \\
c_{36} = {} & -\sin(2kL)(128n(\kappa/k) + 192n^2(\kappa/k) - 64(\kappa/k)) \\
& + \exp(-2\kappa L)\cos(2kL)(64n^2 - 64) \\
& - \exp(-2\kappa L)\sin(2kL)(64n^2 - 64 - 80n^2(\kappa/k)) \\
& - \exp(2\kappa L)\cos(2kL)(144n^2(\kappa/k) - 16(\kappa/k)) \\
& + \exp(2\kappa L)\cos(2kL)(-64n^2 + 64), \\
c_{41} = {} & -(n-1)(\kappa/k)(4(1+n)(\exp(\kappa L)\sin(2kL) \\
& + \exp(-\kappa)\sin(2kL))), \\
c_{42} = {} & -(n-1)(\kappa/k)(-4(1+n)(\exp(\kappa L)\cos(2kL) \\
& + \exp(-\kappa L)\cos(2kL))),
\end{aligned}
$$

$$
\begin{aligned}
c_{43} = {} & \exp(\kappa L)\cos(2kL)(16n^2 - 16) \\
& + \exp(\kappa L)\sin(2kL)(28n^2(\kappa/k) + 4(\kappa/k)) \\
& + \exp(-\kappa L)\sin(2kL)(12n^2(\kappa/k) - 12(\kappa/k) - 32(\kappa/k)n) \\
& + \exp(-\kappa L)\cos(2kL)(16n^2 - 16) \\
& + \exp(-\kappa L)\cos(2kL)(-16n^2 + 16) \\
& - \exp(-\kappa L)\sin(2kL)(-4n^2(\kappa/k) + 4(\kappa/k)) \\
& + \exp(\kappa L)\cos(2kL)(16n^2 - 16) \\
& - \exp(\kappa L)\sin(2kL)(12n^2(\kappa/k) - 12(\kappa/k)) + 32n^2 + 32 + 64n, \\
c_{44} = {} & -\exp(\kappa L)\sin(2kL)(16n^2 - 16) \\
& + \exp(\kappa L)\cos(2kL)(28n^2(\kappa/k) + 4(\kappa/k)) \\
& + \exp(-\kappa L)\cos(2kL)(12n^2(\kappa/k) - 12(\kappa/k) - 32(\kappa/k)n) \\
& - \exp(-\kappa L)\sin(2kL)(16n^2 - 16) \\
& + \exp(-\kappa L)\sin(2kL)(-16n^2 + 16) \\
& + \exp(-\kappa L)\cos(2kL)(-4n^2(\kappa/k) + 4(\kappa/k)) \\
& + \exp(\kappa L)\sin(2kL)(16n^2 - 16) \\
& + \exp(\kappa L)\cos(2kL)(12n^2(\kappa/k) - 12(\kappa/k)) \\
& + 64n(\kappa/k) + 16(\kappa/k) - 16n^2(\kappa/k), \\
c_{45} = {} & -(n-1)(\kappa/k)(4(1+n)(\exp(\kappa)\cos(2kL) - \exp(-\kappa L)\cos(2kL)) \\
& + 4\exp(-\kappa L)\sin(2kL)), \\
c_{46} = {} & -(n-1)(\kappa/k)(4(1+n)(\exp(\kappa L)\sin(2kL) - \exp(-\kappa L)\sin(2kL)) \\
& - 4\exp(-\kappa L)\cos(2kL)).
\end{aligned}
$$

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