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Nonlinear fractional relaxation

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Abstract. We define a nonlinear model for fractional relaxation phenomena. We use ε-expansion method to analyse this model. By studying the fundamental solutions of this model we find that when $t \to 0$ the model exhibits a fast decay rate and when $t \to \infty$ the model exhibits a power-law decay. By analysing the frequency response we find a logarithmic enhancement for the relative ratio of susceptibility.

Keywords. Nonlinear fractional equation; nonlinear fractional relaxation; ε-expansion.

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1. Introduction

Fractional differential equation is a convenient tool to address the dynamics of complex systems, such as polymers, biopolymers, biological cells and porous materials [1–5]. Transport phenomena in the complex media have often exhibit peculiar behaviour. For instance, relaxation in a complex system is slow (super-slow) [6]. In recent years there has been extensive research on linear relaxation phenomena [7–10].

Our objective in this paper is to explore the issue of nonlinear fractional relaxation. For simplicity, we consider complex media with low level fractionality. In these media the order of fractional derivative α is very close to an integer, i.e., $\alpha = n - \varepsilon$, with small positive ε and integer n. It is possible to use a perturbation method to investigate the dynamics in such media [10–12]. Another treatment of nonlinear fractional differential equation utilizes variational iteration method [13].

This paper is organized as follows. In §2 we briefly review the ε-expansion method. In §3 we discuss a specific model for the nonlinear relaxation phenomena. We show that in the asymptotic regions the nonlinear relaxation phenomena and linear fractional relation phenomena behave similarly. In §4 we study the frequency response of this model. By performing a harmonic analysis we find an expression for the complex susceptibility. We also find a logarithmic enhancement for the relative ratio of susceptibility, and finally in §5 we present our conclusions.

2. ε-Expansion

There are several definitions [1–5] like Riemann–Liouville, Weyl, Riesz and Caputo for the fractional derivative and in ref. [11] an expansion method has been formulated for most of them. However, in this paper we only use the Caputo fractional derivative as it is easier to apply the initial conditions in this type. The left (forward) Caputo fractional derivative for $\alpha > 0$ is defined by

$$\begin{aligned} D^{\alpha} f(t) &= {}_0^C D_t^{\alpha} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau, \quad n-1 < \alpha < n \quad \text{and} \quad t > 0, \end{aligned} \tag{1}$$

where n is an integer and $f^{(n)}(\tau)$ denotes the nth derivative of the function $f(\tau)$.

To construct a solution for a process described by an equation with Caputo fractional derivative, one needs the initial conditions that can be written as

$$f^{(k)} = c_k, \quad k = 0, 1, ..., (n-1). \tag{2}$$

In this work we seek causal solutions. Hence we require $f(t) = 0$ for $t < 0$ [6].

Now if the order of fractional derivative α is close to a positive integer, namely $\alpha = n - \varepsilon$ with small positive ε, we have [11]

$$D^{n-\varepsilon} f(t) = f^{(n)}(t) + \varepsilon D_1^n f(t) + \cdots, \tag{3}$$

where

$$D_1^n f(t) = f^{(n)}(0) \ln t + \gamma f^{(n)}(t) + \int_0^t f^{(n+1)}(\tau) \ln(t-\tau) \mathrm{d}\tau. \tag{4}$$

γ is the Euler constant.

This expansion is valid when $\varepsilon \ll 1$ and $\varepsilon t \ll 1$. In the domain $\varepsilon t \gg 1$, one should perform an asymptotic analysis. Following the method of ref. [11] we find that

$$D^{1-\varepsilon} f(t) \simeq -\frac{\varepsilon}{t}, \quad \text{as} \quad t \to \infty. \tag{5}$$

3. Nonlinear fractional relaxation

The relaxation phenomenon is defined by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \frac{x(t)}{\tau} = 0. \tag{6}$$

For simplicity we assume $\tau = 1$ throughout this paper. With $x(0) = 1$ the solution is $x(t) = \exp(-t)$, the so called exponential decay rate.

The linear fractional relaxation is described by

$$\frac{\mathrm{d}x^{\alpha}(t)}{\mathrm{d}t^{\alpha}} + x(t) = 0, \quad \text{where} \quad 0 < \alpha < 1. \tag{7}$$

In this case the solution is expressed in terms of Mittag–Leffler functions [6]. The solution in this case exhibits a fast decay rate (in comparison to exponential decay) for small values of t and a slow decay rate for large values of t.

Nonlinear relaxation arises in various situations, such as in magnetic fluids [14], anelastic solids [15] and excited closed physical systems [16]. Nonlinear dielectric response also has been considered within the context of fractional kinetic equation [17].

In order to discuss nonlinear relaxation we consider a medium with low-level fractionality. In such a medium our model is defined by

$$\frac{\mathrm{d}x^{\alpha}(t)}{\mathrm{d}t^{\alpha}} + 2x(t) - x^2(t) = 0, \quad \text{where} \quad \alpha = 1 - \varepsilon. \tag{8}$$

We can utilize a perturbation expansion of our dynamical variable [10–12]

$$x(t) = x_0(t) + \varepsilon x_1(t), \tag{9}$$

where $x_0(t)$ denotes the solution of integer differential equation. By choosing the initial condition $x_0(0) = 1$ we find that

$$x_0(t) = 1 - \tanh(t), \tag{10}$$

and by substituting the perturbation expansion of $x(t)$ in eq. (8) for the perturbation term we find

$$x_1'(t) + 2[1 - x_0(t)]x_1(t) + D_1^1 x_0(t) = 0, \tag{11}$$

where

$$D_1^1 x_0(t) = -\ln(t) - \frac{\gamma}{\cosh^2(t)} + 2\int_0^t \frac{\sinh(\tau)}{\cosh^3(\tau)} \ln(t-\tau)\mathrm{d}\tau. \tag{12}$$

Generally, the solution of eq. (11) can be obtained by numerical techniques. But when $t \to 0$ and in the leading order we find

$$x_1'(t) + 2tx_1(t) \simeq \gamma + \ln(t), \tag{13}$$

with approximate solution

$$x_1(t) \simeq t(\gamma + \ln(t) - 1), \quad \text{as} \quad t \to 0. \tag{14}$$

Therefore the total solution from eq. (9) is

$$x(t) \simeq 1 - t + \varepsilon t[\gamma + \ln(t) - 1], \quad \text{as} \quad t \to 0. \tag{15}$$

But the expression inside the bracket is negative. Therefore, our model has a fast decay rate in comparison with the exponential decay when $t \to 0$.

Next we consider the limiting case of $t \to \infty$. From eqs (5) and (8) we obtain

$$-\frac{\varepsilon}{t} + 2x(t) - x^2(t) \simeq 0, \quad \text{as} \quad t \to \infty, \tag{16}$$

with solution

$$x(t) \simeq \frac{\varepsilon}{2t}, \quad \text{as} \quad t \to \infty. \tag{17}$$

This proves that the model has an algebraic decay when $t \to \infty$.

4. Frequency response

In this section, we consider the response of this model under the influence of a harmonic deriving force. This case is of interest for experimental research on complex systems. The dynamics of the system is described by

$$\frac{\mathrm{d}x^{\alpha}(t)}{\mathrm{d}t^{\alpha}} + 2x(t) - x^2(t) = \exp(i\omega t), \quad \text{where} \quad \alpha = 1 - \varepsilon. \tag{18}$$

For the integer solution, we have

$$\frac{\mathrm{d}x_0(t)}{\mathrm{d}t} + 2x_0(t) - x_0^2(t) = \exp(i\omega t). \tag{19}$$

In order to find the solution of eq. (19) we perform a harmonic analysis, namely we consider

$$x_0(t) = \sum_{l=1}^{\infty} \eta_l(\omega) \exp(il\omega t). \tag{20}$$

The coefficients of expansion are given by

$$\eta_1(\omega) = \frac{1}{i\omega + 2} \tag{21}$$

and

$$\eta_k(\omega) = \frac{1}{ik\omega + 2} \sum_{l=1}^{k-1} \eta_l(\omega)\eta_{k-l}(\omega), \quad \text{for} \quad k \geq 2. \tag{22}$$

But for $\omega \gg 1$

$$\eta_k(\omega) \simeq \omega^{-(2k-1)}, \quad \text{for} \quad k \geq 1. \tag{23}$$

Hence higher harmonics are suppressed. The perturbation term $x_1(t)$ is determined from

$$x_1'(t) + 2\left[1 - \sum_{l=1}^{\infty} \eta_l(\omega)\, \exp(il\omega t)\right] x_1(t) = -D_1^1 \sum_{l=1}^{\infty} \eta_l(\omega) \exp(il\omega t). \tag{24}$$

In order to calculate the left-hand side of eq. (24) we consider $D_1^1 \exp(il\omega t)$. The result is (for details of the calculation, see [12])

$$D_1^1 \exp(i\omega t) = \omega[Si(\omega t) - i\ln(\omega) + iCi(\omega t)] \exp(i\omega t), \tag{25}$$

where $Si(t)$ and $Ci(t)$ are sine and cosine integrals respectively [18] defined by

$$Si(t) = \int_0^t \frac{\sin(x)}{x}, \quad Ci(t) = -\int_t^{\infty} \frac{\cos(x)}{x}. \tag{26}$$

But in the steady state $Si(\omega t) \to 0$ and $Ci(\omega t) \to \pi/2$. Therefore,

$$D_1^1 \exp(i\omega t) = [-i\omega \ln(i\omega)] \exp(i\omega t). \tag{27}$$

The harmonic representation of the perturbation term is

$$x_1(t) = \sum_{l=1}^{\infty} \xi_l(\omega) \exp(il\omega t). \tag{28}$$

Upon substituting in eq. (24) we find

$$\xi_1(\omega) = \frac{i\omega\eta_1(\omega)}{i\omega + 2} = \frac{i\omega \ln(i\omega)}{(i\omega + 2)^2}, \tag{29}$$

and

$$\xi_k(\omega) = \frac{1}{ik\omega + 2}\left[ik\omega\eta_k \ln(ik\omega) + 2\sum_{l=1}^{k-1} \xi_l(\omega)\eta_{k-l}(\omega)\right], \quad \text{for} \quad k \geq 2. \tag{30}$$

The complete solution is

$$x(t) = x_0(t) + \varepsilon x_1(t) = \sum_{l=1}^{\infty} \chi_l(\omega) \exp(il\omega t), \tag{31}$$

where

$$\chi_l(\omega) = \eta_l(\omega) + \varepsilon\xi_l(\omega). \tag{32}$$

To study the general feature of the response function, we consider the harmonics of order l.

We find that at high frequencies the following relation holds:

$$\frac{\chi_l(\omega) - \eta_l(\omega)}{\eta_l(\omega)} \simeq \varepsilon \ln(\omega). \tag{33}$$

So we have a logarithmic enhancement for the relative ratio of the susceptibility of the nonlinear fractional relaxation with respect to nonlinear integer relaxation. The first harmonic of the integer solution $\eta_1(\omega)$ is similar to Debye's susceptibility. The other harmonics of the integer solutions are due to nonlinearity and hence they show deviation from the results of the linear case. The functions $\xi_l(\omega)$ appear because of the fractal nature of the media and also because of the nonlinear nature of our model.

5. Conclusions

We have used analytical techniques to investigate a specific model for the nonlinear fractional relaxation phenomena in a medium with low-level fractionality. We found the fundamental solutions and we discussed it's behaviour in two limiting cases. We also studied the frequency response of the model. We found that at high frequencies only the first few harmonics were relevant.

It would be interesting to study other nonlinear dynamical systems. In this paper we did not study the case where $\alpha = 1 + \varepsilon$ with small positive ε. It is interesting to study this case and find it's relation to the model presented in this work. We hope to report on these issues in the future.

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