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Energy–momentum localization for Bianchi type-IV Universe in general relativity and teleparallel gravity

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Abstract. In this study we have investigated the energy–momentum distributions for homogeneous and anisotropic Bianchi type-IV in B class Universe. For this purpose, we have used energy–momentum complexes of Einstein, Bergmann–Thomson, Landau–Lifshitz (LL), Papapetrou, Tolman and Møller in general relativity (GR) as also Einstein, Bergmann–Thomson, Landau–Lifshitz and Møller in teleparallel gravity (TG). From the obtained results we have found that Einstein and Bergmann–Thomson distributions are exactly giving the same results in GR and TG but the Landau–Lifshitz, Papapetrou Tolman and Møller energy–momentum distributions do not provide the same results with Einstein and Bergmann–Thomson in GR and TG. Furthermore, Einstein, Bergmann–Thomson and LL results are the same in different gravitation theories and we get that both GR and TG are equivalent theories for these prescriptions. From the obtained solutions, we could say that these are equivalent theories. Also, Møller energy–momentum distributions do not give the same results in GR and TG. However, we have found that all energy prescriptions are negative and our results agree with Nester *et al*.

Keywords. Bianchi type-IV Universe; energy–momentum localization; teleparallel gravity.

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1. Introduction

There are many studies in the literature about energy–momentum localization problem but primary investigation about energy–momentum localization problem has been made by Einstein [1]. Later, different energy–momentum complexes were built, for example, Papapetrou [2], Tolman [3], Bergmann–Thomson [4], Møller [5], Landau–Lifshitz [6], Weinberg [7], Qadir–Sharif [8] and the teleparallel gravity versions of the Einstein, Landau–Lifshitz, Bergmann–Thomson [9] and Møller energy–momentum definitions [10,11]. According to Xulu [12] "Virbhadra and his collaborators have showed that different energy–momentum complexes give same and reasonable results for a given

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space-time” [11,13,14]. Many authors have tried to solve energy–momentum localization problem using different space-time metrics and different energy–momentum prescriptions and they evaluated some important results in gravity theories in GR and TG [15–35]. Favata [36] has investigated energy localization invariance of tidal work in general relativity. He investigated the formulation of covariant conservation laws and calculated the tidal work using various energy–momentum definitions in GR. However, various authors have investigated with different definitions and different gravitation theories to solve energy–momentum localization using the Bianchi models. Radinschi [29] has obtained Landau–Lifshitz and Papapetrou energy–momentum solutions for the Bianchi type-$\mathrm{VI_o}$ space-time. Later, she obtained the same solutions for the Bianchi type-$\mathrm{VI_o}$ Universe using the Tolman, Bergmann–Thomson and Møller energy–momentum complexes [30]. For Bianchi type-I Universe, Xulu [37] has investigated Weinberg, Landau–Lifshitz and Papapetrou energy–momentum prescriptions and he found that the overall energy is zero. However, the solutions of this study agrees with the solution in a paper by Banerjee and Sen. Banerjee–Sen [38], for Bianchi type-I Universe, using the Einstein prescription, investigated the energy–momentum problem and they found that the total energy is zero. Radinschi [28] has evaluated the total energy due to the matter and gravitational field to be zero using the Møller energy–momentum definition for Bianchi type-I metric. Aydoğdu and Saltı [39], using the energy definition in Møller’s tetrad theory of gravity, have obtained total energy of the Universe in Bianchi type-I cosmological models which includes both the matter and gravitational fields as zero. Loi and Vargas [40] have studied energy localization for Bianchi I and II type Universes in TG. Aydoğdu [41] has investigated Einstein and LL energy definitions for Bianchi type-II Universe in GR and TG. Nevertheless, Aydoğdu and Saltı [42] have obtained Einstein and Bergmann–Thomson prescriptions for Bianchi type-V metric in general relativity and teleparallel gravity. Also Aydoğdu and Saltı [43] have investigated Bianchi type-II space-time using the Bergmann–Thomson energy–momentum prescription in gravitation theories. Nester *et al* [44] have investigated energy distributions of Bianchi class A and B models and they found that the energy for all class B models are negative.

In this study, we have investigated energy–momentum distributions of Bianchi type-IV Universe using different definitions in GR and TG. The plan of the paper is as follows. In the next section, we briefly present Bianchi type-IV space-time model and carry out some necessary calculations for this model. In §3, we introduce energy–momentum definitions of Einstein, Bergmann–Thomson, Landau–Lifshitz, Papapetrou, Tolman and Møller in GR and then compute the energy–momentum densities. In §4, we introduce energy–momentum definitions of Einstein, Bergmann–Thomson, Landau–Lifshitz and Møller in TG and then compute the energy–momentum densities. The last section is devoted to discussion and conclusion. In this paper, all indices are from 0 to 3 and $G = 1, c = 1$ in gravitational units.

2. The Bianchi type-IV Universe

Bianchi-type models are spatially homogeneous and anisotropic Universe models. According to Di Pietro and Demaret [45]; “these models are nine in number but their classification permits to split them in two classes. There are six models in class A (I, II, VI_1, VII_o, VIII and IX) and five in class B (III, IV, V, VI_h and VII_h)" [46]. The Bianchi-type space-times are generally defined by the following metric:

$$ds^2 = -dt^2 + dl^2 \quad \text{where} \quad dl^2 = g_{ab}\, dx^a\, dx^b.$$

Here dl^2 is the three-dimensional line element. According to MacCallum [47]; "spatially homogeneous cosmological models play an important role in attempts to understand the structure and properties of the space of all cosmological solutions of Einstein field equations." The spatially homogeneous Bianchi type-IV Universe is given by

$$ds^2 = A^2 dx^2 + B^2 e^{2x} dy^2 + C^2 e^{2x} (x\, dy + dz)^2 - dt^2, \tag{1}$$

where A, B, C are functions of t only [48,49]. For the line element (1), $g_{\mu\nu}$ is defined by

$$(g_{\mu\nu}) = \begin{pmatrix} A^2 & 0 & 0 & 0 \\ 0 & e^{2x}(B^2 + C^2x^2) & C^2 x e^{2x} & 0 \\ 0 & C^2 x e^{2x} & e^{2x} C^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

The inverse of $g_{\mu\nu}$ is given by

$$(g^{\mu\nu}) = \begin{pmatrix} 1/A^2 & 0 & 0 & 0 \\ 0 & 1/e^{2x} B^2 & -x/e^{2x} B^2 & 0 \\ 0 & -x/e^{2x} B^2 & B^2 + C^2 x^2/e^{2x} B^2 C^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3}$$

Introducing the tetrad ω^i ($i = 0, 1, 2, 3$) given by

$$\omega^0 = dt; \quad \omega^1 = A\, dx; \quad \omega^2 = B(e^x) dy; \quad \omega^3 = Ce^x (x\, dy + dz) \tag{4}$$

the metric (1) can be expressed in the simple form

$$ds^2 = -(\omega^0)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2. \tag{5}$$

Using this relation, we obtain the tetrad components

$$h^0{}_0 = 1, \quad h^1{}_1 = A, \quad h^2{}_2 = Be^x, \quad h^3{}_3 = Ce^x, \quad h^2{}_3 = Cxe^x \tag{6}$$

and inverse of $h^\alpha{}_\beta$ is

$$h_0{}^0 = 1, \quad h_1{}^1 = 1/A, \quad h_2{}^2 = 1/Be^x, \quad h_3{}^3 = 1/Ce^x, \quad h_2{}^3 = -x/Be^x. \tag{7}$$

3. Energy–momentum in general relativity

In this section, we introduce Einstein, Bergmann–Thomson, Landau–Lifshitz, Papapetrou, Tolman and Møller energy–momentum definitions and then compute the energy–momentum densities, respectively.

3.1 *Einstein energy–momentum definition in GR*

Energy–momentum prescription of Einstein [1] is given by

$$E_\mu^\nu = \frac{1}{16\pi}\Theta^{\nu\alpha}_{\mu,\alpha}, \tag{8}$$

where

$$\Theta^{\nu\alpha}_\mu = \frac{g_{\mu\beta}}{\sqrt{-g}}\Psi^{\nu\alpha\beta\rho}_{,\rho} \tag{9}$$

and $\Psi^{\nu\alpha\beta\rho}$ is giving below as

$$\Psi^{\nu\alpha\beta\rho} = [-g(g^{\nu\beta}g^{\alpha\rho} - g^{\alpha\beta}g^{\nu\rho})]. \tag{10}$$

E_0^0 is the energy density, E_α^0 are the momentum density components and E_0^α are the components of energy current density. To obtain energy and momentum densities in Einstein's complex associated with the space-time in eq. (1), we evaluate the required components of $\Theta^{\nu\alpha}_\mu$ as

$$\begin{aligned}
\Theta_1^{01} &= 2A\mathrm{e}^{2x}(BC)_t,\\
\Theta_2^{02} &= 2B\mathrm{e}^{2x}(AC)_t,\\
\Theta_3^{03} &= 2C\mathrm{e}^{2x}(AB)_t,\\
\Theta_0^{01} &= \frac{4BC\mathrm{e}^{2x}}{A},
\end{aligned} \tag{11}$$

where t indices describe the derivative with respect to time. Substituting these results into eq. (8), we get Einstein energy–momentum densities in the form

$$\begin{aligned}
E^{00} &= -\frac{BC\mathrm{e}^{2x}}{2\pi A},\\
E^{10} &= \frac{\mathrm{e}^{2x}(BC)_t}{4\pi A},\\
E^{20} &= 0,\\
E^{30} &= 0.
\end{aligned} \tag{12}$$

3.2 *Bergmann–Thomson energy–momentum definition in GR*

Energy–momentum prescription of Bergmann–Thomson [4] is given by

$$\Im^{\beta\nu} = \frac{1}{16\pi}\Xi^{\beta\nu\alpha}_{,\alpha}, \tag{13}$$

where

$$\Xi^{\beta\nu\alpha} = g^{\beta\mu}X_\mu^{\nu\alpha} \tag{14}$$

with

$$X_\mu^{\nu\alpha} = -X_\mu^{\alpha\nu} = \frac{g_{\mu\beta}}{\sqrt{-g}}\Psi^{\nu\alpha\beta\rho}_{,\rho}. \tag{15}$$

Here $\Im_0^0$ is the energy density, $\Im_\mu^0$ are the momentum density components and $\Im_0^\mu$ are the components of energy current density. To obtain energy and momentum densities in Bergmann–Thomson's complex associated with the space-time in eq. (1), we evaluate the required non-vanishing components of $\Xi^{\mu\nu\alpha}$ as

$$\begin{aligned}
\Xi^{101} &= \frac{2\mathrm{e}^{2x}(BC)_t}{A},\\
\Xi^{202} &= \frac{2(AC)_t}{B},\\
\Xi^{203} &= \frac{-2x(AC)_t}{B},\\
\Xi^{302} &= \frac{-2x(AC)_t}{B},\\
\Xi^{303} &= \frac{2(A_tB^2 + A_tC^2x^2 + ABB_t + ACC_tx^2)}{BC},
\end{aligned} \tag{16}$$

and if we substitute these results into eq. (13) we obtain Bergmann–Thomson energy–momentum densities in GR as follows:

$$\begin{aligned}
\Im^{00} &= -\frac{BC\mathrm{e}^{2x}}{2\pi A},\\
\Im^{10} &= \frac{\mathrm{e}^{2x}(BC)_t}{4\pi A},\\
\Im^{20} &= 0,\\
\Im^{30} &= 0.
\end{aligned} \tag{17}$$

The above results are exactly the same with the Einstein's energy and momentum complex in GR.

3.3 *Landau–Lifshitz (LL) energy–momentum definition in GR*

Energy–momentum prescription of Landau–Lifshitz [6] is given by

$$L^{\nu\beta} = \frac{1}{16\pi}\Psi^{\nu\alpha\beta\rho}_{,\alpha\rho}. \tag{18}$$

Here L_0^0 is the energy density, L_ν^0 are the momentum density components and L_0^ν are the components of energy current density. To obtain energy and momentum densities in LL's complex associated with the space-time in eq. (1), we evaluate the required non-vanishing components of $\Psi^{\nu\alpha\beta\rho}$ as

$$\begin{aligned}
\Psi^{1001} &= -\Psi^{1010} = B^2C^2\mathrm{e}^{4x}\\
\Psi^{0202} &= -\Psi^{2002} = -A^2\mathrm{e}^{2x}C^2\\
\Psi^{3002} &= \Psi^{2003} = -A^2C^2\mathrm{e}^{2x}x\\
\Psi^{0303} &= -A^2\mathrm{e}^{2x}(B^2 + C^2x^2)
\end{aligned} \tag{19}$$

and if we take these results into eq. (18), we obtain LL energy–momentum densities in GR as follows:

$$\begin{aligned} L^{00} &= -\frac{B^2C^2\mathrm{e}^{4x}}{\pi} \\ L^{10} &= \frac{\mathrm{e}^{4x}BC(BC)_t}{2\pi} \\ L^{20} &= 0 \\ L^{30} &= 0. \end{aligned} \tag{20}$$

The non-vanishing Landau–Lifshitz energy–momentum densities do not agree with the Einstein and Bergmann–Thomson's energy and momentum complexes in GR.

3.4 *Papapetrou energy–momentum definition in GR*

Papapetrou energy–momentum distribution is given by [2]

$$\Sigma^{\mu\nu} = \frac{1}{16\pi} N^{\mu\nu\alpha\beta}_{,\alpha\beta}, \tag{21}$$

where

$$N^{\mu\nu\alpha\beta} = \sqrt{-g}(g^{\mu\nu}\eta^{\alpha\beta} - g^{\mu\alpha}\eta^{\nu\beta} + g^{\alpha\beta}\eta^{\mu\nu} - g^{\nu\beta}\eta^{\mu\alpha}). \tag{22}$$

Σ^0_0 is the energy density, Σ^0_μ are the momentum density components and Σ^μ_0 are the components of energy current density. Using the line element we have found the required components of $N^{\mu\nu\alpha\beta}$ (eq. (22)) as

$$\begin{aligned} N^{1010} &= -N^{0011} = \frac{CB\mathrm{e}^{2x}(A^2+1)}{A}, \\ N^{2020} &= -N^{0022} = \frac{AC(\mathrm{e}^{2x}B^2+1)}{B}, \\ N^{3030} &= -N^{0033} = \frac{A(\mathrm{e}^{2x}C^2B^2 + B^2 + C^2x^2)}{CB}, \\ N^{0023} &= N^{0032} = -N^{0203} = \frac{ACx}{B}. \end{aligned} \tag{23}$$

Using these components in eq. (21) we have obtained Papapetrou energy–momentum distribution for Bianchi type-IV metric as

$$\begin{aligned} \Sigma^{00} &= -\frac{BC(A^2+1)\mathrm{e}^{2x}}{4\pi A}, \\ \Sigma^{10} &= \frac{\mathrm{e}^{2x}(A^2BCA_t + A^3BC_t + A^3CB_t - A_tCB + C_tAB + CAB_t)}{8\pi A^2}, \\ \Sigma^{20} &= 0, \\ \Sigma^{30} &= 0. \end{aligned} \tag{24}$$

3.5 *Tolman energy–momentum definition in GR*

The energy–momentum complex of Tolman is given by [3]

$$\Lambda_k^i = \frac{1}{8\pi}\Upsilon_{k,j}^{ij}, \tag{25}$$

where

$$\begin{aligned}\Upsilon_k^{ij} &= \sqrt{-g}\Big[-g^{pi}\Big(-\Gamma_{kp}^j + \frac{1}{2}g_k^j\Gamma_{ap}^a + \frac{1}{2}g_p^j\Gamma_{ak}^a\Big)\\ &\quad + \frac{1}{2}g_k^i g^{pm}\Big(-\Gamma_{pm}^j + \frac{1}{2}g_p^j\Gamma_{am}^a + \frac{1}{2}g_m^j\Gamma_{ap}^a\Big)\Big],\end{aligned} \tag{26}$$

where Λ_0^0 is the energy density and Λ_0^α are the components of momentum density. Using the line element we have found the required components of Υ_k^{ij} as

$$\begin{aligned}\Upsilon_0^{01} &= \frac{e^{2x}C(2B^2+2C^2-xC^2-2x^2C^2)}{A\sqrt{B^2+C^2-C^2x^2}},\\ \Upsilon_1^{00} &= \frac{1}{2}\frac{Ae^{2x}C(2B^2+2C^2-xC^2-2x^2C^2)}{\sqrt{B^2+C^2-C^2x^2}},\\ \Upsilon_2^{02} &= \frac{1}{2}\frac{e^{2x}(A_tCB^2-ABCB_t+A_tC^3-A_tC^3x^2+C_tAB^2)}{\sqrt{B^2+C^2-C^2x^2}},\\ \Upsilon_2^{03} &= -\frac{Ae^{2x}xB(C_tB-CB_t)}{B^2+C^2-C^2x^2},\\ \Upsilon_3^{03} &= \frac{1}{2}\frac{e^{2x}(A_tB^2C-AB^2C_t+A_tC^3-A_tC^3x^2+ABCB_t)}{\sqrt{B^2+C^2-C^2x^2}}.\end{aligned} \tag{27}$$

Using these components in eq. (25) we have obtained Tolman energy–momentum distribution for Bianchi type-IV metric as follows:

$$\begin{aligned}\Lambda^{00} &= -\frac{e^{2x}}{8\pi}\frac{\begin{array}{c}C(4C^4x^4+4C^4x^3-8C^4x^2-4xC^4+3C^4-8B^2C^2x^2\\ -4xB^2C^2+7B^2C^2+4B^4)\end{array}}{A(B^2+C^2-C^2x^2)^{3/2}}\\ \Lambda^{10} &= \frac{e^{2x}}{8\pi}\frac{\begin{array}{c}(2C^4C_t(2x^4+x^3-4x^2-x+2)+C^3BB_t(x+2-2x^2)\\ -3C^2B^2C_t(2x^2+x-2)+2B^3(BC)_t)\end{array}}{A(B^2+C^2-C^2x^2)^{3/2}}\\ \Lambda^{20} &= 0,\\ \Lambda^{30} &= 0.\end{aligned} \tag{28}$$

3.6 *Møller energy–momentum definition in GR*

The energy–momentum complex of Møller [5] is given by

$$M_{\mu}^{\nu} = \frac{1}{8\pi}\lambda_{\mu,\alpha}^{\nu\alpha}, \tag{29}$$

where the antisymmetric superpotential $\lambda_{\mu}^{\nu\alpha}$ is

$$\lambda_{\mu}^{\nu\alpha} = \sqrt{-g}[g_{\mu\beta,\gamma} - g_{\mu\gamma,\beta}]g^{\nu\gamma}g^{\alpha\beta} \tag{30}$$

and M_0^0 is the energy density and M_α^0 are the momentum density components. Using the metric in eq. (1), we have found the required components of $\lambda_{\mu}^{\nu\alpha}$ as

$$\begin{aligned}
\lambda_1^{01} &= -2BC\mathrm{e}^{2x}A_t,\\
\lambda_2^{02} &= -2A\mathrm{e}^{2x}CB_t,\\
\lambda_3^{03} &= -2A\mathrm{e}^{2x}BC_t.
\end{aligned} \tag{31}$$

Using these components in eq. (29), we get the Møller energy and momentum densities for Bianchi type-IV metric as follows:

$$\begin{aligned}
M_0^0 &= 0,\\
M_1^0 &= -\frac{BC\mathrm{e}^{2x}A_t}{2\pi},\\
M_2^0 &= 0,\\
M_3^0 &= 0.
\end{aligned} \tag{32}$$

4. Energy–momentum definitions in teleparallel gravity

In this section, we introduce Einstein, Bergmann–Thomson, Landau–Lifshitz and Møller energy–momentum definitions and then compute the energy–momentum densities, respectively.

4.1 *Einstein, Bergmann–Thomson and Landau–Lifshitz energy–momentum definitions in TG*

Teleparallel gravity is called the alternative theory of gravitation theory [9,50]. In this section, we introduce Einstein, Bergmann–Thomson and Landau–Lifshitz energy–momentum definitions in TG and then compute the energy–momentum densities. The energy–momentum complexes of Einstein, Bergmann–Thomson and Landau–Lifshitz in teleparallel gravity are respectively given by the following definitions [9]:

$$hE^{\mu}{}_{\nu} = \frac{1}{4\pi}\partial_{\lambda}(U_{\nu}{}^{\mu\lambda}), \tag{33}$$

$$h\Im^{\mu\nu} = \frac{1}{4\pi}\partial_\lambda (g^{\mu\beta} U_\beta{}^{\nu\lambda}), \tag{34}$$

$$hL^{\mu\nu} = \frac{1}{4\pi}\partial_\lambda (hg^{\mu\beta} U_\beta{}^{\nu\lambda}), \tag{35}$$

where $h = \mathtt{det}(h^a{}_\mu)$ and $U_\beta{}^{\nu\lambda}$ is the Freud's superpotential, which is given by

$$U_\beta{}^{\nu\lambda} = hF_\beta{}^{\nu\lambda}. \tag{36}$$

Here $F^{\mu\nu\lambda}$ is the tensor

$$F^{\mu\nu\lambda} = \epsilon_1 T^{\mu\nu\lambda} + \frac{\epsilon_2}{2}(T^{\nu\mu\lambda} - T^{\lambda\mu\nu}) + \frac{\epsilon_3}{2}(g^{\mu\lambda}T^{\beta\nu}{}_\beta - g^{\nu\mu}T^{\beta\lambda}{}_\beta) \tag{37}$$

with ϵ_1, ϵ_2 and ϵ_3 the three dimensionless coupling constants of teleparallel gravity [51]. For the teleparallel equivalent of general relativity, the specific choice of these three constants are

$$\epsilon_1 = \frac{1}{4}, \qquad \epsilon_2 = \frac{1}{2}, \qquad \epsilon_3 = -1. \tag{38}$$

To calculate this tensor, first we must calculate Weitzenböck connection

$$\Gamma^\alpha{}_{\mu\nu} = h_a{}^\alpha \partial_\nu h^a{}_\mu \tag{39}$$

and after this calculation we get the torsion of the Weitzenböck connection

$$T^\mu{}_{\nu\lambda} = \Gamma^\mu{}_{\lambda\nu} - \Gamma^\mu{}_{\nu\lambda}. \tag{40}$$

The energy and momentum distributions in the complexes of Einstein, Bergmann–Thomson and Landau–Lifshitz in the teleparallel gravity are given respectively by the following equations [9]:

$$P_\mu{}^E = \int_\Sigma hE^0{}_\mu \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z, \tag{41}$$

$$P_\mu{}^\Im = \int_\Sigma hB^0{}_\mu \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z, \tag{42}$$

$$P_\mu{}^L = \int_\Sigma hL^0{}_\mu \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z, \tag{43}$$

where P_μ for $\mu = 1, 2, 3$ gives the momentum components, while P_0 gives the energy and the integration hypersurface Σ is described by $x^0 = t =$ constant [9]. Using the above tetrad and its inverse in eqs (6) and (7), we get the following non-vanishing Weitzenböck connection components:

$$\begin{aligned}
\Gamma^1{}_{10} &= \frac{A_t}{A}, \\
\Gamma^2{}_{20} &= \frac{B_t}{B}, \\
\Gamma^3{}_{30} &= \frac{C_t}{C}, \\
\Gamma^3{}_{20} &= \frac{x(BC_t - B_tC)}{BC}, \\
\Gamma^2{}_{21} &= \Gamma^3{}_{21} = \Gamma^3{}_{31} = 1
\end{aligned} \tag{44}$$

and the corresponding non-vanishing torsion components are obtained as

$$\begin{aligned}
T^1_{10} &= -T^1_{01} = -\frac{A_t}{A}, \\
T^2_{12} &= T^3_{12} = T^3_{13} = 1 \\
T^2_{21} &= T^3_{31} = T^3_{21} = -1, \\
T^2_{20} &= -T^2_{02} = -\frac{B_t}{B}, \\
T^3_{03} &= -T^3_{30} = \frac{C_t}{C}, \\
T^3_{20} &= -T^3_{02} = -\frac{x(BC_t - B_tC)}{BC}.
\end{aligned} \tag{45}$$

Using these components in eq. (37), we get the required non-vanishing components of the tensor $F^{\mu\nu\lambda}$ as

$$\begin{aligned}
F^{110} &= -F^{101} = -\frac{(BC)_t}{2A^2BC}, \\
F^{123} &= -F^{132} = -\frac{\mathrm{e}^{-2x}}{4B^2A^2}, \\
F^{212} &= -F^{221} = -\frac{\mathrm{e}^{-2x}}{2A^2B^2}, \\
F^{213} &= -F^{231} = \frac{\mathrm{e}^{-2x}(2x+1)}{A^2B^2}, \\
F^{220} &= -F^{202} = -\frac{\mathrm{e}^{2x}(A_tB^2x^2 + A_tC^2 + B_tABx^2 + C_tAC)}{AB^2C^2}, \\
F^{230} &= -F^{203} = -\frac{x\mathrm{e}^{-2x}(AC)_t}{AB^2C}, \\
F^{313} &= -F^{331} = \frac{-\mathrm{e}^{-2x}(B^2 + C^2x^2 + C^2x)}{A^2B^2C^2}, \\
F^{010} &= -F^{001} = -\frac{1}{A^2}.
\end{aligned} \tag{46}$$

From eq. (36) the required non-vanishing components of Freud's superpotential are found as

$$
\begin{aligned}
U_1{}^{10} &= -U_1{}^{01} = -\frac{Ae^{2x}(BC)_t}{2},\\
U_1{}^{23} &= -U_1{}^{32} = -\frac{AC}{4B},\\
U_2{}^{12} &= -U_2{}^{21} = \frac{Ce^{2x}(C^2x-2B^2)}{AB},\\
U_2{}^{13} &= -U_2{}^{31} = -\frac{e^{2x}C(C^2x^2-B^2)}{4AB},\\
U_2{}^{20} &= -U_2{}^{02} = -\frac{Be^{2x}(AC)_t}{2},\\
U_3{}^{12} &= -U_3{}^{21} = \frac{e^{2x}(C^3)}{4AB},\\
U_3{}^{13} &= -U_3{}^{31} = \frac{Be^{-2x}(2C^2+B^2x)}{4AC},\\
U_3{}^{30} &= U_3{}^{03} = -\frac{Ce^{-2x}(AB)_t}{2},\\
U_0{}^{10} &= -U_0{}^{01} = \frac{-Be^{-2x}C}{A}.
\end{aligned}
\tag{47}
$$

Substituting eq. (47) into eqs (41), (42) and (43), we get Einstein, Bergmann–Thomson and Landau–Lifshitz energy–momentum densities in teleparallel gravity, respectively.

$$hE^{00} = -\frac{e^{2x}BC}{2\pi A},\quad hE^{10} = \frac{e^{2x}(BC)_t}{4\pi A},\quad hE^{20} = hE^{30} = 0. \tag{48}$$

$$h\Im^{00} = -\frac{e^{2x}BC}{2\pi A},\quad h\Im^{10} = \frac{e^{2x}(BC)_t}{4\pi A},\quad h\Im^{20} = h\Im^{30} = 0. \tag{49}$$

$$hL^{00} = -\frac{e^{4x}B^2C^2}{\pi},\quad hL^{10} = \frac{e^{4x}BC(BC)_t}{2\pi},\quad hL^{20} = hL^{30} = 0. \tag{50}$$

4.2 *Møller Energy–momentum definition in TG*

Møller modified general relativity by constructing a new field theory in teleparallel space [5]. The goal of this theory was to overcome the problem of the energy–momentum complex that appears in Riemannian space [5,21]. The superpotential of Møller in teleparallel gravity is given by Mikhail *et al* [10].

$$\Re_{\mu}^{\nu\beta} = \frac{(-g)^{1/2}}{2\kappa} \Upsilon^{\tau\nu\beta}_{\chi\rho\sigma} \left[\Phi^{\rho} g^{\sigma\chi} g_{\mu\tau} - \xi g_{\tau\mu} \gamma^{\chi\rho\sigma} - (1-2\xi) g_{\tau\mu} \gamma^{\sigma\rho\chi} \right], \tag{51}$$

where

$$\Upsilon^{\tau\nu\beta}_{\chi\rho\sigma} = \delta^{\tau}_{\chi} g^{\nu\beta}_{\rho\sigma} + \delta^{\tau}_{\rho} g^{\nu\beta}_{\sigma\chi} - \delta^{\tau}_{\sigma} g^{\nu\beta}_{\chi\rho} \tag{52}$$

with $g^{\nu\beta}_{\rho\sigma}$ being a tensor defined by

$$g^{\nu\beta}_{\rho\sigma} = \delta^{\nu}_{\rho} \delta^{\beta}_{\sigma} - \delta^{\nu}_{\sigma} \delta^{\beta}_{\rho} \tag{53}$$

and $\gamma_{\mu\nu\beta}$ is the con-torsion tensor given by

$$\gamma_{\mu\nu\beta} = h_{i\mu} h^{i}_{\nu;\beta}, \tag{54}$$

where the semicolon denotes covariant differentiation with respect to Christoffel symbols, g is the determinant of $g_{\mu\nu}$ and Φ_{μ} is the basic vector field defined by

$$\Phi_{\mu} = \gamma^{\rho}_{\mu\rho} \tag{55}$$

κ is the Einstein constant and ξ is the free dimensionless parameter [35]. The energy may be expressed by the surface integral

$$E^{\text{Moller}}_{\text{TG}} = \lim_{r\to\infty} \int_{r=\text{constant}} \Re^{0\alpha}_{0} n_{\alpha} \mathrm{d}S, \tag{56}$$

where n_{α} is the unit three-vector normal to the surface element dS [35]. Taking the results which are given by (6) and (7) into eq. (54) we get some non-vanishing components of $\gamma_{\mu\nu\beta}$ as

$$\begin{aligned} \gamma_{011} &= -\gamma_{101} = AA_t, \qquad \gamma_{022} = -\gamma_{202} = B\mathrm{e}^{-2x} B_t, \\ \gamma_{023} &= -Bx\mathrm{e}^{-2x} B_t, \qquad \gamma_{123} = \gamma_{132} = \frac{B^2 \mathrm{e}^{-2x}(2x-1)}{2}, \\ \gamma_{231} &= \frac{B^2 \mathrm{e}^{-2x}(Bx^2 - C)}{2C}, \qquad \gamma_{331} = -\frac{B^2 \mathrm{e}^{-2x} x}{2}. \end{aligned} \tag{57}$$

Using these results, we find the following non-vanishing component of basic vector field:

$$\begin{aligned} \Phi^0 &= \frac{A_t BC^2 + B_t AC^2 - AB^2 x^2 C_t + ABCC_t}{ABC^2} \\ \Phi^1 &= \frac{4C^2 - BCx + B^2 x - 2CBx^2}{2A^2 C^2}. \end{aligned} \tag{58}$$

From eq. (51), with the results which are given in eqs (57) and (58), we find the required components of Møller's superpotentials as follows:

$$
\begin{aligned}
\Re_0^{01} &= \frac{B\mathrm{e}^{-2x}(BCx-4C^2-B^2x+2BCx^2)}{AC\kappa},\\
\Re_0^{20} &= \frac{A_tB^2C^2x^2+A_tC^4+5B_tAC^2Bx^2-5AB^3x^4C_t-5ABx^2C_tC^2+AC^3C_t}{2BC^3\kappa},\\
\Re_0^{30} &= \frac{A_tBC^2+B_tAC^2-5AB^2x^2C_t}{2C^3\kappa},\\
\Re_1^{10} &= \frac{A\mathrm{e}^{-2x}(B^2x^2C_t-C^2B_t-C_tBC)}{C\kappa},\\
\Re_1^{30} &= \frac{A^2(-A_tBC^2-B_tAC^2+5AB^2x^2C_t)}{2C^3\kappa},\\
\Re_2^{01} &= \frac{B^3\mathrm{e}^{-4x}(x-1)(BCx-4C^2-B^2x+2CBx^2)}{AC\kappa},\\
\Re_2^{02} &= \frac{B\mathrm{e}^{-2x}(A_tC^2+C_tAC+9B_tABx^2+4ABx^2C_t)}{C\kappa},\\
\Re_2^{30} &= -\frac{\begin{array}{c}B^2\mathrm{e}^{-2x}(-xA_tBC^2+8xB_tAC^2+5AB^2x^3C_t+5xC_tAC^2\\ +A_tBC^2+B_tAC^2-5AB^2x^2C_t)\end{array}}{2C^3\kappa},\\
\Re_3^{01} &= \frac{B\mathrm{e}^{-4x}(4C^2-BCx+B^2x-2CBx^2)(B^2x^2+C^2-B^2x)}{4AC\kappa},\\
\Re_3^{02} &= -\frac{A\mathrm{e}^{-2x}x(9B^2x^2B_t+4C^2B_t+9B^2x^2C_t+C_tBC+9C^2C_t)}{2C\kappa},\\
\Re_3^{03} &= -\frac{\mathrm{e}^{-2x}(9B^2x^2B_tA+10AB^2x^2C_t-A_tBC^2-B_tAC^2)}{2C\kappa}.
\end{aligned}
\tag{59}
$$

Substituting these results into eq. (56) we get the Møller energy and momentum distributions in teleparallel gravity as

$$
\begin{aligned}
\kappa M_0^0 &= -\frac{B\mathrm{e}^{-2x}(4BCx^2-BC+B^2-8C^2-2BCx)}{2AC},\\
\kappa M_1^0 &= -\frac{\begin{array}{c}\mathrm{e}^{-2x}(B^2C(2x+1)-4B^2x^2(C+C_t)+B^3(2x-1)\\ +4C^2(B_t+2B)+4BC_t(C+Bx))\end{array}}{4C},\\
\kappa M_2^0 &= \frac{\begin{array}{c}B^3\mathrm{e}^{-4x}(2BCx(1-4x^2)+4C^2(4x-5)+2Bx^2(2+5C)\\ +B^2(1-6x)-CB)\end{array}}{AC},\\
\kappa M_3^0 &= -\frac{\begin{array}{c}B\mathrm{e}^{-4x}(B^2C^2(16x^2-20x+3)+x^2B^3(12Cx+C+4Bx-7B-8x^2C)\\ +BC^3(1-8x^2)+2xB^3(B-C)+16C^4)\end{array}}{4AC}.
\end{aligned}
\tag{60}
$$

5. Summary and discussions

In this study, we have explored the energy–momentum distributions of the homogeneous and anisotropic Bianchi type-IV Universe. For this, we have used Einstein, Bergmann–Thomson, Landau–Lifshitz, Papapetrou, Tolman and Møller energy–momentum complexes in GR and Einstein, Bergmann–Thomson, Landau–Lifshitz and Møller in TG. We found that the results of the energy–momentum distributions of Einstein and Bergmann–Thomson are exactly same but the results of Landau–Lifshitz, Papapetrou, Tolman and Møller energy–momentum distributions are different from Einstein and Bergmann–Thomson in GR and also Møller energy density is zero for Bianchi type-IV Universe. We also found that the second and third momentum densities are same and zero in GR and TG for all definitions for Bianchi type-IV Universe except Møller energy–momentum density in TG. We have found that energy and momentum densities are well defined and non-vanishing for Møller definition in TG. From eqs (12), (17), (20), (24), (28), (48)–(50) and (60) it can be seen that the energy densities are finite, well defined and negative. The results of this paper also support the Cooperstock's hypothesis [52]. According to Cooperstock "that energy is localized to the region where the energy–momentum tensor is non-vanishing" [52–54]. Nester *et al* [44] have investigated energy distributions of Bianchi class A and B models and they found that the energy for all class B models are negative. Also we found negative energy density for Einstein, Bergmann–Thomson and LL, Papapetrou and Tolman prescriptions and these results agree with the study of Nester *et al* [44]. At the end, we have found that these two gravitational theories give the same results for the energy (E) and momentum (M) distributions in GR and TG for Einstein, Bergmann–Thomson and LL definitions:

$$E_{\rm GR} = E_{\rm TG}, \qquad M_{\rm GR} = M_{\rm TG}.$$

From these results we show that clearly Einstein, Bergmann–Thomson and Landau–Lifshitz energy–momentum prescriptions are equal not only in general relativity but also in teleparallel gravity and these are equivalent theories. But we have found different energy and momentum results for Møller energy–momentum distribution in GR and TG

$$E_{\rm GR} \neq E_{\rm TG}, \qquad M_{\rm GR} \neq M_{\rm TG}.$$

We show these results in tables 1 and 2.

Here X and Z give respectively

$$\begin{aligned} X =\ & C(4C^4x^4 + 4C^4x^3 - 8C^4x^2 - 4xC^4 + 3C^4 - 8B^2C^2x^2 \\ & -4xB^2C^2 + 7B^2C^2 + 4B^4) \end{aligned} \tag{61}$$

and

$$\begin{aligned} Z =\ & 2C^4C_t(2x^4 + x^3 - 4x^2 - x + 2) + C^3BB_t(x + 2 - 2x^2) \\ & -3C^2B^2C_t(2x^2 + x - 2) + 2B^3(BC)_t \end{aligned} \tag{62}$$

in table 1.

Table 1. Non-vanishing energy and momentum distributions of Bianchi type-IV Universe in general relativity.

General relativity	Energy density in GR	Momentum density in GR
Einstein	$E^{00} = -\frac{e^{2x} BC}{2\pi A}$	$E^{10} = \frac{e^{2x} (BC)_t}{4\pi A}$
Bergmann–Thomson	$\Im^{00} = -\frac{e^{2x} BC}{2\pi A}$	$\Im^{10} = \frac{e^{2x} (BC)_t}{4\pi A}$
Landau–Lifshitz	$L^{00} = -\frac{e^{4x} B^2 C^2}{\pi}$	$L^{10} = \frac{e^{4x} BC(BC)_t}{2\pi}$
Papapetrou	$\Sigma^{00} = -\frac{BC(A^2+1)e^{2x}}{4\pi A}$	$\Sigma^{10} = \frac{e^{2x}(A^2 BCA_t + A^3 BC_t + A^3 CB_t - A_t CB + C_t AB + CAB_t)}{8\pi A^2}$
Tolman	$\Lambda^{00} = -\frac{e^{2x}}{8\pi} \frac{X}{A(B^2 + C^2 - C^2 x^2)^{3/2}}$	$\Lambda^{10} = \frac{e^{2x}}{8\pi} \frac{Z}{A(B^2 + C^2 - C^2 x^2)^{3/2}}$
Møller	$M_0^0 = 0$	$M_1^0 = -\frac{BCA_t e^{2x}}{2\pi}$

Table 2. Non-vanishing energy and momentum distributions of Bianchi type-IV Universe in teleparallel gravity.

Teleparallel gravity	Energy density in TG	Momentum density in TG
Einstein	$hE^{00} = -\dfrac{e^{2x}BC}{2\pi A}$	$hE^{10} = \dfrac{e^{2x}(BC)_t}{4\pi A}$
Bergmann–Thomson	$h\Im^{00} = -\dfrac{e^{2x}BC}{2\pi A}$	$h\Im^{10} = \dfrac{e^{2x}(BC)_t}{4\pi A}$
Landau–Lifshitz	$hL^{00} = -\dfrac{e^{4x}B^2C^2}{\pi}$	$hL^{10} = \dfrac{e^{4x}BC(BC)_t}{2\pi}$
Møller	$\kappa M_0^0 = -\dfrac{Be^{-2x}(4BCx^2 - BC + B^2 - 8C^2 - 2BCx)}{2AC}$	$\kappa M_1^0 \neq 0, \kappa M_2^0 \neq 0, \kappa M_3^0 \neq 0$

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