PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 78, No. 4
April 2012
pp. 513–529

Exact solutions of some physical models using the (G'/G)-expansion method

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MS received 9 February 2011; revised 12 October 2011; accepted 8 November 2011

Abstract. The (G'/G)-expansion method and its simplified version are used to obtain generalized travelling wave solutions of five nonlinear evolution equations (NLEEs) of physical importance, viz. the (2+1)-dimensional Maccari system, the Pochhammer–Chree equation, the Newell–Whitehead equation, the Fitzhugh–Nagumo equation and the Burger–Fisher equation. A variety of special solutions like periodic, kink–antikink solitons, bell-type solitons etc. can easily be derived from the general results. Three-dimensional profile plots of some of the solutions are also drawn.

Keywords. Nonlinear evolution equation; soliton; kink–antikink solution.

PACS Nos **02.30.Jr; 05.45.Yv**

1. Introduction

Nonlinear evolution equations are frequently used to describe many problems of solid-state physics, fluid mechanics, plasma physics, population dynamics, chemical kinetics, nonlinear optics, protein chemistry, theory of Bose–Einstein condensates etc. [1,2]. The basic strategy one may adopt to predict, control and quantify the underlying features of a system under investigation is to model the system in terms of some mathematical equations, which are generally nonlinear, and then find exact analytic solutions of such model equations using a suitable method. By the aid of exact solutions, when they exist, the phenomena modelled by these NLEEs can be better understood.

In the last few decades, considerable efforts have been made to obtain exact analytical solutions of such nonlinear equations and a number of powerful and efficient methods have been developed for obtaining explicit travelling wave solutions [1,3–18].

Recently, a new powerful technique called (G'/G)-expansion method [19] was developed for a reliable treatment of nonlinear wave equations. This method is straightforward, concise and capable of producing new applications. Moreover, the solutions obtained by this method are of general nature and a number of specific solutions can be deduced by putting conditions on arbitrary constants present in the general solutions. Thereafter, a

DOI: 10.1007/s12043-011-0253-6; ***e*Publication: 16 February 2012**

number of applications of this method have also been reported [20–27]. A generalized and simplified version of (G'/G)-expansion method is also reported [28–30].

Keeping in view the importance of this method, we also exploited it to obtain some interesting results of a number of equations of physical relevance [26]. With a motivation to further expand the domain of applications of (G'/G)-expansion method, here in the present work, we investigate five nonlinear equations of physical importance, namely the (2+1)-dimensional Maccari system, the Pochhammer–Chree equation, the Newell–Whitehead equation, the Fitzhugh–Nagumo equation and the Burger–Fisher equation.

The organization of the paper is as follows: In §2, a brief account of the (G'/G)-expansion method for finding the travelling wave solutions of nonlinear equations is given. Section 3 comprises solutions of these five problems. Finally concluding remarks are presented in §4.

2. The (G'/G)-expansion method

Here we briefly describe the main steps of the (G'/G)-expansion method. Consider a nonlinear partial differential equation (PDE) of the form

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{1}$$

where $u = u(x, t)$ is an unknown function and P is a polynomial in $u(x, t)$ and its partial derivatives, in which higher-order derivatives and nonlinear terms are involved. In order to solve (1) using (G'/G)-expansion method, the following steps are to be considered.

Step 1: To find the travelling wave solution of (1), introduce the wave variable $\xi = (x - ct)$, so that $u(x, t) = u(\xi)$ and we use the following changes:

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi}, \quad \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \xi^2}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2}, \tag{2}$$

and so on for other derivatives. With the help of (2), PDE (1) changes to an ordinary differential equation (ODE) as

$$P(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, ...) = 0, \tag{3}$$

where u_ξ, $u_{\xi\xi}$, etc. denote derivative of u with respect to ξ.

Next integrate the ODE (3) as many times as possible and set the constants of integration to be zero for simplicity.

Step 2: The solution of (3) can be expressed by a polynomial in (G'/G) as

$$u(\xi) = \alpha_m\left(\frac{G'}{G}\right)^m + \alpha_{m-1}\left(\frac{G'}{G}\right)^{m-1} + \cdots, \tag{4}$$

where $G = G(\xi)$ satisfies the second-order linear ODE of the form

$$G'' + \lambda G' + \mu G = 0, \tag{5}$$

where $\alpha_m, \alpha_{m-1}, \ldots, \alpha_0$, λ and μ are constants to be determined later and $\alpha_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms appearing in ODE (3), after using (4).

Step 3: Substitute (4) into (3) and use (5) and from the resultant expression collect all terms with the same order of (G'/G) together and then equating each coefficient of the resulting polynomial to zero yields a set of algebraic equations for α_m, $\alpha_{m-1}, \ldots, \alpha_0$, c, λ and μ.

Step 4: Substitute $\alpha_i (i = 0, 1, 2, \ldots, m)$, c, λ and μ obtained in Step 3 and the general solutions of (5) into (4), we obtain travelling wave solutions of the nonlinear PDE (1). The general solutions of (5) are given as

$$\left(\frac{G'}{G}\right) = \begin{cases} \dfrac{\sqrt{\lambda^2 - 4\mu}}{2}\left(\dfrac{A_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi + A_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi}{A_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi + A_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi}\right) \\ \quad -\dfrac{\lambda}{2}, \ \lambda^2 - 4\mu > 0, \\ \dfrac{\sqrt{4\mu - \lambda^2}}{2}\left(\dfrac{-A_1 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\right)\xi + A_2 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\right)\xi}{A_1 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\right)\xi + A_2 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\right)\xi}\right) \\ \quad -\dfrac{\lambda}{2}, \ \lambda^2 - 4\mu < 0, \\ \dfrac{A_2}{A_1 + A_2\xi} - \dfrac{\lambda}{2}, \quad \lambda^2 - 4\mu = 0. \end{cases} \tag{6}$$

These results can further be written in some more simplified forms [30] depending upon the conditions on the ratio of A_1 and A_2 as

$$\left(\frac{G'}{G}\right) = \begin{cases} \dfrac{\sqrt{\lambda^2 - 4\mu}}{2}\tanh\left(\dfrac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + \xi_0\right) - \dfrac{\lambda}{2}, \\ \quad \lambda^2 - 4\mu > 0, \tanh(\xi_0) = \dfrac{A_2}{A_1}, \left|\dfrac{A_2}{A_1}\right| < 1, \\ \dfrac{\sqrt{\lambda^2 - 4\mu}}{2}\coth\left(\dfrac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + \xi_0\right) - \dfrac{\lambda}{2}, \\ \quad \lambda^2 - 4\mu > 0, \coth(\xi_0) = \dfrac{A_2}{A_1}, \left|\dfrac{A_2}{A_1}\right| > 1, \\ \dfrac{\sqrt{4\mu - \lambda^2}}{2}\cot\left(\dfrac{\sqrt{4\mu - \lambda^2}}{2}\xi + \xi_0\right) - \dfrac{\lambda}{2}, \\ \quad \lambda^2 - 4\mu < 0, \cot(\xi_0) = \dfrac{A_2}{A_1}, \\ \dfrac{A_2}{A_1 + A_2\xi} - \dfrac{\lambda}{2}, \lambda^2 - 4\mu = 0. \end{cases} \tag{7}$$

After the brief description of the (G'/G)-expansion method, we now apply it for solving some PDEs of physical importance.

3. Applications

Here we apply the (G'/G)-expansion method to find travelling wave solutions for the (2+1)-dimensional Maccari system, the Pochhammer–Chree equation, the Newell–Whitehead equation, the Fitzhugh–Nagumo equation and the Burger–Fisher equation. These equations have many applications in different fields.

3.1 *The Maccari system*

Recently, by using asymptotically exact reduction method based on Fourier expansion and spatio-temporal rescaling, Maccari derived the following new integrable (2+1)-dimensional nonlinear system from the KP equation [31]:

$$\begin{aligned} &\imath u_t + u_{xx} + uv = 0, \\ &v_t + v_y + (|u|^2)_x = 0. \end{aligned} \tag{8}$$

This system is a kind of nonlinear evolution equations that are often presented to describe the motion of the isolated waves, localized in a small part of space, in many fields such as hydrodynamics, plasma physics, nonlinear optics, etc. The integrability property was explicitly demonstrated and Lax pairs were also obtained. Zhao [32] constructed some general travelling wave solutions of system (8). Also, several periodic and soliton solutions of the above system have recently been reported [33–35].

To apply the (G'/G)-expansion method on the Maccari system, we use the wave variables

$$u = \mathrm{e}^{\imath\theta} U(\xi), \quad v = V(\xi), \quad \theta = px + qy + rt, \quad \xi = x + y + ct, \tag{9}$$

which transform (8) into ODEs as

$$\begin{aligned} &U'' - (r + p^2)U + UV = 0, \\ &(c+1)V' + 2UU' = 0. \end{aligned} \tag{10}$$

On integrating the second equation of (10) with respect to ξ we obtain

$$V = \frac{(A - U^2)}{c+1}, \tag{11}$$

where A is the integration constant. Substituting (11) into the first equation of (10), we derive

$$(c+1)U'' - [(c+1)(r+p^2) - A]U - U^3 = 0. \tag{12}$$

Now, balancing the terms of U'' with U^3 gives $m = 1$ and thus we write

$$U(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0, \tag{13}$$

where $G = G(\xi)$ satisfies the second-order linear ODE (5). Again substituting (13) into (12) and equating the coefficients of same powers of G'/G to zero, yields a set of simultaneous algebraic equations among $\alpha_0, \alpha_1, \lambda$ and μ. The solutions of such an equation are given as

$$\alpha_0 = \pm\sqrt{\frac{c+1}{2}}\lambda, \quad \alpha_1 = \pm\sqrt{2(c+1)}, \tag{14}$$

with a restriction $\lambda^2 - 4\mu = -2(r+p^2) + 2A/(c+1)$. So substitution of α_0 and α_1 from (14) and the general solution of second-order linear ODE (5) into (13), we derive three types of travelling wave solutions of the Maccari system.

Case 1: When $\lambda^2 - 4\mu > 0$.

We get hyperbolic travelling wave solutions as

$$\begin{aligned} u(\xi) &= \pm\sqrt{\frac{c+1}{2}}\sqrt{\lambda^2-4\mu} \\ &\times \left(\frac{A_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + A_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{A_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + A_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi} \right) e^{\iota\theta}, \end{aligned} \tag{15a}$$

$$\begin{aligned} v(\xi) &= \frac{(4\mu-\lambda^2)}{2} \\ &\times \left(\frac{A_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + A_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{A_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + A_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi} \right)^2. \end{aligned} \tag{15b}$$

In particular, if $A_1 \neq 0$, $A_2 = 0$, then u and v become

$$u(\xi) = \pm\sqrt{\frac{c+1}{2}}\sqrt{\lambda^2-4\mu}\,\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) e^{\iota\theta}, \tag{16a}$$

$$v(\xi) = \frac{4\mu-\lambda^2}{2}\tanh^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right). \tag{16b}$$

The solutions obtained in (16a) give periodic (real and imaginary parts) solution and kink soliton solution (modulus form). However, solution (16b) represents bell-type soliton solution. The plots of these results are shown in figures 1 and 2 for some particular values of constants.

However, when $A_1 = 0$ and $A_2 \neq 0$, the solutions for u and v become

$$u(\xi) = \pm\sqrt{\frac{c+1}{2}}\sqrt{\lambda^2-4\mu}\,\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) e^{\iota\theta}, \tag{17a}$$

$$v(\xi) = \frac{4\mu-\lambda^2}{2}\coth^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right). \tag{17b}$$

The above particular solutions are similar to the solutions obtained by other methods [33]. Note that singular nature of solutions in (17a) and (17b) make them less acceptable in physical terms. In subsequent examples, we do not consider such categories of solutions.

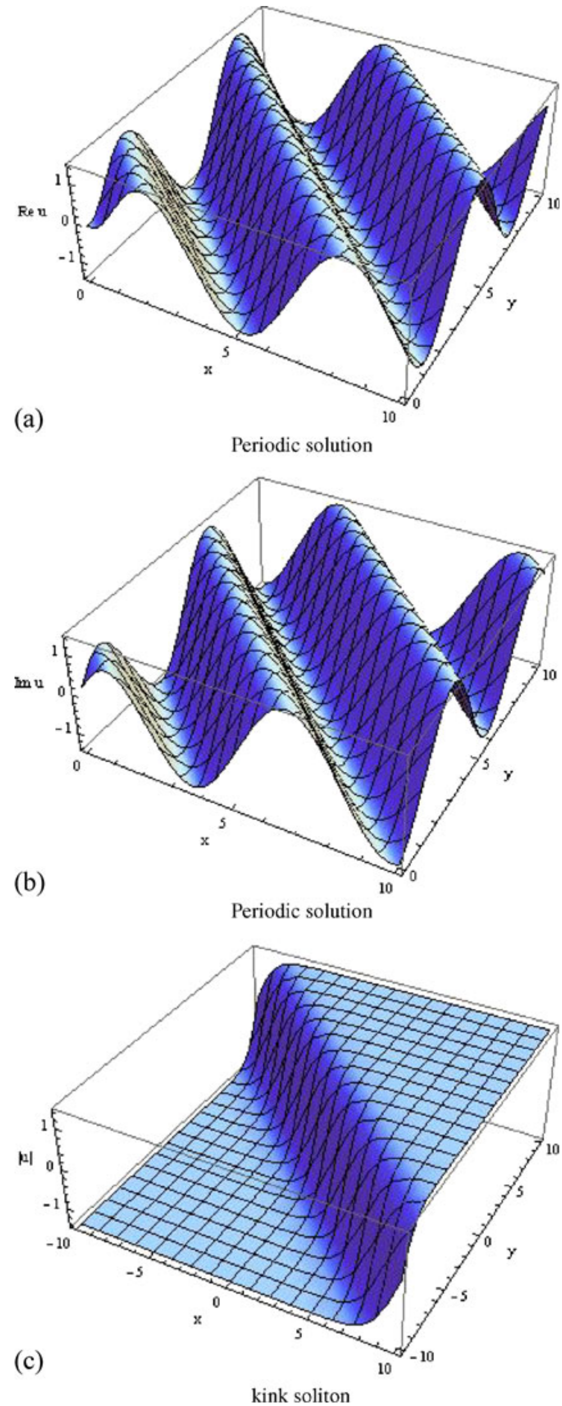


Figure 1. 3D plots of eq. (16a). (**a**) Real part, (**b**) imaginary part and (**c**) modulus part when $p = 1, q = 1, r = -2, c = 1, t = 0.2, \lambda = \sqrt{2}$ and $\mu = 0$.

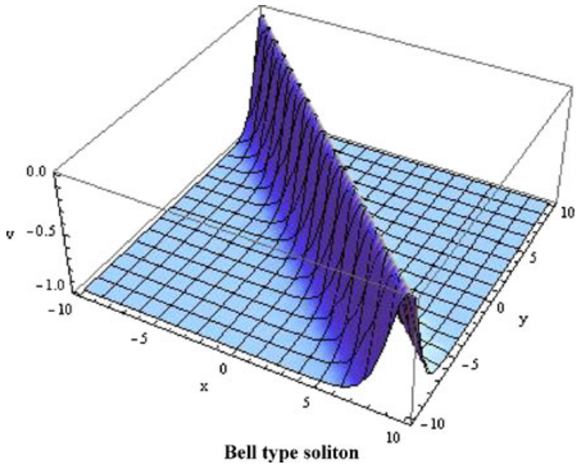


Figure 2. 3D plot of eq. (16b) when $p = 1, q = 1, r = -2, c = 1, t = 0.2, \lambda = \sqrt{2}$ and $\mu = 0$.

Again using (7), the general solutions for $u(\xi)$ and $v(\xi)$ in simplified forms are written as

$$u(\xi) = \pm\sqrt{\frac{c+1}{2}}\sqrt{\lambda^2-4\mu}\,\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\xi_0\right)\mathrm{e}^{\iota\theta}, \tag{18a}$$

$$v(\xi) = \frac{(4\mu-\lambda^2)}{2}\,\tanh^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\xi_0\right), \tag{18b}$$

when $|A_2/A_1| < 1$ and $\xi_0 = \tanh^{-1}(A_2/A_1)$.

Case 2: When $\lambda^2 - 4\mu < 0$.

The trigonometric solutions of the Maccari system are given as

$$u(\xi) = \pm\sqrt{\frac{c+1}{2}}\sqrt{4\mu-\lambda^2} \times\left(\frac{-A_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi+A_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}{A_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi+A_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}\right)\mathrm{e}^{\iota\theta}, \tag{19a}$$

$$v(\xi) = \frac{(\lambda^2-4\mu)}{2} \times\left(\frac{-A_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi+A_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}{A_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi+A_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}\right)^2. \tag{19b}$$

It is to be noted that, from these general results, some specific solutions can be obtained for different choices of A_1 and A_2.

Case 3: When $\lambda^2 - 4\mu = 0$.

This condition provides rational solutions as

$$u(\xi) = \pm\sqrt{2(c+1)}\left(\frac{A_2}{A_1 + A_2\xi}\right)\mathrm{e}^{\iota\theta}, \tag{20a}$$

$$v(\xi) = -2\left(\frac{A_2}{A_1 + A_2\xi}\right)^2, \tag{20b}$$

with a condition $r = -p^2$. The real and imaginary parts of (20a) are decaying periodic solutions.

3.2 *The Pochhammer–Chree equation*

The Pochhammer–Chree equation (PCE) is written as

$$u_{tt} - u_{ttxx} - (\alpha u + \beta u^{n+1} + \gamma u^{2n+1})_{xx} = 0, \quad n \geq 1. \tag{21}$$

This equation represents a nonlinear model of longitudinal wave propagation of elastic rods. A model was investigated in [36,37] for $n = 1$ and 2 where explicit solitary wave solutions and kink solutions were derived. In this work, we study the PCE when $n = 1$ by (G'/G)-method. Thus we have

$$u_{tt} - u_{ttxx} - (\alpha u + \beta u^2 + \gamma u^3)_{xx} = 0. \tag{22}$$

Using the wave variable $\xi = x - ct$, eq. (22) turns to an ODE

$$c^2u'' - c^2u'''' - (\alpha u + \beta u^2 + \gamma u^3)'' = 0, \tag{23}$$

which on integrating twice leads to

$$c^2u'' - (c^2 - \alpha)u + \beta u^2 + \gamma u^3 + A\xi + B = 0, \tag{24}$$

where A and B are integration constants. The homogeneous balance between u'' and u^3 in (24) gives $m = 1$. Thus the solution of (24) is of the form

$$u(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0, \tag{25}$$

where $G = G(\xi)$ satisfies the second-order linear ODE (5). The solutions for α_0 and α_1 for the PCE case are given as

$$\alpha_1 = \pm\sqrt{\frac{-2}{\gamma}}c, \quad \alpha_0 = \pm\frac{\left(3\lambda c \mp \beta\sqrt{\frac{-2}{\gamma}}\right)}{3\sqrt{-2\gamma}}, \tag{26}$$

with $c = \pm\sqrt{\frac{9\alpha\beta\gamma - 2\beta^2 - 27A\gamma^2\xi - 27B\gamma^2}{9\beta\gamma}}$. Now, substituting the values obtained in (26) and the general solution of second-order linear ODE (5) into (25), we get the following three travelling wave solutions of (21) for $n = 1$:

Case 1: When $\lambda^2 - 4\mu > 0$.

For this case, we get the hyperbolic travelling wave solution

$$u(\xi) = \pm\frac{1}{2}\sqrt{\frac{-2}{\gamma}}c\sqrt{\lambda^2-4\mu}$$
$$\times\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi+A_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{A_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi+A_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right)-\frac{\beta}{3\gamma}. \tag{27}$$

In particular, if $A_1 \neq 0$, $A_2 = 0$, $\lambda > 0$, $\mu = 0$, then $u(\xi)$ becomes

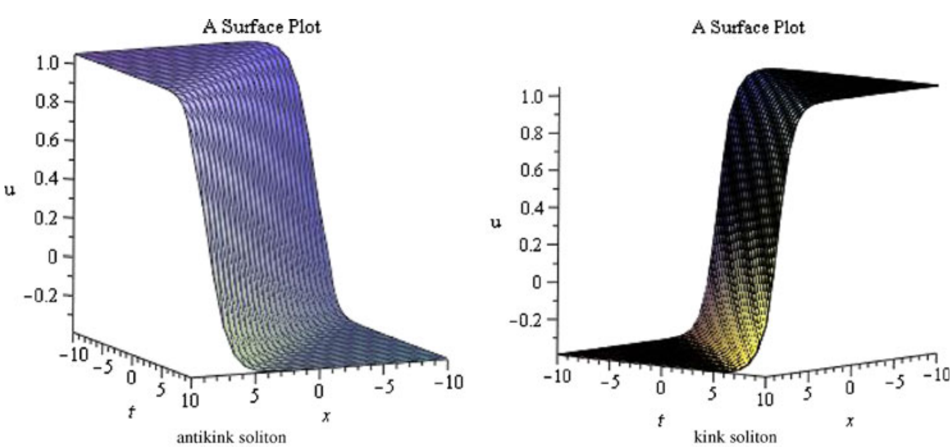
$$u(\xi) = \pm\frac{1}{2}\lambda c\sqrt{\frac{-2}{\gamma}}\tanh\left(\frac{\lambda}{2}\xi\right)-\frac{\beta}{3\gamma}. \tag{28}$$

The solution (28) represents kink or antikink solitons (depending upon the choice of sign) which are shown graphically in figure 3.

Again using (7), we derive the general solution for $u(\xi)$ in simplified form as

$$u(\xi) = \pm\frac{c}{2}\sqrt{\frac{-2}{\gamma}}\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\xi_0\right)-\frac{\beta}{3\gamma}, \tag{29}$$

when $|A_2/A_1| < 1$ and $\xi_0 = \tanh^{-1}(A_2/A_1)$.



(a) (b)

Figure 3. 3D plots of eq. (28). (**a**) When +sign is taken, (**b**) when −ve sign is taken, when $\alpha = 1$, $\beta = 1$, $\gamma = -1$, $\mu = 0$ and $c = (+)$ value.

Case 2: When $\lambda^2 - 4\mu < 0$.

The trigonometric solution is given as

$$u(\xi) = \pm\frac{c}{2}\sqrt{\frac{-2}{\gamma}}\sqrt{4\mu - \lambda^2} \times \left(\frac{-A_1 \sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi + A_2 \cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}{A_1 \cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi + A_2 \sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi} \right) - \frac{\beta}{3\gamma}. \tag{30}$$

Case 3: When $\lambda^2 - 4\mu = 0$.

The solution in rational form is given as

$$u(\xi) = \pm\sqrt{\frac{-2}{\gamma}}c\left(\frac{A_2}{A_1 + A_2\xi}\right) - \frac{\beta}{3\gamma}. \tag{31}$$

These are the travelling wave solutions of the PCE under different assumptions.

3.3 *The Newell–Whitehead equation*

The Newell–Whitehead equation is given by

$$u_t = u_{xx} + au - bu^3, \tag{32}$$

where a and b are constants. This equation describes the dynamical behaviour near the bifurcation point for the Rayleigh–Benard convection of binary fluid mixtures [38]. Benard's problem is a hydrodynamic problem in which water contained between two plates is heated from below. It exhibits patterns like rolls, hexagons or rectangles if the bifurcation parameter which is related to the temperature difference between the plates is above a certain threshold. This equation is derived to describe the envelope of modulated roll-solutions in systems with two large extended or unbounded space directions. If $a = b = -4$, eq. (32) becomes the Allen–Cahn equation which serves as a model for the study of phase separation in isothermal, isotropic and binary mixtures such as molten alloys [39].

The wave variable transformation $\xi = x - ct$ leads (32) to an ODE

$$u'' + cu' + au - bu^3 = 0. \tag{33}$$

Now, balancing u'' with u^3 in (33), we get $m = 1$. Then we suppose that

$$u(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0. \tag{34}$$

The coefficients α_1 and α_0 are computed as

$$\alpha_1 = \pm\sqrt{\frac{2}{b}}, \quad \alpha_0 = \pm\frac{(3\lambda - c)}{3\sqrt{2b}}, \quad c = \pm 3\sqrt{\frac{a}{2}}. \tag{35}$$

Now, substituting (35) into (34) and the general solution of second-order linear ODE (5) into (34), we have three types of travelling wave solutions of (32).

Case 1: When $\lambda^2 - 4\mu > 0$.

For this condition, the hyperbolic travelling wave solution is given as

$$u(\xi) = \pm\sqrt{\frac{\lambda^2 - 4\mu}{2b}} \times \left(\frac{A_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi + A_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi}{A_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi + A_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi} \right) - \frac{1}{2}\sqrt{\frac{a}{b}}. \tag{36}$$

In particular, if $A_1 \neq 0$, $A_2 = 0$, $\lambda > 0$, $\mu = 0$, then the general solution (36) turns out to be kink/antikink solitary wave solutions

$$u(\xi) = \pm\frac{\lambda}{\sqrt{2b}} \tanh\frac{\lambda}{2}\xi - \frac{1}{2}\sqrt{\frac{a}{b}}. \tag{37}$$

The plots of eq. (37) are shown in figure 4 which represent kink and antikink solitons depending on the choice of sign. These particular solutions are similar to the solutions obtained by other methods [40,41].

Again using (7), we derive the general solution for $u(\xi)$ in simplified form as

$$u(\xi) = \pm\sqrt{\frac{\lambda^2 - 4\mu}{2b}} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + \xi_0\right) - \frac{1}{2}\sqrt{\frac{a}{b}}, \tag{38}$$

when $|A_2/A_1| < 1$ and $\xi_0 = \tanh^{-1}(A_2/A_1)$.

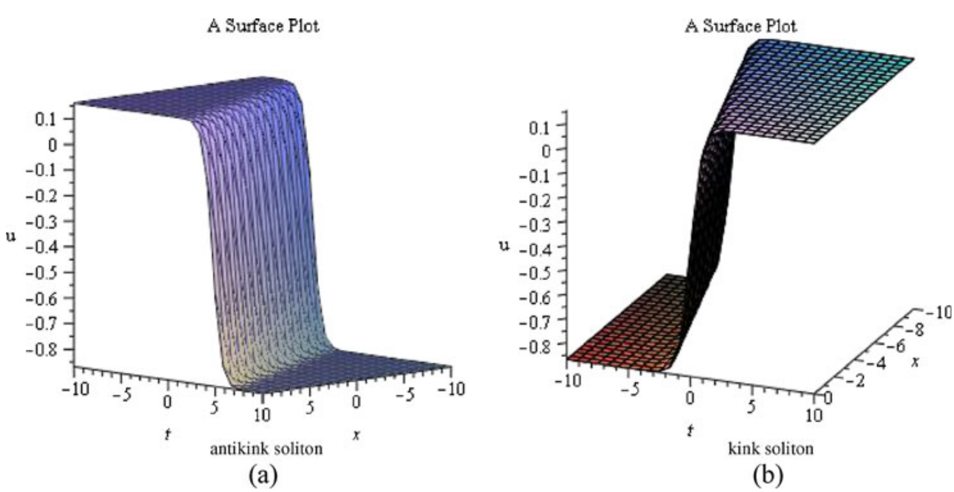


Figure 4. 3D plots of eq. (37). (**a**) When +sign is taken, (**b**) when −ve sign is taken, when $a = 1$, $b = 1$, then $c \pm 2.12$ (+ve only), $\lambda = 1.12$ (+ve only) and $\mu = 0$.

Case 2: When $\lambda^2 - 4\mu < 0$.

In this case we get trigonometric solution as

$$u(\xi) = \pm\sqrt{\frac{4\mu - \lambda^2}{2b}} \times \left(\frac{-A_1 \sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi + A_2 \cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}{A_1 \cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi + A_2 \sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi} \right) - \frac{1}{2}\sqrt{\frac{a}{b}}. \tag{39}$$

Case 3: When $\lambda^2 - 4\mu = 0$.

The rational form of solution is given as

$$u(\xi) = \pm\sqrt{\frac{2}{b}}\left(\frac{A_2}{A_1 + A_2\xi}\right). \tag{40}$$

3.4 *The Fitzhugh–Nagumo equation*

We consider the Fitzhugh–Nagumo equation

$$u_t = u_{xx} - u(1-u)(1-u). \tag{41}$$

This equation is used to model the phenomenon of control of the electrical potential across cell membranes which is the same as the Hodgkin–Huxley model. This control is done by the change of flow of the ionic channels of the cell membrane. This results in a change in potential which is used to send electrical signals between the cells. This is readily observed in muscles and other excitable cells, e.g., electrical waves of the heart. The travelling form of eq. (41) has been discussed in detail by Hereman and Takaoka [6].

The system (41) can be converted to an ODE, by using $\xi = x - ct$, as

$$u'' + cu' - u + 2u^2 - u^3 = 0. \tag{42}$$

Now, balancing the terms of u'' with u^3 we find, $m = 1$. Hence, we have

$$u(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0. \tag{43}$$

For this system, the solutions for α_1 and α_0 are derived as

$$\alpha_1 = \pm\sqrt{2}, \quad \alpha_0 = \frac{\pm(3\lambda - c) + 2\sqrt{2}}{3\sqrt{2}}, \tag{44}$$

with the restrictions on arbitrary constants $\lambda^2 - 4\mu = -\frac{1}{3}(c^2 - 2)$ and $c = \mp\sqrt{2}$, $\pm(1/\sqrt{2})$. These two restrictions here lead to only two different solutions (i.e. for $\lambda^2 - 4\mu > 0$ when $c = \pm(1/\sqrt{2})$ and $\lambda^2 - 4\mu = 0$ when $c = \mp\sqrt{2}$) which are as follows.

Case 1: When $\lambda^2 - 4\mu > 0$.

The hyperbolic travelling wave solution is given as

$$u(\xi) = \pm\sqrt{\frac{\lambda^2-4\mu}{2}} \times \left(\frac{A_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + A_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{A_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + A_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right) + \frac{1}{2}. \tag{45}$$

Here $\xi = x \pm (1/\sqrt{2})t$, A_1 and A_2 are arbitrary constants. Equation (41) represents the general solution of (41) which under certain restrictions gives some interesting results shown below:

If $A_1 \neq 0$, $A_2 = 0$, $\lambda > 0$, $\mu = 0$, then $u(\xi)$ becomes

$$u(\xi) = \pm\frac{\lambda}{\sqrt{2}} \tanh\frac{\lambda}{2}\xi + \frac{1}{2}, \tag{46}$$

which represents kink (antikink) soliton solutions of (41). These results are similar to the results obtained in [41,42].

Again using (7), we derive the general solution for $u(\xi)$ in simplified form as

$$u(\xi) = \pm\sqrt{\frac{\lambda^2-4\mu}{2}} \tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \xi_0\right) + \frac{1}{2}, \tag{47}$$

when $|A_2/A_1| < 1$ and $\xi_0 = \tanh^{-1}(A_2/A_1)$.

Case 2: When $\lambda^2 - 4\mu = 0$.

The rational solution is given as

$$u(\xi) = \pm\sqrt{2}\left(\frac{A_2}{A_1 + A_2\xi}\right) + 1. \tag{48}$$

Here $\xi = x \mp \sqrt{2}$t.

3.5 *The Burger–Fisher equation*

The Burger–Fisher equation is given by

$$u_t = u_{xx} + auu_x + u(1-u). \tag{49}$$

This equation arises in genetics, biology, and heat and mass transfer. This equation also acts as a prototype model for describing the interaction between the reaction mechanism, convection effect and diffusion transport [43]. The solitary wave solutions of (49) are obtained in [4] by using the Hirota method. Using the wave variable $\xi = x - ct$, eq. (49) is carried to an ODE

$$u'' + cu' + auu' + u(1-u) = 0. \tag{50}$$

Now balancing u'' with uu' in (50), we get $m = 1$ and we assume that

$$u(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0. \tag{51}$$

The solutions for α_1 and α_0 are given as

$$\alpha_1 = \frac{2}{a}, \quad \alpha_0 = \frac{1}{2a}(2\lambda + a), \tag{52}$$

with $c = -(1/2a)(a^2 + 4)$ and $(\lambda^2 - 4\mu) = (a^2/4)$. Now, substituting (52) into (51) we get the solution of (49) as

$$u(\xi) = \frac{2}{a}\left(\frac{G'}{G}\right) + \frac{1}{2a}(2\lambda + a). \tag{53}$$

Substituting the general solution of second-order linear ODE (5) into (53), we get three types travelling wave solutions of (49) as follows:

Case 1: When $\lambda^2 - 4\mu > 0$.

The hyperbolic travelling wave solution is written as

$$u(\xi) = \frac{\sqrt{\lambda^2 - 4\mu}}{a} \times \left(\frac{A_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi + A_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi}{A_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi + A_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)\xi}\right) + \frac{1}{2}. \tag{54}$$

Again if $A_1 \neq 0$, $A_2 = 0$, $\lambda > 0$, $\mu = 0$, then $u(\xi)$ becomes

$$u(\xi) = \frac{\lambda}{a}\tanh\frac{\lambda}{2}\xi + \frac{1}{2}. \tag{55}$$

The plot of (55) is shown in figure 5. These results are similar to the results obtained in [42,43].

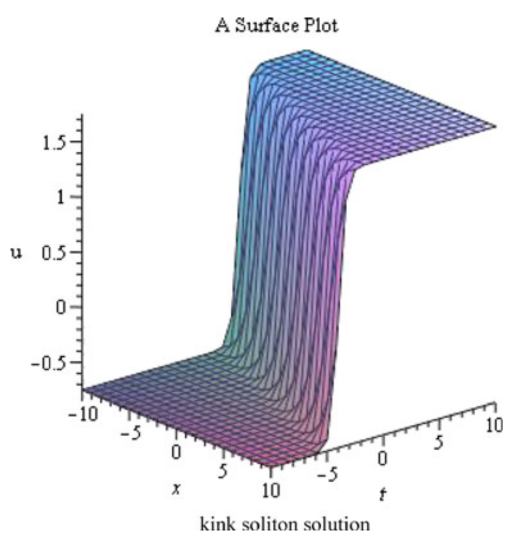


Figure 5. 3D plots of eq. (55), when $a = 1$, $\lambda = \pm 1.25$ (+ve only) and $\mu = 0$.

Again using (7), we derive the general solution for $u(\xi)$ in simplified form as

$$u(\xi) = \frac{\sqrt{\lambda^2 - 4\mu}}{a} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + \xi_0\right) + \frac{1}{2}, \tag{56}$$

when $|A_2/A_1| < 1$ and $\xi_0 = \tanh^{-1}(A_2/A_1)$.

Case 2: When $\lambda^2 - 4\mu < 0$.

We construct a trigonometric solution as

$$u(\xi) = \frac{\sqrt{4\mu - \lambda^2}}{a} \times \left(\frac{-A_1 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\right)\xi + A_2 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\right)\xi}{A_1 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\right)\xi + A_2 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\right)\xi}\right) + \frac{1}{2}. \tag{57}$$

Case 3: When $\lambda^2 - 4\mu = 0$.

Finally the rational form of the solution is given as

$$u(\xi) = \frac{2}{a}\left(\frac{A_2}{A_1 + A_2\xi}\right) + \frac{1}{2}. \tag{58}$$

4. Conclusion

With a view to further expand the catalogue of applications of the (G'/G)-expansion method, in this paper we obtained exact solutions, in general forms, of five nonlinear evolution equations, namely, the (2+1)-dimensional Maccari system, the Pochhammer–Chree equations, the Newell–Whitehead equation, the Fitzhugh–Nagumo equation and the Burger–Fisher equation. The general travelling wave solutions can be solitonic or periodic solutions depending upon different parametric restrictions. We have also derived the general results of these systems by applying the simplified version of (G'/G)-expansion method. It is interesting to note that from the general results, one can easily recover numerous solutions like kink, antikink, bell-shaped solitons etc. which are obtained by other methods. These solutions may be important for explaning some practical physical phenomena which are modelled by these equations. Three-dimensional plots of some of the investigated solutions are also drawn to visualize the underlying dynamics of such results.

It is worth to mention that we have also applied (G'/G)-expansion method for the coupled Higgs equation, $u_{tt} - u_{xx} + |u|^2 u - 2uv = 0$, $v_{tt} + v_{xx} - (|u|^2)_{xx} = 0$, and the (2+1)-dimensional nonlinear Schrödinger equation, $\iota u_t + u_{xx} + \sigma_d u_{yy} + \sigma_n |u|^2 u = 0$. But for these two cases, with the assumption of real profile functions, the obtained solutions came out to be purely imaginary. Thus, this method does not yield solutions for these two systems. This observation highlights the shortcoming of the present method. This direct, concise and computerizable method can further be used to explore its applicability for more nonlinear evolution equations.

Acknowledgements

The authors are very thankful to the referee for the valuable suggestions for improving the quality of this paper.

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