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New exact solutions to the generalized KdV equation with generalized evolution

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Abstract. In this paper, by using a transformation and an application of Fan subequation, we study a class of generalized Korteweg–de Vries (KdV) equation with generalized evolution. As a result, more types of exact solutions to the generalized KdV equation with generalized evolution are obtained, which include more general single-hump solitons, multihump solitons, kink solutions and Jacobian elliptic function solutions with double periods.

Keywords. Improved Fan subequation method; bifurcation method; generalized KdV equation; soliton solution; kink solution; periodic solution.

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1. Introduction

Studies of various physical structures of nonlinear dispersive equations had attracted much attention in connection with the important problems that arise in scientific applications. Mathematically, these physical structures have been studied by using various analytical methods, such as inverse scattering method Hirota bilinear method and so on [1–3]. Practically, there is no unified technique that can be employed to handle all types of nonlinear differential equations.

Recently, Fan [4] presented a Fan subequation. More recently, Feng and Luo [5] presented an improved Fan subequation method. We shall study a class generalized KdV equation with generalized evolution using the improved Fan subequation method. As a result, more types of exact solutions to the generalized KdV equation with generalized evolution are obtained.

2. The simple introduction of the improved Fan subequation method

For a given nonlinear partial differential equation in $q(x, t)$

$$P(q, q_t, q_x, q_{xx}, q_{xxx}, ...) = 0, \tag{2.1}$$

where P is, in general, a polynomial function of its arguments, and the subscripts denote the partial derivatives. Under the travelling wave transformation $q = q(\xi),\ \xi = x - \upsilon t$ is reduced to an ordinary differential equation with constant coefficients

$$Q(q, q', q'', q''', ...) = 0. \tag{2.2}$$

A transformation presented in the form

$$q(\xi) = \sum_{i=0}^{m} a_i \phi^i, \tag{2.3}$$

was presented by Fan [4] with the new variable $\phi = \phi(\xi)$ satisfying

$$\phi'(\xi) = \epsilon\sqrt{c_0 + c_1\phi + c_2\phi^2 + c_3\phi^3 + c_4\phi^4}, \tag{2.4}$$

where $\epsilon = \pm 1$ and a_i, c_j are constants.

Substituting eq. (2.3) into (2.2) along with (2.4), we can determine the parameter m by balancing the highest-order derivative term with the highest-order nonlinear terms. And then by substituting eqs (2.3) and (2.4) with the concrete m into eq. (2.2) and setting all the coefficients of those terms like $\phi^i(\phi')^j (i = 0, 1, 2, \ldots; j = 0, 1)$ to zero, we obtain a system of algebraic equations with respect to the parameters a_i, c_j, υ. By solving the system, if available, we may determine these parameters. If we know the solutions of (2.4), we can obtain the corresponding solutions of (2.2), i.e., (2.1), using (2.3). It is so difficult to give the general solutions of (2.4) that Fan [4] just gave several solutions for certain special values of c_j. Since they did not study the general cases for general values of c_j, many other exact solutions of (2.4) were missed. To make up for the loss and to find more exact solutions, we use the improved subequation method and consider exact solutions of (2.4) for general case (not for special case) by using bifurcation method of dynamical systems. Therefore, we can obtain not only many new exact solutions of (2.4) but also their dynamical behaviour.

Obviously, eq. (2.4) is equivalent to the two-dimensional systems as follows:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{1}{2}(c_1 + 2c_2\phi + 3c_3\phi^2 + 4c_4\phi^3), \tag{2.5}$$

which has the Hamiltonian function

$$H(\phi, y) = y^2 - (c_1\phi + c_2\phi^2 + c_3\phi^3 + c_4\phi^4) = c_0. \tag{2.6}$$

One can easily find that c_0 corresponds to the Hamiltonian constant and (2.4) is equivalent to the Hamiltonian system (2.5). Thus, in order to search for the exact solutions of (2.4) we need to only discuss (2.5). For a fixed c_0, eq. (2.6) determines a set of orbits of (2.5). As c_0 varies, eq. (2.6) defines different families of orbits of (2.5) which have different dynamical behaviour. Below we shall first study the bifurcation of phase portraits of (2.5) using bifurcation method of dynamical systems and with the aid of Maple, the computer symbolic system. Then according to the obtained bifurcation and the Hamiltonian function (2.6), we shall gain many new exact solutions of (2.4) for all possible parameters c_j.

3. Application to generalized KdV equation with generalized evolution

In this section, we illustrate the method developed in §2 to the generalized KdV equation with generalized evolution.

Recently, Ismail and Biswas [6] studied the following generalized Korteweg–de Vries equation with generalized evolution (the GKdV(l, n) equation in short)

$$(q^l)_t + a(q^{n+1})_x + b[q(q^n)_{xx}]_x + cq(q^n)_{xxx} = 0. \tag{3.1}$$

The first term of eq. (3.1) represents the generalized evolution. The special case with $l = 1$ is the regular evolution term. The coefficients of a are the nonlinear terms while the coefficients of b and c are the nonlinear dispersion terms. Ismail and Biswas [6] obtained the topological 1-soliton solution using the solitary wave ansatz method. More recently, Sturdevant and Biswas [7] studied eq. (3.1) and obtained the topological 1-soliton solution using another solitary wave ansatz method. This equation with $l = 1$ first appeared in 2004 [8]. It was studied by Wazwaz [8] with $l = 1$ and a number of soliton solutions were obtained. Here, our method will construct more types of new exact solutions of (3.1).

Substituting $q(x,t) = q(x - \upsilon t) = q(\xi)$ into eq. (3.1) and applying the transformation $q(\xi) = (u(\xi))^{-1/(n-l+1)}$, eq. (3.1) is reduced to

$$s_1(u')^3 + s_2uu'u'' + s_3u^2u' + s_4u^2u''' + s_5\upsilon u^3u' = 0, \tag{3.2}$$

where

$$\begin{aligned}
s_1 &= \frac{1}{(n-l+1)^3}(-2bl^2n + 7bln^2 - 6bn^3 - 2cl^2n + 7cln^2 - 6cn^3 \\
&\qquad + 5bln - 9bn^2 + 4cln - 7cn^2 - 3bn - 2cn),\\
s_2 &= \frac{1}{(n-l+1)^2}(-3cln + 6cn^2 + 4bn + 3cn + 6bn^2 - 3bln),\\
s_3 &= \frac{-a(n+1)}{n-l+1},\\
s_4 &= \frac{-n(b+c)}{n-l+1},\\
s_5 &= \frac{l\upsilon}{n-l+1}.
\end{aligned}$$

Substituting eqs (2.3) and (2.4) into eq. (3.2) and balancing u^2u''' with u^3u' in eq. (3.2), we obtain $m = 2$. Thus we can assume that eq. (3.2) has the following formal solutions:

$$u = a_0 + a_1\phi + a_2\phi^2, \tag{3.3}$$

where a_0, a_1 and a_2 are constants to be determined later and ϕ satisfies eq. (2.4) (i.e., (2.6)). Substituting (3.3) into (3.2) with (2.4), setting all the coefficients of $\phi^i(\phi')^j(i =$

0, 1, 2, ...; $j = 0, 1$) to zero, and solving the obtained algebraic equations, we find the following sets of solutions:

(1)

$$a_0 = \frac{\mu_2}{\upsilon}\left(\frac{3c_3^2}{8c_4} + 6\mu_1\right), \quad a_1 = \frac{3\mu_2 c_3}{\upsilon}, \quad a_2 = \frac{6\mu_2 c_4}{\upsilon},$$
$$c_0 = \frac{[(3c_3^2/8c_4) + 6\mu_1]^2}{36c_4}, \quad c_1 = \frac{c_3}{6c_4}\left(\frac{3c_3^2}{8c_4} + 6\mu_1\right),$$
$$c_2 = \frac{3c_3^2}{8c_4} + 2\mu_1, \quad c_2 + 4\mu_1 \neq 0, \tag{3.4}$$

where

$$\mu_1 = \frac{a(n+1)(n-l+1)^2}{4n^2(bn+cn+b)},$$
$$\mu_2 = \frac{n(n+l-1)(bl+cl-c)}{3l(n-l+1)^2}.$$

(2)

$$a_0 = \frac{3\mu_2 c_3^2}{8\upsilon c_4}, \quad a_1 = \frac{6\mu_2 c_3}{\upsilon}, \quad a_2 = \frac{6\mu_2 c_4}{\upsilon},$$
$$c_0 = \frac{c_3^2}{96c_4^2}\left(\frac{3c_3^2}{8c_4} - 6\mu_1\right), \quad c_1 = \frac{c_3}{6c_4}\left(\frac{3c_3^2}{8c_4} - 3\mu_1\right),$$
$$c_2 = \frac{3c_3^2}{8c_4} - \mu_1, \quad c_4 = c_4 \neq 0, \quad \upsilon = \upsilon \neq 0. \tag{3.5}$$

(3)

$$a_0 = \frac{\mu_2}{2\upsilon}(c_2 + 4\mu_1), \quad a_1 = \frac{3\mu_2 c_3}{2\upsilon}, \quad a_2 = c_4 = 0,$$
$$c_0 = \frac{1}{27c_3^2}(c_2 + 4\mu_1)^2(c_2 - 8\mu_1), \quad c_1 = \frac{1}{3c_3}(c_2^2 - 16\mu_1^2),$$
$$c_2 = c_2, \quad c_3 = c_3 \neq 0, \quad \upsilon = \upsilon \neq 0. \tag{3.6}$$

(4)

$$a_0 = c_0 = c_1 = 0, \quad a_1 = \frac{3\mu_2 c_3}{\upsilon}, \quad a_2 = \frac{6\mu_2 c_4}{\upsilon},$$
$$c_3 = 4\sqrt{-\mu_1 c_4}, \quad c_2 = -4\mu_1. \tag{3.7}$$

Next we discuss the bifurcations of phase portraits of the Hamiltonian system (2.5) for the above four sets of parametric cases and seek for the exact solutions of eq. (3.1).

3.1 *Exact solutions of eq.* (3.1) *for the first case* (3.4)

For the first parametric case (3.4), the Hamiltonian system (2.5) becomes

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{1}{2}\left(\phi + \frac{c_3}{4c_4}\right)\left(4c_4\phi^2 + 2c_3\phi + \frac{c_3^2}{4c_4} + 4\mu_1\right) \tag{3.8}$$

and the Hamiltonian function (2.6) reduces to

$$\begin{aligned} H_1(\phi, y) \;=\; y^2 - &\left(\frac{c_3}{2c_4}\left(\frac{c_3^2}{8c_4} + 2\mu_1\right)\phi + \left(\frac{3c_3^2}{8c_4} + 2\mu_1\right)\phi^2\right. \\ &\left. + \; c_3\phi^3 + c_4\phi^4\right) = c_0. \end{aligned} \tag{3.9}$$

3.1.1 *Phase portraits and bifurcations of system* (3.8). Now we discuss the bifurcations of phase portraits of (3.8). Obviously all the equilibrium points of (3.8) lie in the ϕ-axis and their abscissas are the real zeros of $f(\phi) = \frac{1}{2}\left[\frac{c_3}{6c_4}\left(\frac{3c_3^2}{8c_4} + 6\mu_1\right) + \left(\frac{3c_3^2}{4c_4} + 4\mu_1\right)\phi + 3c_3\phi^2 + 4c_4\phi^3\right]$. Thus we have the following proposition on the distribution of the equilibrium points of (3.8).

Proposition 3.1

(1.1) For $\Delta_1 = -c_4\mu_1 > 0$, eq. (3.8) has three equilibria at $E_{10}(-\frac{c_3}{4c_4}, 0)$, $E_{11}(\phi_{11}, 0)$ and $E_{12}(\phi_{12}, 0)$, where

$$\phi_{11} = \frac{-c_3 + 8\sqrt{\Delta_1}}{4c_4}, \qquad \phi_{12} = \frac{-c_3 - 8\sqrt{\Delta_1}}{4c_4}.$$

(1.2) For $\Delta_1 = -c_4\mu_1 \leq 0$, eq. (3.8) has a unique equilibrium at $E_{10}(-\frac{c_3}{4c_4}, 0)$.

Notice that we need only consider the case $c_3 \geq 0$ because of the invariance of eq. (3.8) under the transformation $\phi \to -\phi$, $y \to -y$, $c_3 \to -c_3$.

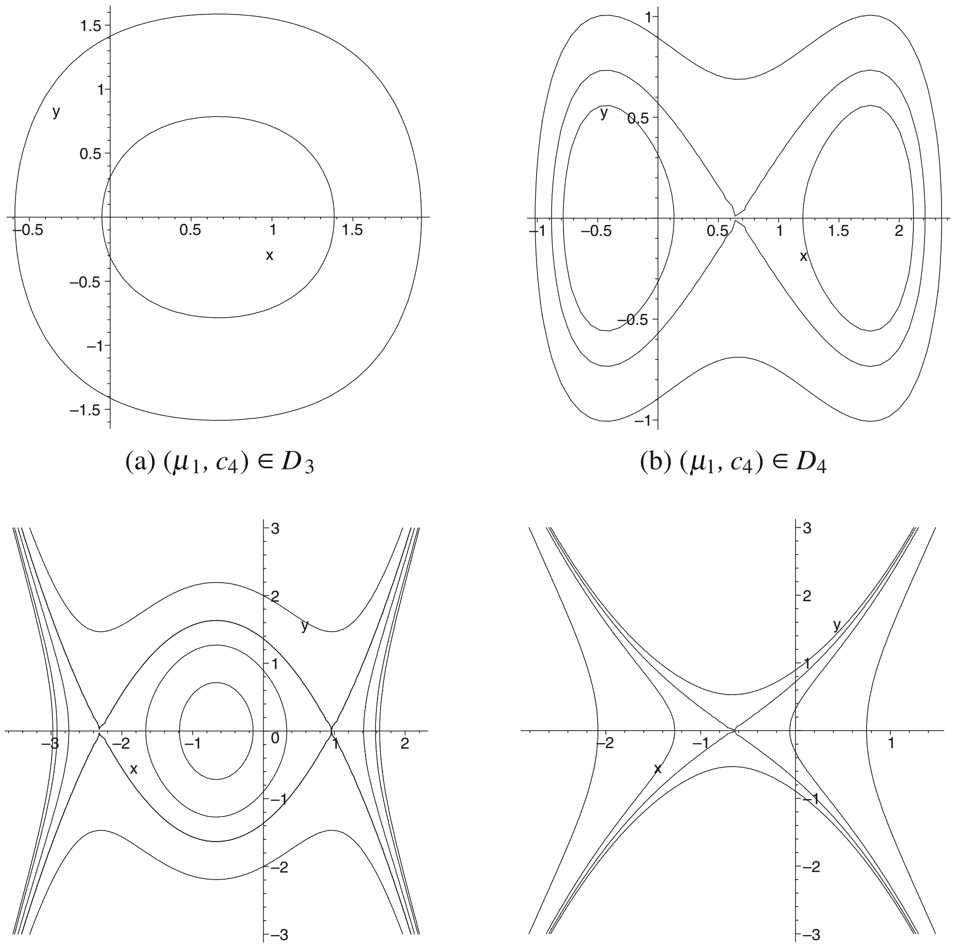
Using the bifurcation theory of dynamical systems [9–11], we can obtain the bifurcation curves and phase portraits under various parameter conditions shown as follows.

For a fixed $c_3 > 0$, there are two bifurcation curves L_1: $c_4 = 0$ and L_2: $\mu_1 = 0$ which divides the (μ_1, c_4)-parametric plane into four subregions

$$D_1: \{\mu_1 > 0, c_4 > 0\}\,; \;\; D_2: \{\mu_1 < 0, c_4 > 0\}\,;$$
$$D_3: \{\mu_1 < 0, c_4 < 0\}\,; \;\; D_4: \{\mu_1 > 0, c_4 < 0\}\,.$$

The phase portraits of eq. (3.8) are shown in figure 1.

3.1.2 *Exact solutions of eq.* (3.1) *determined by phase portraits in figure* 1. In this subsection, we shall give explicit and exact solutions of eq. (3.1).



(a) $(\mu_1, c_4) \in D_3$ (b) $(\mu_1, c_4) \in D_4$

(c) $(\mu_1, c_4) \in D_2$ (d) $(\mu_1, c_4) \in D_1$

Figure 1. The phase portraits of eq. (3.8) when $c_4 \neq 0$ and $c_3 > 0$.

For the function defined by eq. (3.9), we denote that

$$h_{1i} = H_1(\phi_{1i}, 0) = \phi_{1i}^2 \left(\frac{9c_3^2}{8c_4} + 6\mu_1 + 5c_3\phi_{1i} + 7c_4\phi_{1i}^2 \right), \qquad i = 0, 1, 2.$$

Notice that for the first parametric case (3.4), transformation (3.3) becomes

$$u = \frac{\mu_2}{\upsilon} \left(\frac{3c_3^2}{8c_4} + 6\mu_1 \right) + \frac{3\mu_2 c_3}{\upsilon}\phi + \frac{6\mu_2 c_4}{\upsilon}\phi^2. \tag{3.10}$$

By using the transformation (3.10) and travelling wave system (3.8) along with the Hamiltonian function (3.9) to do calculations, we have the following results:

Case I. $c_4 < 0$.

(I) Suppose that $(\mu_1, c_4) \in D_3$, i.e., $c_2 < (3c_3^2/8c_4)$, $c_3 > 0$ (see figure 1a). For $c_0 > h_{10}$, system (3.8) has a family of doubly periodic solutions

$$\phi_1(\xi) = -\frac{c_3}{4c_4} + \frac{k_1}{4c_4}\mathrm{cn}\left(\sqrt{\frac{2\mu_1}{2k_1^2-1}}\xi, k_1\right), \tag{3.11}$$

where

$$\alpha_1 = -16\mu_1 c_4 + \sqrt{\left(\tfrac{1}{2}c_3^2 + 8\mu_1 c_4\right)^2 - 64c_0c_4^3},$$

$$\beta_1 = -16\mu_1 c_4 - \sqrt{\left(\tfrac{1}{2}c_3^2 + 8\mu_1 c_4\right)^2 - 64c_0c_4^3},$$

$k_1^2 = \alpha_1/(\alpha_1 - \beta_1)$ and $\mathrm{cn}(x, k)$ and below $\mathrm{sn}(x, k)$ are Jacobian elliptic functions with modulus k [12]. Then eq. (3.1) has one family of smooth periodic solutions with double periods

$$q_1(x,t) = \left[\frac{6\mu_1\mu_2}{\upsilon} + \frac{3\mu_2 k_1^2}{8\upsilon c_4}\mathrm{cn}^2\left(\sqrt{\frac{2\mu_1}{2k_1^2-1}}(x-\upsilon t), k_1\right)\right]^{-1/(n-l+1)}. \tag{3.12}$$

(II) Suppose that $(\mu_1, c_4) \in D_4$, i.e., $c_2 > (3c_3^2/8c_4)$, $c_3 > 0$ (see figure 1b).

(1) For $c_0 = h_{10}$, system (3.8) has two soliton solutions with peak form and valley form, respectively,

$$\phi_2(\xi) = -\frac{c_3}{4c_4} \pm \frac{\sqrt{-2\mu_1 c_4}}{-c_4}\mathrm{sech}\left(\sqrt{2\mu_1}\xi\right). \tag{3.13}$$

Then eq. (3.1) has one family of soliton solutions for $n - l + 1 < 0$

$$q_2(x,t) = \left[\frac{6\mu_1\mu_2}{\upsilon}\left(1 - 2\,\mathrm{sech}^2\left(\sqrt{2\mu_1}(x-\upsilon t)\right)\right)\right]^{-1/(n-l+1)}. \tag{3.14}$$

Because of the limitation of length, we omit the expression of $q(x,t)$, beginning from here.

(2) For $c_0 \in (h_{11}, h_{10})$, system (3.8) has two families of doubly periodic solutions

$$\phi_3(\xi) = -\frac{c_3}{4c_4} \pm \frac{k_1}{c_4}\sqrt{\frac{2\mu_1 c_4}{1-2k_1^2}}\mathrm{dn}\left(k_1\sqrt{\frac{2\mu_1}{2k_1^2-1}}\xi, \frac{1}{k_1}\right). \tag{3.15}$$

(3) For $c_0 > h_{10}$, system (3.8) has a family of doubly periodic solutions

$$\phi_4(\xi) = -\frac{c_3}{4c_4} + \frac{k_1}{c_4}\sqrt{\frac{2\mu_1 c_4}{1-2k_1^2}}\mathrm{cn}\left(\sqrt{\frac{2\mu_1}{1-2k_1^2}}\xi, k_1\right). \tag{3.16}$$

Case II. $c_4 > 0$.

Suppose that $(\mu_1, c_4) \in D_2$, i.e., $c_2 < (3c_3^2/8c_4),\ c_3 > 0$ (see figure 1c).

(1) For $c_0 = h_{11}$, system (3.8) has two smooth kink wave solutions

$$\phi_5(\xi) = -\frac{c_3}{4c_4} \pm 2\frac{\sqrt{-\mu_1 c_4}}{c_4} \tanh\left(2\sqrt{-\mu_1}\xi\right). \tag{3.17}$$

(2) For $c_0 \in (h_{10}, h_{11})$, system (3.8) has a family of doubly periodic solutions

$$\phi_6(\xi) = -\frac{c_3}{4c_4} + \frac{k_2}{c_4}\sqrt{\frac{-2\mu_1 c_4}{1+k_2^2}}\mathrm{sn}\left(\sqrt{\frac{-2\mu_1}{1+k_2^2}}\xi, k_2\right), \tag{3.18}$$

where $k_2^2 = \beta_1/\alpha_1$.

Remark 1. To the best of our knowledge, solutions (3.12), (3.14)–(3.18) obtained for eq. (1.1) have not been reported in literature.

3.2 *Exact solutions of eq.* (3.1) *for the second case* (3.5)

For the second parametric case (3.5), the Hamiltonian system (2.5) becomes

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{1}{2}\left(\phi + \frac{c_3}{4c_4}\right)\left(4c_4\phi^2 + 2c_3\phi + \frac{c_3^2}{4c_4} - 2\mu_1\right) \tag{3.19}$$

and the Hamiltonian function (2.6) reduces to

$$H_2(\phi, y) = y^2 - \left(\frac{c_3}{6c_4}\left(\frac{3c_3^2}{8c_4} - 3\mu_1\right)\phi + \left(\frac{3c_3^2}{8c_4} - \mu_1\right)\phi^2 + c_3\phi^3 + c_4\phi^4\right) = c_0. \tag{3.20}$$

3.2.1 *Phase portraits and bifurcations of system* (3.19).

Denote that $\Delta_2 = c_4\mu_1$. Then system (3.19) has three equilibria at $E_{20}(-\frac{c_3}{4c_4}, 0),\ E_{21}(\phi_{21}, 0)$ and $E_{22}(\phi_{22}, 0)$ for $\Delta_2 = c_4\mu_1 > 0$, where

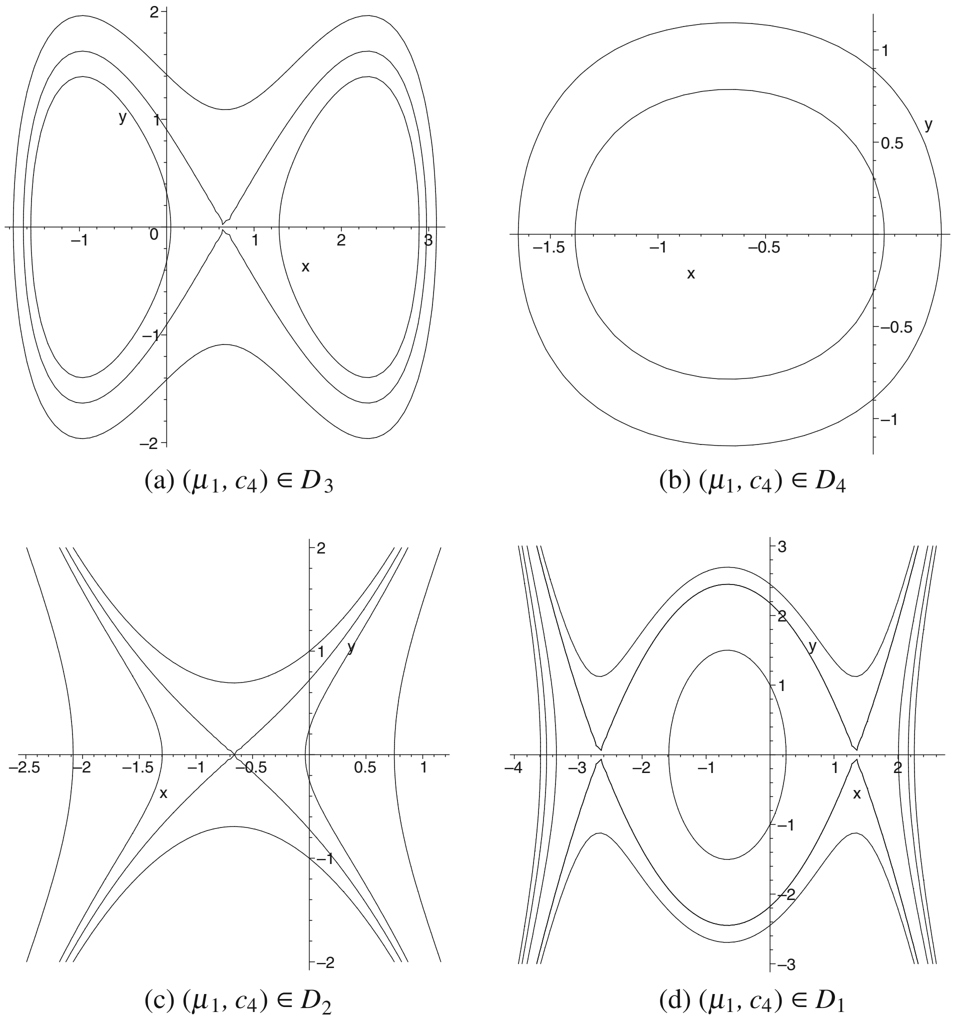
$$\phi_{21} = \frac{-c_3 + 2\sqrt{2\Delta_2}}{4c_4},$$
$$\phi_{22} = \frac{-c_3 - 2\sqrt{2\Delta_2}}{4c_4}.$$

For $\Delta_2 \le 0$, eq. (3.19) has a unique equilibrium at $E_{20}(-\frac{c_3}{4c_4}, 0)$.

For a fixed $c_3 > 0$, there are two bifurcation curves L_1: $c_4 = 0$ and L_2: $\mu_1 = 0$ which divide the (μ_1, c_4)-parametric plane into four subregions

$$D_1: \{\mu_1 > 0, c_4 > 0\};\ D_2: \{\mu_1 < 0, c_4 > 0\};$$
$$D_3: \{\mu_1 < 0, c_4 < 0\};\ D_4: \{\mu_1 > 0, c_4 < 0\}.$$

The phase portraits of (3.19) are shown in figure 2.



(a) $(\mu_1, c_4) \in D_3$ (b) $(\mu_1, c_4) \in D_4$

(c) $(\mu_1, c_4) \in D_2$ (d) $(\mu_1, c_4) \in D_1$

Figure 2. The phase portraits of eq. (3.19) when $c_4 \neq 0$ and $c_3 > 0$.

3.2.2 *Exact solutions of eq.* (3.1) *determined by phase portraits in figure* 2. Let $h_{2i} = H_2(\phi_{2i}, 0)$, $i = 0, 1, 2$. Notice that for the first parametric case (3.5), transformation (3.3) becomes

$$u = \frac{3\mu_2}{\upsilon}\left(\frac{c_3^2}{8c_4} + 2c_3\phi + 2c_4\phi^2\right). \tag{3.21}$$

We shall obtain the exact solutions of eq. (3.1) by using the transformation (3.21) and subsystem (3.19) along with the Hamiltonian function (3.20) to do calculations, and we have the following results.

Case I. $c_4 < 0$.

(I) Suppose that $(\mu_1, c_4) \in D_3$, i.e., $c_2 > (3c_3^2/8c_4)$, $c_3 > 0$ (see figure 2a).

(1) For $c_0 = h_{20}$, system (3.19) has two soliton solutions with peak form and valley form, respectively,

$$\phi_7^{\pm}(\xi) = -\frac{c_3}{4c_4} \pm \sqrt{\frac{c_3^2 - 8\mu_1 c_4}{8c_4^2}} \operatorname{sech}\left(\sqrt{\frac{c_3^2 - 8c_4}{8c_4^2}}\xi\right). \tag{3.22}$$

(2) For $c_0 \in (h_{21}, h_{20})$, system (3.19) has two families of doubly periodic solutions

$$\phi_8(\xi) = -\frac{c_3}{4c_4} \pm \frac{k_1}{c_4}\sqrt{\frac{\mu_1 c_4}{2k_1^2 - 1}} \operatorname{dn}\left(k_1\sqrt{\frac{\mu_1}{1 - 2k_1^2}}\xi, \frac{1}{k_1}\right). \tag{3.23}$$

(3) For $c_0 > h_{20}$, system (3.19) has a family of doubly periodic solutions

$$\phi_9(\xi) = -\frac{c_3}{4c_4} + \frac{k_1}{c_4}\sqrt{\frac{\mu_1 c_4}{2k_1^2 - 1}} \operatorname{cn}\left(\sqrt{\frac{\mu_1}{1 - 2k_1^2}}\xi, k_1\right). \tag{3.24}$$

(II) Suppose that $(\mu_1, c_4) \in D_4$, i.e., $c_2 < (3c_3^2/8c_4)$, $c_3 > 0$ (see figure 2b). For $c_0 > h_{20}$, system (3.19) has a family of doubly periodic solutions

$$\phi_{20}(\xi) = -\frac{c_3}{4c_4} + \frac{k_1}{4c_4} \operatorname{cn}\left(\sqrt{\frac{\mu_1}{1 - 2k_1^2}}\xi, k_1\right). \tag{3.25}$$

Case II. $c_4 > 0$.

Suppose that $(\mu_1, c_4) \in D_1$, i.e., $c_2 < (3c_3^2/8c_4)$, $c_3 > 0$ (see figure 2d).

(1) For $c_0 = h_{21}$, system (3.19) has two smooth kink wave solutions

$$\phi_{11}(\xi) = -\frac{c_3}{4c_4} \pm \frac{\sqrt{2\mu_1 c_4}}{2c_4} \tanh\left(\sqrt{\frac{\mu_1}{2}}\xi\right). \tag{3.26}$$

(2) For $c_0 \in (h_{20}, h_{21})$, system (3.19) has a family of doubly periodic solutions

$$\phi_{12}(\xi) = -\frac{c_3}{4c_4} + \frac{k_2}{c_4}\sqrt{\frac{\mu_1 c_4}{1 + k_2^2}} \operatorname{sn}\left(\sqrt{\frac{\mu_1}{1 + k_2^2}}\xi, k_2\right). \tag{3.27}$$

Remark 2. To the best of our knowledge, solutions (3.22)–(3.27) obtained for eq. (1.1) have not been reported in literature.

3.3 *Exact solutions of eq.* (3.1) *for the third case* (3.6)

Similar to the preceding discussion, for the third parametric case (3.6), we have the following results:

$$\phi_{13}(\xi) = \frac{4|\mu_1| - c_2}{3c_3} - \frac{4|\mu_1|}{c_3}\operatorname{sech}^2(\sqrt{|\mu_1|}\xi), \tag{3.28}$$

$$\phi_{14}(\xi) = \phi_m + (\phi_l - \phi_m)\operatorname{sn}^2\left(\frac{1}{2}\sqrt{c_3(\phi_M - \phi_m)}\xi,\ \sqrt{\frac{\phi_l - \phi_m}{\phi_M - \phi_m}}\right), \tag{3.29}$$

where $\phi_M > \phi_l > \phi_m$.

Remark 3. To the best of our knowledge, solutions (3.28) and (3.29) obtained for eq. (1.1) have not been reported in literature.

3.4 *Exact solutions of eq.* (3.1) *for the fourth case* (3.7)

Similar to the preceding discussion, for the fourth parametric case (3.7), we have the following results:

$$\phi_{15}(\xi) = \frac{4\mu_1}{3}\left(1 \pm \tanh\frac{4}{3}\sqrt{-\mu_1}\xi\right). \tag{3.30}$$

$$\phi_{16}(\xi) = \frac{\left(\frac{4\mu_1}{3} - b_2\right)(b_2 - b_1) - 2b_1\left(\frac{4\mu_1}{3} - b_1\right)\operatorname{sn}^2\left(\frac{c_3\sqrt{(b_1+b_2)(b_2-b_1)}}{8\sqrt{-\mu_1}}\xi, k_3\right)}{(b_2 - b_1) - 2b_1\operatorname{sn}^2\left(\frac{c_3\sqrt{(b_1+b_2)(b_2-b_1)}}{8\sqrt{-\mu_1}}\xi, k_3\right)}, \tag{3.31}$$

where

$$b_1 = 2\frac{\sqrt{4\mu_1^2 + c_3\sqrt{\mu_1 c_0}}}{c_3}, \quad b_2 = 2\frac{\sqrt{4\mu_1^2 - c_3\sqrt{\mu_1 c_0}}}{c_3}.$$

$$\phi_{17}(\xi) = \frac{4\mu_1}{c_3}\left(1 \pm \sqrt{2}\operatorname{sech}\left(\frac{c_3}{4\sqrt{\mu_1}}\xi\right)\right). \tag{3.32}$$

$$\phi_{18}(\xi) = \frac{\beta(\alpha - \gamma) - \gamma(\alpha - \beta)\operatorname{sn}^2(\Omega\xi, k)}{\alpha - \gamma - (\alpha - \beta)\operatorname{sn}^2(\Omega\xi, k)}, \tag{3.33}$$

$$\phi_{19}(\xi) = \frac{\gamma(\beta - \delta) + \beta(\gamma - \delta)\operatorname{sn}^2(\Omega\xi, k)}{-(\beta - \delta) + (\beta - \delta)\operatorname{sn}^2(\Omega\xi, k)}, \tag{3.34}$$

where

$$\Omega = \frac{c_3\sqrt{(\alpha - \gamma)(\beta - \delta)}}{8\sqrt{\mu_1}},\ k^2 = \frac{(\alpha - \beta)(\gamma - \delta)}{(\alpha - \gamma)(\beta - \delta)}.$$

$$\phi_{19}(\xi) = \frac{4\mu_1}{c_3} + b_1\operatorname{cn}(\Omega_4\xi, k_4), \tag{3.35}$$

where

$$\Omega_4 = \frac{c_3}{4}\sqrt{\frac{b_1^2 + b_2^2}{\mu_1}},$$

$$k_4^2 = \frac{b_1^2}{b_1^2 + b_2^2}.$$

Remark 4. To the best of our knowledge, solutions (3.30)–(3.35) obtained for eq. (1.1) have not been reported in literature.

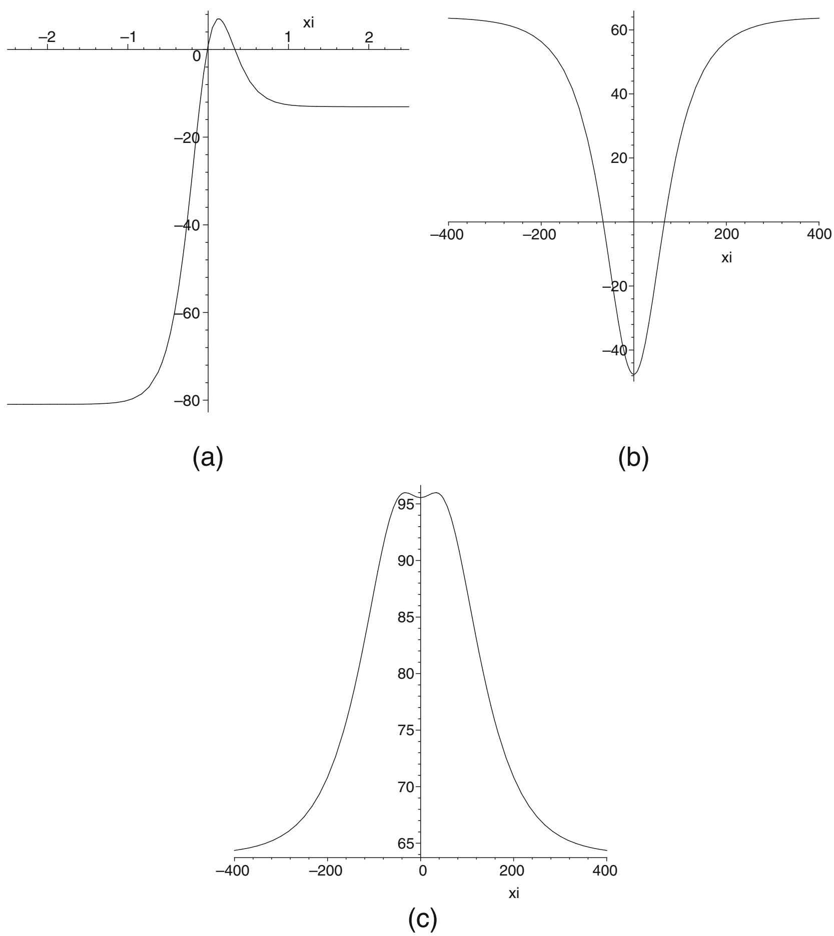


Figure 3. The wave profiles of solitary wave of eq. (3.1). (**a**) a kink wave $q_5(\xi)$ with $\mu_1 = -2, c_3 = 4, c_4 = 4, l = 5, n = 3, \mu_2 = 1$; (**b**) a single-hump soliton $q_7^+(\xi)$ with $\mu_1 = -2, c_3 = 4, c_4 = -\frac{3}{8}, l = 3, n = 1, \mu_2 = -2$; (**c**) a multihump soliton $q_7^-(\xi)$ with $\mu_1 = -2, c_3 = 4, c_4 = -\frac{3}{8}, l = 6, n = 4, \mu_2 = -2$, where $q_i = a_0 + a_1\phi_i + a_2\phi_i^2, i = 5, 7$.

4. Conclusions and summary

In this paper, a transformation and an improved Fan subequation method is used to construct exact travelling wave solutions of eq. (3.1). By using bifurcation theory of dynamical systems, we have succeeded in obtaining the bifurcations and phase portraits of the subequations involving all possible parametric conditions. Then many types of exact travelling wave solutions are obtained after solving the subequations. On the other hand, since all the parameters c_j $(j = 0, 1, 2, 3, 4)$, in the representations of exact solutions are free variables, the solutions obtained show more complex dynamical behaviour. For example, a continuum of kink solution, single-hump soliton and multihump soliton of eq. (3.1) are gained (see figure 3).

Since this method can help us find all exact solutions of the subequations involving all possible parameters and it is concise and efficient, it must be widely applied to other types of nonlinear dispersion partial differential equations.

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