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Modulation instability of an intense laser beam in an unmagnetized electron–positron–ion plasma

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Abstract. The modulation instability of an intense circularly polarized laser beam propagating in an unmagnetized, cold electron–positron–ion plasma is investigated. Adopting a generalized Karpman method, a three-dimensional nonlinear equation is shown to govern the laser field. Then the conditions for modulation instability and the temporal growth rate are obtained analytically. In order to compare with the usual electron–ion plasmas, the effect of positron concentration is considered. It is found that the increase in positron-to-electron density ratio shifts the instability region towards higher vertical wave numbers but does not cause displacement along the parallel wave number direction, and the growth rate increases as the positron-to-electron density ratio increases.

Keywords. Modulation instability; nonlinear dispersion relation; nonlinear governing equation; electron–positron–ion plasma.

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1. Introduction

Electron–positron (e–p) plasmas have gathered a great deal of interest and been widely studied theoretically and experimentally in the last decade because of the fact that the e–p plasmas were found in the early Universe, in some astrophysical objects such as pulsar magnetospheres, active galactic nuclei and supernova remnants [1–5], and in inertial confinement fusion (ICF) schemes using ultra-intense lasers [6]. Electron–positron plasmas can be created in large tokamaks through collisions between multi-MeV electrons and thermal particles [7]. However, many astrophysical and experimental plasmas contain ions besides the electrons and positrons (electron–positron–ion (e–p–i) plasma). Many authors have investigated the basic collective phenomena, linear and nonlinear effects in an e–p–i plasma [8–11]. It has been found that the physics of the plasma consisting of electrons and ions is different from the physics of plasmas containing electrons, positrons and ions.

Modulation instability (MI) is one of the most ubiquitous nonlinear effects in a plasma and it is a consequence of the interplay between nonlinear and dispersive effects. The MI of different waves in an e–p–i plasma has received great attention due to its importance in wave propagation [12–22]. As an example, Bains *et al* [12] derived a three-dimensional (3D) nonlinear Schrödinger equation by using the standard reductive perturbation technique for studying the MI of nonlinear ion-acoustic wave in a magnetized quantum e–p–i plasma. Moreover, Jehan *et al* [14] studied the nonlinear amplitude modulation of low-frequency electrostatic ion waves propagating in a collisionless magnetized e–p–i plasma, in which the MI of both ion-acoustic and ion-cyclotron-like modes were examined.

In laser–plasma interaction, the ponderomotive force originated from the electromagnetic (EM) waves arouses low-frequency perturbations of density, then they interact with the high-frequency EM waves so that the amplitude of the pump wave becomes modulated, and the MI of EM waves occurs. The development of MI would induce a variety of nonlinear processes such as field collapse and localization, filamentation, envelope solitons, envelope shocks, etc. [15–17]. The modulation instabilities of EM waves in both e–p–i and e–p plasmas were studied extensively [14,18–22]. Shatashvili *et al* [20] investigated the nonlinear wave dynamics in two-temperature e–p–i plasmas, in which the one-dimensional (1D) propagation of a circularly polarized EM wave was analysed and its MI was discussed. Kourakis *et al* [21] have considered the propagation of nonlinear amplitude-modulated EM wave in a pair plasma and e–p–i plasmas embedded in a uniform magnetic field, respectively. They analysed the MI based on the 1D nonlinear Schrödinger-type equation and found that the presence of the background ion species affected the stability profile of the modulated wave packets. Shukla *et al* [22] have taken into account the nonrelativistic temperature of an e–p–i plasma, and a pair of 3D nonlinear governing equations which admit a novel MI as well as 1D envelope electromagnetic solitons were derived and the modulated growth rate of the CPEM wave was obtained. Although the early works have investigated the MI of EM waves in relatively complicated plasmas (e.g., a plasma-embedded magnetic field, a plasma with temperature), these works mostly considered the 1D nonlinear governing equations in which transverse variations of laser field amplitude were ignored and very few papers have systematically explored a two- or three-dimensional case.

It is well known that the 3D nonlinear governing equation for the envelope of laser field can be obtained straightforwardly by Karpman method from the nonlinear dispersion relation [23]. However, this model is limited to the wave field with one polarization. Thus, it will not be appropriate for the circularly or elliptically polarized EM waves with two polarizations, whereas, Li [24] has generalized Karpman method to the wave field with more than one polarization. Chen *et al* [25] employed it to investigate the MI of an elliptical polarization laser in a magnetized electron–ion plasma. Subsequently, this investigation was extended to an e–p plasma [26].

In this paper, we aim at studying the MI of an intense right-handed circularly polarized laser beam in a collisonless, unmagnetized and cold e–p–i plasma. The effect of the presence of positrons on the MI is considered. For this purpose, we adopt the generalized Karpman method to acquire the 3D nonlinear governing equation. The organization of the paper is as follows: In §2, the nonlinear dispersion relation for the laser radiation in the e–p–i plasma by means of Lorentz transformation is derived. In §3 the 3D nonlinear governing equation for the slowly-varying envelope of the laser electric field is obtained via the generalized Karpman method. Based on the nonlinear governing equation, the MI is analysed in §4. In §5, the discussion and conclusion are presented.

2. Nonlinear dispersion relation

The analysis in this section is based on the Lorentz transformation employed in ref. [27]. We consider the propagation of an intense, right-handed circularly polarized laser beam in a plasma comprising electrons, positrons and ions (suppose $|q_\mathrm{e}| = q_\mathrm{p}, q_\mathrm{i}$; $m_\mathrm{e} = m_\mathrm{p}$). The electric field is represented by

$$\mathbf{E} = E_0(\mathbf{r}, t)(\mathbf{e}_x + i\mathbf{e}_y)\exp[i(k_0 z - \omega_0 t)], \tag{1}$$

where $E_0(\mathbf{r}, t)$ is the complex, slowly-varying electric field, k_0 and ω_0 are the central wave number and frequency of the laser beam, respectively. The equations of motion for the particles are

$$\frac{\mathrm{d}(\gamma_j m_j \mathbf{v}_j)}{\mathrm{d}t} = q_j\left[\mathbf{E} + \frac{1}{c}\mathbf{v}_j \times \mathbf{B}\right], \tag{2}$$

where m_j, q_j are the rest mass and the charge of the particle specie j (for electrons, $j = \mathrm{e}$; for ions $j = \mathrm{i}$; for positrons $j = \mathrm{p}$), $\mathbf{v}_j$ is the particle quiver velocity induced by the EM waves, $\gamma_j = (1 - \mathbf{v}_j^2/c^2)^{-1/2}$ is the relativistic factor, $\mathbf{B}$ and c are the wave magnetic field and the speed of light in vacuum, respectively.

Let us suppose that an inertial reference frame S' moves relative to the laboratory frame S with velocity nc along z-axis, where n is the refractive index. From here on, we use the ‘ $'$ ’ for the corresponding quantities in the frame S'. In the S' system, the space-time variable $\xi(= t - nz/c)$ becomes the transformed time variable $t'(=(1 - n^2)^{-1/2}\xi)$. As a result, the wave fields have no spatial dependence in the frame S'. Hence, from the Faraday's law in Maxwell equations, we can conclude that the magnetic field $\mathbf{B}'$ is a constant vector in the frame S' and here we assume $\mathbf{B}' = 0$ for simplicity, and correspondingly, the Ampere theorem can be expressed as

$$\dot{\mathbf{E}}' = -4\pi\,|q_\mathrm{e}|\,[N_\mathrm{e}'(\mathbf{v}_\mathrm{i}' - \mathbf{v}_\mathrm{e}') + N_\mathrm{p}'(\mathbf{v}_\mathrm{p}' - \mathbf{v}_\mathrm{i}')]. \tag{3}$$

Here we have used $|q_\mathrm{e}| = q_\mathrm{p}, q_\mathrm{i}$ and consequently $N_\mathrm{e}' = N_\mathrm{p}' + N_\mathrm{i}'$ is satisfied in an e–p–i plasma in terms of $\nabla' \cdot \mathbf{E}' = 4\pi \sum_j \rho_j' = 0$, and eq. (2) may be written as

$$-m_\mathrm{e}\dot{\mathbf{u}}_\mathrm{e}' = \frac{|q_\mathrm{e}|\mathbf{E}'}{c} = m_\mathrm{p}\dot{\mathbf{u}}_\mathrm{p}' = m_\mathrm{i}\dot{\mathbf{u}}_\mathrm{i}', \tag{4}$$

where $\mathbf{u}_j' = \gamma_j'\mathbf{v}_j'/c$ are the reduced velocities of the particles. Then the equation of the motion of an electron in the S' system can be expressed as

$$\ddot{\mathbf{u}}_\mathrm{e}' + \left[\frac{\omega_\mathrm{e}'^2}{\gamma_\mathrm{e}'} + \frac{\omega_\mathrm{p}'^2}{\gamma_\mathrm{p}'} + \mu_1\left(\frac{\omega_\mathrm{e}'^2}{\gamma_\mathrm{i}'} - \frac{\omega_\mathrm{p}'^2}{\gamma_\mathrm{i}'}\right)\right]\dot{\mathbf{u}}_\mathrm{e}' = \left[u_2'\frac{\omega_\mathrm{p}'^2}{\gamma_\mathrm{p}'} + u_1'\left(\frac{\omega_\mathrm{e}'^2}{\gamma_\mathrm{i}'} - \frac{\omega_\mathrm{p}'^2}{\gamma_\mathrm{i}'}\right)\right]\mathbf{e}_{z'}, \tag{5}$$

where the condition $m_{\rm e} = m_{\rm p}$ is utilized and $\mu_1 = m_{\rm e}/m_{\rm i}$; u'_1 and u'_2 are constants; $\omega'^2_j = 4\pi N'_j q_j^2/m_j$ is the squared plasma frequency of the jth particles in the S' system; $\mathbf{e}_{z'}$ is a unit vector parallel to z'. We make an ansatz for the solution of eq. (5) in the following form:

$$\mathbf{u}'_{\rm e} = \mathbf{u}'_{\perp}{\rm e}^{-i\omega' t'} + u'_{z'}\mathbf{e}_{z'}, \tag{6}$$

where $\mathbf{u}'_{\perp}$, $u'_{z'}$ are the velocities perpendicular and parallel to z'-axis, respectively, and they are all slowly varied compared to the fluctuation at frequency ω'. Substituting eq. (6) into eq. (5), we obtain the dispersion relation in the S' system as

$$\omega'^2 = \frac{\omega'^2_{\rm e}}{\gamma'_{\rm e}} + \frac{\omega'^2_{\rm p}}{\gamma'_{\rm p}} + \mu_1\left(\frac{\omega'^2_{\rm e}}{\gamma'_{\rm i}} - \frac{\omega'^2_{\rm p}}{\gamma'_{\rm i}}\right). \tag{7}$$

Meanwhile, substituting eq. (6) into eq. (4), we have

$$\mathbf{E}' = \mathbf{E}'_{\perp}{\rm e}^{-i\omega' t'}, \tag{8}$$

in which $\mathbf{E}'_{\perp} = i m_{\rm e} c\omega' \mathbf{u}'_{\perp}/|q_{\rm e}|$. If we do transformation as follows:

$$\mathbf{E}_{\perp} = \eta\mathbf{E}'_{\perp}, \quad E_z = E'_{z'}, \quad \omega = \eta\omega', \quad \eta = (1-n^2)^{-1/2}, \tag{9}$$

the quantities in the S' system can be transformed to the laboratory frame S. Finally, we obtain the nonlinear dispersion relation in the S frame from eq. (7):

$$\omega^2 = k^2c^2 + [(\omega^2_{\rm pe}+\omega^2_{\rm pp})(1+u^2_{\perp})^{-1/2} + \omega^2_{\rm pi}(1+\mu_1^2 u^2_{\perp})^{-1/2}]\left(1-\frac{\upsilon_0^2}{c^2}\right), \tag{10}$$

where the quasineutrality condition in the equilibrium state $N^0_{\rm p} + N^0_{\rm i} = N^0_{\rm e}$ is utilized, $\omega^2_{{\rm p}j} = 4\pi e^2 N^0_j/m_j$, N^0_j are the squared plasma frequency and number density of the jth particles in the laboratory frame, respectively, υ_0 is the electron flow velocity in the z direction and $u^2_{\perp} = (q_{\rm e}/m_{\rm e}c\omega)^2\,|\mathbf{E}_{\perp}|^2$. Taking $\upsilon_0^2/c^2 \ll 1$ into account, eq. (10) can be reduced to

$$\omega^2 = k^2c^2 + \left[\left(\omega^2_{\rm pe}+\omega^2_{\rm pp}\right)\left(1+\frac{q_{\rm e}^2E_0^2}{m_{\rm e}^2c^2\omega^2}\right)^{-1/2} + \omega^2_{\rm pi}\left(1+\mu_1^2\frac{q_{\rm e}^2E_0^2}{m_{\rm e}^2c^2\omega^2}\right)^{-1/2}\right]. \tag{11}$$

Equation (11) is the nonlinear dispersion relation for the intense circularly polarized laser beam in an e–p–i plasma. If the terms associated with the laser intensity in the square brackets of eq. (11) is absent, eq. (11) would reduce to the linear dispersion relation $\omega^2 = k^2c^2 + \omega^2_{\rm pe} + \omega^2_{\rm pp} + \omega^2_{\rm pi}$.

On the other hand, transforming eq. (8) from the S' system to the S system, we have

$$\mathbf{E} = \mathbf{\Psi}\exp[-i(\omega_0 t - k_0 z)], \tag{12}$$

where $\mathbf{\Psi}(=\mathbf{E}_{\perp}\exp\{-i[(\omega-\omega_0)t-(k-k_0)z]\})$ is the complex, slowly-varying envelope of the electric field and is governed by the nonlinear evolution equation that will be derived in the next section.

3. 3D nonlinear governing equation for the envelope of laser electric field

The envelope of the pump laser electric field can be represented by

$$\mathbf{\Psi}(\mathbf{r}, t) = \mathbf{a}(\mathbf{r}, t) \exp[i\phi(\mathbf{r}, t)], \tag{13}$$

where $\phi(\mathbf{r}, t) = \mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t + \varphi(\mathbf{r}, t)$. In the short wavelength approximation,

$$-\phi_t \equiv \omega = \omega_0 - \varphi_t, \quad \nabla\phi \equiv \mathbf{k} = \mathbf{k}_0 + \nabla\varphi, \tag{14}$$

where $\mathbf{k}_0$ $(=k_0\mathbf{e}_z)$ is the wave vector of the pump wave and subscript t denotes the total differential of time. We expand the dispersion relation $\omega(k)$ in a Taylor series about k_0 and retain terms up to third order

$$\omega(k) - \omega(k_0) = \upsilon_{\mathrm{g}}\left[\varphi_z + \frac{(\nabla_\perp\varphi)^2}{2k_0}\right] + \frac{1}{2}\dot{\upsilon}_{\mathrm{g}}\varphi_z^2, \tag{15}$$

where $\upsilon_{\mathrm{g}}(=(\partial\omega/\partial k)|_{k=k_0})$ and $\dot{\upsilon}_{\mathrm{g}}(=(\partial^2\omega/\partial k^2)|_{k=k_0})$ represent group velocity and group velocity dispersion of the wave envelope, respectively. Here, we have dropped the small term $\nabla\varphi/k_0$. To account for the nonlinear frequency shift of the EM wave, the nonlinear dispersion relation is taken to be [28]

$$\tilde{\omega}(\mathbf{k}, a) = \omega(\mathbf{k}) + \alpha a^2, \tag{16}$$

where a^2 is the normalized laser intensity and it is straightforward to find that the nonlinear frequency shift grows when the normalized laser intensity goes stronger. α is the nonlinear coefficient which has contributions from ponderomotive force and relativistic particle mass varition. By means of eq. (11), one finds

$$\upsilon_{\mathrm{g}} = \frac{k_0c^2}{\omega_0}, \quad \dot{\upsilon}_{\mathrm{g}} = \frac{c^2(\omega_{\mathrm{pe}}^2 + \omega_{\mathrm{pp}}^2 + \omega_{\mathrm{pi}}^2)}{\omega_0^3}, \quad \alpha = -\frac{q_{\mathrm{e}}^2(\omega_{\mathrm{pe}}^2 + \omega_{\mathrm{pp}}^2)}{4m_{\mathrm{e}}^2c^2\omega_0^3}, \tag{17}$$

where the term $\mu_1^2 q_{\mathrm{e}}^2\omega_{\mathrm{pi}}^2$ is elided in the expression of α as $\mu_1 \ll 1$. Expanding the modulated wave field $\mathbf{E}(\mathbf{r}, t)$ in a Fourier series yields

$$\tilde{\mathbf{\Psi}}_{\tilde{\omega},\mathbf{k}} = \mathbf{E}_{\tilde{\omega},\mathbf{k}} \exp\{i[(\mathbf{k} - \mathbf{k}_0)\cdot\mathbf{r} - (\tilde{\omega} - \omega_0)t]\}, \tag{18}$$

where $\tilde{\mathbf{\Psi}}_{\tilde{\omega},\mathbf{k}} = \mathbf{\Psi}_{\tilde{\omega},\mathbf{k}}\mathrm{e}^{[i(\mathbf{k}\cdot\mathbf{r}-\tilde{\omega}t)]}$. After multiplying eq. (15) by eq. (18) and integrating over $\mathrm{d}\tilde{\omega}\,\mathrm{d}\mathbf{k}$, and considering $\tilde{\omega} - \omega_0 \to i\partial/\partial t$, $\nabla\varphi = (\mathbf{k} - \mathbf{k_0}) \to -i\nabla$, we have the nonlinear governing equation for waves with one polarization [23]. Note that, for waves with more than one polarization, φ_z^2 and $(\nabla_\perp\varphi)^2$ should be replaced by $(\nabla\times\nabla\times)_z - (\nabla\nabla\cdot)_z$ and $(\nabla\times\nabla\times)_\perp - (\nabla\nabla\cdot)_\perp$ respectively. Then we can obtain [24]

$$i\left(\frac{\partial\mathbf{\Psi}}{\partial t} + \upsilon_{\mathrm{g}}\frac{\partial\mathbf{\Psi}}{\partial z}\right) + \frac{1}{2}\dot{\upsilon}_{\mathrm{g}}[-(\nabla\times\nabla\times)_z\mathbf{\Psi} + (\nabla\nabla\cdot)_z\mathbf{\Psi}]$$
$$+\frac{1}{2}\frac{\upsilon_{\mathrm{g}}}{k_0}[(\nabla\nabla\cdot)_\perp\mathbf{\Psi} - (\nabla\times\nabla\times)_\perp\mathbf{\Psi}] - \alpha\,|\mathbf{\Psi}|^2\,\mathbf{\Psi} = 0, \tag{19}$$

where the fourth term on the left-hand side of eq. (19) represents shift in plasma frequency due to the density fluctuations, i.e., the usual modulation perturbations resulting from density rarefaction by ponderomotive force [29]. Equation (19) describes the evolution of the envelope of the laser electric field and is the desired equation for studying the MI of the circularly polarized laser beam in an e–p–i plasma. Evidently, $-\alpha/\dot{\upsilon}_{\mathrm{g}} > 0$, and thus the MI can be excited. The MI will develop, finally resulting in the formation of a stable structure, namely, the well-known envelope soliton, because of the competition between dispersion and nonlinear effects.

4. Analysis of modulation instability

Now let us turn our attention to the problem of the stability of nonlinear monochromatic waves in the Liapunov sense [30] for eq. (19). Firstly, we make the following substitutions:

$$\tilde{t} = k_0^2 \dot{\upsilon}_{\rm g} t, \quad \tilde{x} = \sqrt{2} k_0 \left(\frac{k_0 \dot{\upsilon}_{\rm g}}{\upsilon_{\rm g}}\right)^{1/2} x, \quad \tilde{y} = \sqrt{2} k_0 \left(\frac{k_0 \dot{\upsilon}_{\rm g}}{\upsilon_{\rm g}}\right)^{1/2} y,$$

$$\tilde{z} = \sqrt{2} k_0 z, \quad \tilde{\mathbf{\Psi}} = \frac{|q_{\rm e}|\mathbf{\Psi}}{m_{\rm e} c \omega_0},$$

then eq. (19) is written as

$$i(\tilde{\mathbf{\Psi}})_{\tilde{t}} + i\beta_1 (\tilde{\mathbf{\Psi}})_{\tilde{z}} + \tilde{\nabla}^2 \tilde{\mathbf{\Psi}} + \beta_2 \tilde{N} \tilde{\mathbf{\Psi}} = 0, \tag{20}$$

in which we have assumed

$$\tilde{N} = -|\tilde{\mathbf{\Psi}}|^2. \tag{21}$$

Adding Laplace operator at the two sides of eq. (21), we obtain

$$\tilde{\nabla}^2 \tilde{N} = -\tilde{\nabla}^2 |\tilde{\mathbf{\Psi}}|^2, \tag{22}$$

where $\beta_1 = \sqrt{2}\upsilon_{\rm g}/k_0\dot{\upsilon}_{\rm g}$, $\beta_2 = \alpha m_{\rm e}^2 c^2 \omega_0^2 / q_{\rm e}^2 k_0^2 \dot{\upsilon}_{\rm g}$, and $\tilde{N}$ is the dimensionless electron density perturbation. It is worth noting that eqs (20)–(22) are not isotropic because the field envelope $\tilde{\mathbf{\Psi}}$ is a slowly varied vector function related to the x, y, z directions. Also, the flowing form of plane wave,

$$\tilde{\mathbf{\Psi}}_{\rm I} = \tilde{\mathbf{\Psi}}_0 \exp[i(\tilde{K}_0 \tilde{z} - \tilde{\Omega}_0 \tilde{t})], \quad \tilde{N}_{\rm I} = 0 \tag{23}$$

is a solution of eqs (20) and (22), provided that $\tilde{\Omega}_0 = \beta_1 \tilde{K}_0 + \tilde{K}_0^2$, $\tilde{\mathbf{K}}_0 \cdot \tilde{\mathbf{\Psi}}_0 \propto \tilde{\mathbf{K}}_0 \cdot \mathbf{e}_0 = 0$, $\tilde{\mathbf{K}}_0 = \tilde{K}_0 \mathbf{e}_{\tilde{z}}$, where $\tilde{\mathbf{\Psi}}_0$, $\tilde{K}_0$ and $\tilde{\Omega}_0$ are the initial dimensionless peak amplitude, wave number and frequency, respectively. In order to investigate the stability of the solution, we assume the presence of small perturbations ($\delta\tilde{\mathbf{\Psi}}$, $\tilde{N}_{\rm II}$) to the solution; if the amplitudes of the disturbance are amplified, then the solution is unstable in the Liapunov sense. Linearizing eqs (20) and (22) with respect to the perturbations yields

$$i(\delta\tilde{\mathbf{\Psi}})_{\tilde{t}} + i\beta_1 (\delta\tilde{\mathbf{\Psi}})_{\tilde{z}} + \tilde{\nabla}^2 (\delta\tilde{\mathbf{\Psi}}) + \beta_2 \tilde{N}_{\rm II} \tilde{\mathbf{\Psi}}_{\rm I} = 0, \tag{24}$$

$$-\tilde{\nabla}^2 \tilde{N}_{\rm II} = \tilde{\nabla}^2 [|\tilde{\mathbf{\Psi}}_{\rm I} \cdot (\delta\tilde{\mathbf{\Psi}})^* + \tilde{\mathbf{\Psi}}_{\rm I}^* \cdot (\delta\tilde{\mathbf{\Psi}})|]. \tag{25}$$

We study the following forms of the perturbations:

$$\delta\tilde{\mathbf{\Psi}} = [(\tilde{\mathbf{\Psi}}_1 + \tilde{\mathbf{\Psi}}_2) \mathrm{e}^{i(\tilde{\mathbf{K}}\cdot\tilde{\mathbf{r}} - \tilde{\Omega}\tilde{t})} + (\tilde{\mathbf{\Psi}}_1^+ + \tilde{\mathbf{\Psi}}_2^+) \mathrm{e}^{-i(\tilde{\mathbf{K}}\cdot\tilde{\mathbf{r}} - \tilde{\Omega}\tilde{t})}] \mathrm{e}^{i(\tilde{K}_0\tilde{z} - \tilde{\Omega}_0\tilde{t})}, \tag{26}$$

$$\tilde{N}_{\rm II} = \frac{\tilde{N}}{2} [\mathrm{e}^{i(\tilde{\mathbf{K}}\cdot\tilde{\mathbf{r}} - \tilde{\Omega}\tilde{t})} + \mathrm{e}^{-i(\tilde{\mathbf{K}}\cdot\tilde{\mathbf{r}} - \tilde{\Omega}\tilde{t})}], \tag{27}$$

where $\tilde{\mathbf{K}}(\tilde{\Omega})$ is the dimensionless perturbed wave vector (frequency). The amplitude of longitudinal perturbations $(\tilde{\boldsymbol{\Psi}}_1, \tilde{\boldsymbol{\Psi}}_1^+)$ and amplitude of the transverse perturbations $(\tilde{\boldsymbol{\Psi}}_2, \tilde{\boldsymbol{\Psi}}_2^+)$ are independent of time-space and they can be expressed as

$$\tilde{\boldsymbol{\Psi}}_1 = \tilde{\psi}_1 \mathbf{e}_1, \qquad \tilde{\boldsymbol{\Psi}}_1^+ = \tilde{\psi}_1^* \mathbf{e}_1^+, \qquad \mathbf{e}_1 \| \tilde{\mathbf{K}}_+, \qquad \mathbf{e}_1^+ \| \tilde{\mathbf{K}}_-, \tag{28}$$

$$\tilde{\boldsymbol{\Psi}}_2 = \tilde{\psi}_2 \mathbf{e}_2, \qquad \tilde{\boldsymbol{\Psi}}_2^+ = \tilde{\psi}_2^* \mathbf{e}_2^+, \qquad \mathbf{e}_2 \perp \tilde{\mathbf{K}}_+, \qquad \mathbf{e}_2^+ \perp \tilde{\mathbf{K}}_-, \tag{29}$$

where $\tilde{\mathbf{K}}_\pm = \tilde{\mathbf{K}} \pm \tilde{\mathbf{K}}_0$, $\tilde{\Omega}_\pm = \tilde{\Omega} \pm \tilde{\Omega}_0$, while $\mathbf{e}_1$, $\mathbf{e}_1^+$, $\mathbf{e}_2$ and $\mathbf{e}_2^+$ are unit vectors. Substituting eqs (23), (26) and (27) into eq. (24) and its conjugate equation yields

$$[\tilde{\Omega}_+ - \beta_1(\tilde{K}_{\tilde{z}} + \tilde{K}_0) - \tilde{K}_+^2](\tilde{\psi}_1 \mathbf{e}_1 + \tilde{\psi}_2 \mathbf{e}_2) = -\frac{\tilde{N}}{2}\beta_2 \tilde{\boldsymbol{\Psi}}_0, \tag{30}$$

$$[-\tilde{\Omega}_- + \beta_1(\tilde{K}_{\tilde{z}} - \tilde{K}_0) - \tilde{K}_-^2](\tilde{\psi}_1 \mathbf{e}_1^+ + \tilde{\psi}_2 \mathbf{e}_2^+) = -\frac{\tilde{N}}{2}\beta_2 \tilde{\boldsymbol{\Psi}}_0^*, \tag{31}$$

where $\tilde{K}_{\tilde{z}} = |\tilde{\mathbf{K}}|\cos\theta$, θ is the angle between $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{K}}_0$; $\tilde{\boldsymbol{\Psi}}_0^*$ is the complex conjugate function of $\tilde{\boldsymbol{\Psi}}_0$. Similarly, one gets from eq. (25)

$$-\frac{\tilde{N}}{2} = \tilde{\boldsymbol{\Psi}}_0 \cdot (\tilde{\psi}_1 \mathbf{e}_1^+ + \tilde{\psi}_2 \mathbf{e}_2^+) + \tilde{\boldsymbol{\Psi}}_0^* \cdot (\tilde{\psi}_1 \mathbf{e}_1 + \tilde{\psi}_2 \mathbf{e}_2). \tag{32}$$

Combining eqs (30), (31) and (32), we obtain the dispersion equation of MI including the longitudinal and transverse perturbations

$$\begin{aligned}\tilde{K}^4 \; - \; (\tilde{\Omega} - \beta_1 \tilde{K}_{\tilde{z}} - 2\tilde{K}_{\tilde{z}}\tilde{K}_0)^2 &= \beta_2 |\tilde{\boldsymbol{\Psi}}_0|^2 [(-\tilde{\Omega} + \beta_1 \tilde{K}_{\tilde{z}} \\ &- \; \tilde{K}^2 + 2\tilde{K}_{\tilde{z}}\tilde{K}_0)F + (\tilde{\Omega} - \beta_1 \tilde{K}_{\tilde{z}} - \tilde{K}^2 - 2\tilde{K}_{\tilde{z}}\tilde{K}_0)G],\end{aligned} \tag{33}$$

with

$$\begin{aligned} F &= (\cos^2\theta_+ + \cos^2\alpha_+), \quad \cos\theta_+ = \mathbf{e}_0 \cdot \mathbf{e}_1, \quad \cos\alpha_+ = \mathbf{e}_0 \cdot \mathbf{e}_2, \\ G &= (\cos^2\theta_- + \cos^2\alpha_-), \quad \cos\theta_- = \mathbf{e}_0 \cdot \mathbf{e}_1^+, \quad \cos\alpha_- = \mathbf{e}_0 \cdot \mathbf{e}_2^+. \end{aligned} \tag{34}$$

One can acquire $\pi/2 - \theta_+ \leq \alpha_+ \leq \pi/2 + \theta_+$ from the geometry relationship of $\mathbf{e}_1$ and $\mathbf{e}_2$. Likewise, we have $\pi/2 - \theta_- \leq \alpha_- \leq \pi/2 + \theta_-$. Accordingly, we can infer $\cos^2\theta_+ \leq F \leq 1$ and $\cos^2\theta_- \leq G \leq 1$. Equation (33) is a general dispersion equation to study the modulation instabilities for the perturbations appearing at any direction, e.g., by considering the transverse perturbations, the reduced dispersion equation can be obtained by replacing F and G by $\cos^2\alpha_+$ and $\cos^2\alpha_-$, respectively, by which one can examine the MI for transverse perturbations. Next, we shall discuss the special case $F = 1$, $G = 1$ in detail. In this situation, the dispersion equation (33) is reduced to

$$\tilde{K}^4 - (\tilde{\Omega} - \beta_1 \tilde{K}_{\tilde{z}} - 2\tilde{\mathbf{K}} \cdot \tilde{\mathbf{K}}_0)^2 = -2\tilde{K}^2 \beta_2 |\tilde{\boldsymbol{\Psi}}_0|^2. \tag{35}$$

We can easily find that if and only if $\tilde{K}^2 + 2\beta_2|\tilde{\boldsymbol{\Psi}}_0|^2 < 0$ and $\beta_2 < 0$, it supports unstable solution with the temporal growth rate $\tilde{\Gamma} = |\mathrm{Im}\,\tilde{\Omega}|$. Taking α, υ_g and $\dot{\upsilon}_\mathrm{g}$ in eq. (17) and introducing the following dimensionless parameters:

$$\hat{\Gamma} = \frac{\Gamma}{\omega_0}, \quad \hat{K}_z = \frac{K_z c}{\omega_0}, \quad \hat{K}_\perp = \frac{K_\perp c}{\omega_0}, \quad A^2 = \frac{q_\mathrm{e}^2 |\Psi_0|^2}{m_\mathrm{e}^2 c^2 \omega_0^2}, \quad \mu_j = \frac{\omega_j}{\omega_0},$$

we get the dimensionless temporal growth rate

$$\hat{\Gamma} = \frac{1}{2}[-(\varpi \hat{K}_z^2 + \hat{K}_\perp^2)^2 + A^2(1+p)\mu_e^2(\varpi \hat{K}_z^2 + \hat{K}_\perp^2)]^{1/2}, \tag{36}$$

and the conditions for which the instability occurs

$$\varpi \hat{K}_z^2 + \hat{K}_\perp^2 < (1+p)\mu_e^2 A^2, \quad \frac{(1+p)}{(1+p)+\mu_1(1-p)} > 0, \tag{37}$$

where we have denoted the positron-to-electron density ratio $p = n_{p0}/n_{e0} (p \le 1)$ and accordingly $n_{i0}/n_{e0} = 1 - p$. Furthermore, $\varpi = (1+p)\mu_e^2 + \mu_1(1-p)\mu_e^2$ is introduced. From eq. (36), one can find that, for $\hat{K}_\perp = 0$, the maximum growth rate $\Gamma_{max} = A^2(1+p)\mu_e^2/4$ when $\hat{K}_z = A\mu_e\sqrt{(1+p)/(2\varpi)}$. Similarly, for $\hat{K}_z = 0$, Γ_{max} is the same as the former, but occurs at $\hat{K}_\perp = A\mu_e\sqrt{(1+p)/2}$. What is more, we can conclude that $\mu_e^2 \lesssim 1/(1+p+\mu_1(1-p))$ in view of the linear dispersion relation $\omega^2 = k^2c^2 + \omega_{pe}^2 + \omega_{pp}^2 + \omega_{pi}^2$. When $p = 0$, it has $\mu_e^2 \lesssim 1$ and is represented as the case of pure electron–ion plasmas. When $p = 1$, it has $\mu_e^2 \lesssim 0.5$ and is the case of pure electron–positron plasmas. For the general cases, e.g., $p = 0.2$, it has $\mu_e^2 \lesssim 1/(1.2 + 0.8\mu_1)$ and is the case of electron–positron–ion plasmas.

To estimate the temporal growth rate with respect to $\hat{K}_z$ and $\hat{K}_\perp$, eq. (36) is numerically calculated. The results are shown in figure 1 for $\omega_0 = 1.88 \times 10^{15}\,\mathrm{s}^{-1}$, $A^2 = 0.734$

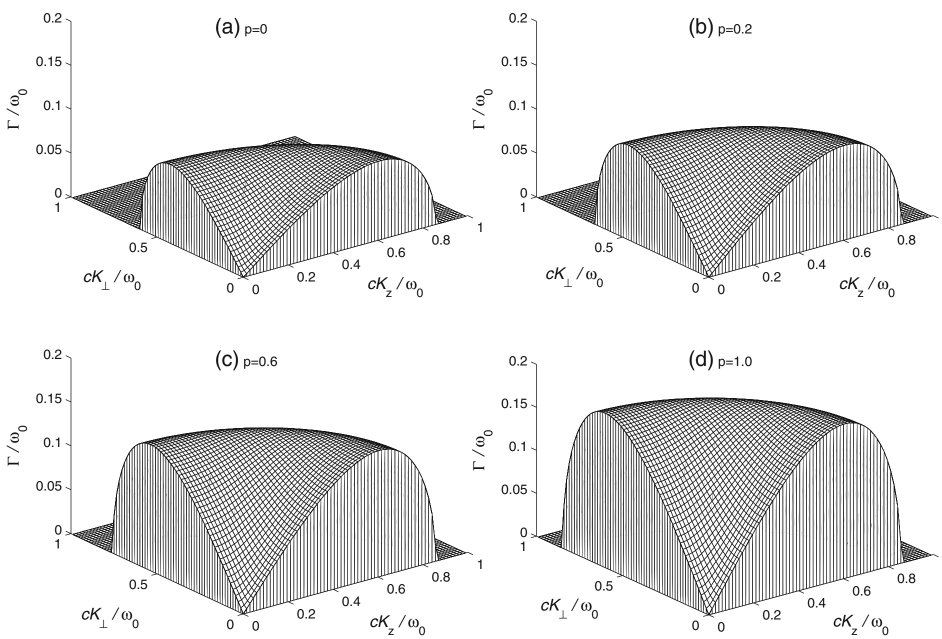

Figure 1. Surface plots of normalized temporal growth rate Γ/ω_0 as a function of $\hat{K}_z$ and $\hat{K}_\perp$, where $\mu_e = 0.7$, $\mu_1 = 1/1836$ and the laser intensity $\approx 10^{18}\,\mathrm{W\cdot cm^{-2}}$ (i.e. $A^2 = 0.734$). The positron-to-electron density ratio is varied for **(a)** $p = 0$, **(b)** $p = 0.2$, **(c)** $p = 0.6$, **(d)** $p = 1$.

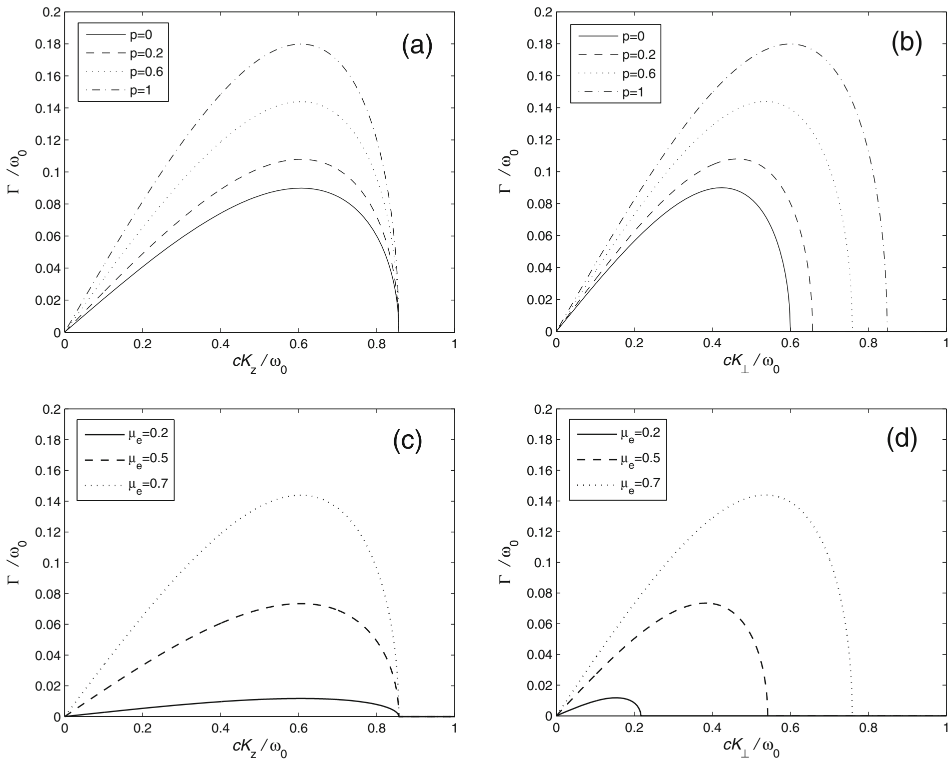

Figure 2. The curves of temporal growth rate Γ/ω_0 against $\hat{K}_z$, $\hat{K}_\perp$, respectively. The parameters are **(a)** $\hat{K}_\perp = 0$, $\mu_e = 0.7$, $p = 0, 0.2, 0.6, 1$, **(b)** $\hat{K}_z = 0$, $\mu_e = 0.7$, $p = 0, 0.2, 0.6, 1$, **(c)** $\hat{K}_\perp = 0$, $p = 0.6$, $\mu_e = 0.2, 0.5, 0.7$ and **(d)** $\hat{K}_z = 0$, $p = 0.6$, $\mu_e = 0.2, 0.5, 0.7$. The other parameters are the same as those used in figure 1.

and $\mu_1 = 1/1836$. We take $\mu_e = 0.7$ according to the range of μ_e^2. Based on the same parameters, some curves of temporal growth rate against $\hat{K}_z$, $\hat{K}_\perp$ respectively are displayed in figures 2a and 2b so as to compare the magnitude of temporal growth rate for different positron-to-electron density ratio p. On the contrary, the curves of temporal growth rate Γ/ω_0 as a function of wave number ($\hat{K}_z$, $\hat{K}_\perp$) with some selected values of μ_e for a given $p = 0.6$ are plotted in figures 2c and 2d. It should be mentioned that the increase of μ_e means the laser gets closer to the critical surface.

It can be observed from figure 1 that the MI occurs in the small wave number region and the increase in positron-to-electron density ratio expands the instability region at the $K_\perp$ direction towards higher wave numbers but does not arouse displacement of instability region along the K_z direction. From figures 2a and 2b we can obviously note that, for any fixed $\hat{K}_z$ or $\hat{K}_\perp$, there is an increase in the growth rate if the positron-to-electron density ratio p increases. In other words, the instability growth rate is shown to suppress with increasing ion concentration. We also can see from figures 2c and 2d that, for any fixed $\hat{K}_z$ or $\hat{K}_\perp$, the temporal growth rate increases as μ_e increases. That means the growth rate of MI goes up significantly when laser approaches the critical surface.

5. Conclusion

In this paper, we have derived the nonlinear dispersion relation for the intense circularly polarized laser radiation in an unmagnetized, cold e–p–i plasma. Then the 3D nonlinear governing equation for the envelope of laser electric field is obtained and the MI has been analysed. In the process of analysis, the effect of positron concentration on the temporal growth rate of MI is investigated. From the above study, we arrive at the following conclusions:

(1) It is found that the presence of positron modifies the instability region, the increase in positron-to-electron density ratio shifts the instability region towards higher vertical wave numbers but does not cause displacement of instability region along the parallel wave number direction. This is determined by the conditions for which MI occurs.

(2) The growth rate increases with the increase in positron-to-electron density ratio until the ion concentration decreases to zero which behaves as a pure electron–positron plasma. That means, the presence of the positron stimulates the occurrence of MI. The physical explanation regarding this is as follows: in an electron–positron–ion plasma, the ions are almost immobile and the effect of relativistic laser on it is negligible, but the effect is great on electrons and positrons whose mass increase due to the relativistic effect. We all know that the nonlinearities arising from the variation of mass of the relativistic particles may excite the MI, and thus the electrons and positrons have significant contributions on the MI.

(3) According to the nonlinear governing equations (20) and (22), it can be seen that the transverse plasmons are modulated by the low-frequency density variation. Near the critical surface in the laser–plasma, it is hard for the transverse plasmons to escape because of the very small group velocities. Thus the interaction between the transverse plasmons and low-frequency density variations is very strong, and the growth rate of MI increases remarkably.

It should be pointed out that the Karpman method is valid only for the weakly nonlinear regime; it cannot be applied directly to the ultrarelativistic regime. For a uniform pump field, the MI would lead to the formation of localized field with different longitudinal and transverse characteristic scales. It may result in filamentation, envelope solitons and other nonlinear phenomena in the laser–plasma. Therefore, the results of our investigation will be helpful in understanding the nonlinear propagation and instability behaviour of high-frequency electromagnetic waves in astrophysical and laboratory plasmas.

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