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Multiple (G'/G)-expansion method and its applications to nonlinear evolution equations in mathematical physics

JUNCHAO CHEN* and BIAO LI
Department of Mathematics, Ningbo University, Ningbo 315211, People's Republic of China
*Corresponding author. E-mail: junchaochen@yahoo.com.cn

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Abstract. In this paper, an extended multiple (G'/G)-expansion method is proposed to seek exact solutions of nonlinear evolution equations. The validity and advantages of the proposed method is illustrated by its applications to the Sharma–Tasso–Olver equation, the sixth-order Ramani equation, the generalized shallow water wave equation, the Caudrey–Dodd–Gibbon–Sawada–Kotera equation, the sixth-order Boussinesq equation and the Hirota–Satsuma equations. As a result, various complexiton solutions consisting of hyperbolic functions, trigonometric functions, rational functions and their mixture with parameters are obtained. When some parameters are taken as special values, the known double solitary-like wave solutions are derived from the double hyperbolic function solution. In addition, this method can also be used to deal with some high-dimensional and variable coefficients' nonlinear evolution equations.

Keywords. Nonlinear evolution equation; extended multiple (G'/G)-expansion method; complexiton solutions; double solitary-like wave solution.

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1. Introduction

In mathematical physics, it is well known that many nonlinear evolution equations (NLEEs) are widely used as models to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid-state physics, plasma physics, plasma waves, biology and so on. Searching for exact solutions of NLEEs using various methods has become extremely valuable, and many powerful and efficient methods such as the inverse scattering transform [1], the Darboux transformation [2], Bäcklund transformation method [3], Hirota bilinear method [4], the tanh method [5], symmetry method [6], Painlevé expansion [7], exp-function method [8] and so on to construct exact solutions of NLEEs have been established and developed by a diverse group of scientists. In recent years, due to the availability of symbolic computation systems like *Mathematica* or *Maple*, direct methods to search for exact solutions of NLEEs have attracted more and more attention.

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Most recently, Wang *et al* [9] introduced an expansion technique called the (G'/G)-expansion method and they demonstrated that it is a powerful technique for seeking analytic solutions of NLEEs. Making use of this method, some useful studies of various equations also appeared in [10–21]. A generalization of the method was given by Zhang *et al* [22,23]. Also, Zhang *et al* [24] made further extension of the method for the evolution equations with variable coefficients. Zhang *et al* [25–28] devised an algorithm for using the method to solve nonlinear differential-difference equations. A new application of this method to some special NLEEs, the balance numbers of which are not positive integers, have been explored [29]. Several other extended methods which supposed different forms of solution of a given NLEE and depended on different auxiliary linear ordinary differential equation (LODE) were also developed [30–32]. However, we find that (G'/G)-expansion method can be improved to construct multiple solitary wave solutions of some NLEEs.

The rest of this paper is organized as follows. In §2, a description of the extended multiple (G'/G)-expansion method is given in detail. In §3, the applications of our method to the Sharma–Tasso–Olver equation, the sixth-order Ramani equation, the generalized shallow water wave equation, the Caudrey–Dodd–Gibbon–Sawada–Kotera equation, the sixth-order Boussinesq equation and the Hirota–Satsuma equations are illustrated. Conclusions are presented in §4.

2. The extended multiple (G'/G)-expansion method

The objective of this section is to outline the use of extended multiple (G'/G)-expansion method for solving certain NlEEs. For a given NLEE with independent variables $X = (x, y, z, ..., t)$ and dependent variable u:

$$F(u, u_t, u_x, u_y, u_z, ..., u_{tt}, u_{xx}, u_{yy}, u_{zz}, u_{xt}, u_{yt}, u_{zt}, ...) = 0, \tag{1}$$

where F is a polynomial in $u = u(X) = u(x, y, z, ..., t)$ and its partial derivatives. We suppose the solution of eq. (1) can be expressed by a polynomial as follows:

$$u = \alpha_0(X) + \sum_{k=1}^{n}\sum_{i+j=k} \alpha_{ij}(X)\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)^i \left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right)^j,$$
$$\sum_{i+j=n} \alpha_{ij}(X)\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)^i \left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right)^j \neq 0, \tag{2}$$

where $\alpha_0(X), \alpha_{ij}(X), \xi_1, \xi_2\ (i, j = 1, 2, ..., n)$ are all functions of X to be determined later, n is an undetermined integer and $G_1(\xi_1)$, $G_2(\xi_2)$ satisfy the auxiliary LODEs

$$G_i''(\xi_i) + \lambda_i G_i'(\xi_i) + \mu_i G_i(\xi_i) = 0, \quad i = 1, 2, \tag{3}$$

where $G_i''(\xi_i) = \mathrm{d}^2 G_i(\xi_i)/\mathrm{d}\xi_i^2$, $G_i'(\xi_i) = \mathrm{d}G_i(\xi_i)/\mathrm{d}\xi_i\ (i = 1, 2)$, and $\lambda_i, \mu_i\ (i = 1, 2)$ are constants. To determine u explicitly, we take the following four steps:

Step 1: Determine the integer n by balancing the highest-order nonlinear term(s) and the highest-order partial derivative of u in eq. (1).

Step 2: With the aid of symbolic computation, substituting (2) along with eq. (3) into eq. (1), yields a partial differential equation. Since terms $\left(G_1'(\xi_1)/G_1(\xi_1)\right)^i \left(G_2'(\xi_2)/G_2(\xi_2)\right)^j$

$(i = 0, 1, 2...; j = 0, 1, 2...)$ in the partial differential equation are linearly independent, a set of over-determined partial differential equations about the unknown variables $\{\alpha_0(X), \alpha_{ij}(X), \xi_1, \xi_2\ (i, j = 1, 2, ..., n)\}$ can be derived by vanishing the coefficients of the terms $\left(G_1'(\xi_1)/G_1(\xi_1)\right)^i \left(G_2'(\xi_2)/G_2(\xi_2)\right)^j$ $(i = 0, 1, 2...; j = 0, 1, 2...)$.

Step 3: Solve the system of over-determined partial differential equations obtained in Step 2 for the unknown variables $\{\alpha_0(X), \alpha_{ij}(X), \xi_1, \xi_2\ (i, j = 1, 2, ..., n)\}$.

Step 4: Utilizing the results obtained in the previous steps, a series of explicitly exact solutions of eq. (1) are obtained immediately.

Remark 1. It is necessary to point out that our method is different from the general (G'/G)-expansion method. In our method, we introduce two independent variables ξ_1 and ξ_2, and therefore, double solitary-like wave solutions and some other complexiton solutions for certain NLEEs can be obtained. However, if we set $\alpha_{ij}(X) = 0,\ i = 1, 2, ..., n$ when $j = 1, 2, ..., n$, our method immediately reduces to the general (G'/G)-expansion method.

Remark 2. Some new developments of our method are listed as follows:

(I) The form of solution (2) of NLEE may be assumed a more generalized form:

$$u = \alpha_0(X) + \sum_{k=1}^{n} \sum_{i+j=k} \alpha_{ij}(X) \left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)^i \left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right)^j + \sum_{k=1}^{n} \sum_{i+j=k} \beta_{ij}(X) \left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)^{-i} \left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right)^{-j}, \tag{4}$$

where $\alpha_0(X), \alpha_{ij}(X), \beta_{ij}, \xi_1, \xi_2,\ i, j = 1, 2, ..., n$ are all functions of X.

(II) The form of solution (2) of NLEE may also be further developed into another generalized form:

$$u = \sum_{k=0}^{n} \sum_{i_1+i_2+\cdots+i_m=k} \alpha_{i_1 i_2 \ldots i_m}(X) \left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)^{i_1} \cdots \left(\frac{G_m'(\xi_m)}{G_m(\xi_m)}\right)^{i_m}, \tag{5}$$

where $\alpha_0(X), \alpha_{ij}(X), \xi_1, \xi_2,\ i, j = 1, 2, ..., n$ are all functions of X. Then by using (5), multisoliton solutions and other multicomplexiton solutions of NLEEs will be derived.

(III) For the auxiliary LODEs (3), there are several other choices, such as Jacobi elliptic equation $[G'(\xi)]^2 = h_0 + h_1 G^2(\xi) + h_2 G^4(\xi)$ [31], the third-order linear ordinary differential equation $G'''(\xi) = k_0 G(\xi) + k_1 G'(\xi) + k_2 G''(\xi)$ [32], general Jacobi elliptic equations $[G'(\xi)]^2 = l_0 + l_1 G(\xi) + l_2 G^2(\xi) + l_3 G^3(\xi) + h_4 G^4(\xi)$ [33] and so on. Thus, we can construct other novel exact solutions of NLEEs, such as multiple Jacobi elliptic function solutions, multiple Weierstrass elliptic function solutions and other types of complexiton solutions.

3. Applications of the extended multiple (G'/G)-expansion method

In this section, we shall apply the extended multiple (G'/G)-expansion method to construct exact solutions for some NLEEs in mathematical physics as follows.

For simplification, in the following subsections, we uniformly denote $\begin{cases}\Delta_1 = \lambda_1^2 - 4\mu_1, \\ \Delta_2 = \lambda_2^2 - 4\mu_2,\end{cases}$ and $\mu_1 \neq \mu_2$.

3.1 *The Sharma–Tasso–Olver equation*

We start with the Sharma–Tasso–Olver (STO) equation [32,34–36] of the form

$$u_t + 3\alpha u^2 u_x + \frac{3}{2}\alpha u_{xx}^2 + \alpha u_{xxx} = 0, \tag{6}$$

where α is an arbitrary real constant, which is a prominent double nonlinear dispersive model and comprises of the linear dispersive term αu_{xxx} and the double nonlinear terms $3\alpha u^2 u_x$ and $\frac{3}{2}\alpha u_{xx}^2$.

Balancing the highest-order partial derivative term and the highest-order nonlinear term of eq. (6), we suppose that the solution of eq. (6) is of the form

$$u = A_1\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + A_0 + a_1\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \tag{7}$$

where $\xi_1, \xi_2, A_1, A_0, a_1$ are functions of $\{x, t\}$ to be determined, and $G_1(\xi_1), G_2(\xi_2)$ satisfy eq. (3).

With the aid of symbolic computation, substituting eq. (7) along with eq. (3) into eq. (6) yields a partial differential equation. Then vanishing the coefficients of terms $\left(G_1'(\xi_1)/G_1(\xi_1)\right)^i\left(G_2'(\xi_2)/G_2(\xi_2)\right)^j$ $(i = 0, 1, 2...; j = 0, 1, 2...)$ of the partial differential equation, yields a set of over-determined partial differential equations with respect to $\{\xi_1, \xi_2, A_1, A_0, a_1\}$. Solving these over-determined partial differential equations, the following results are obtained:

$$A_1 = F_1, \quad a_1 = F_2, \quad A_0 = \frac{\lambda_1 F_1 + \lambda_2 F_2}{2}, \tag{8}$$

$$\xi_1 = F_1 x - \frac{\alpha F_1\left(F_1^2\Delta_1 + 3F_2^2\Delta_2\right)t}{4} + F_3, \tag{9}$$

$$\xi_2 = F_2 x - \frac{\alpha F_2\left(F_2^2\Delta_2 + 3F_1^2\Delta_1\right)t}{4} + F_4, \tag{10}$$

where F_1, F_2, F_3 and F_4 are arbitrary constants.

Therefore, the general form of solution of STO equation (6) can be expressed by

$$u = F_1\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + \frac{\lambda_1 F_1 + \lambda_2 F_2}{2} + F_2\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \tag{11}$$

where F_1 and F_2 are arbitrary constants, and ξ_1 and ξ_2 are determined using eqs (9) and (10).

In what follows, from the explicit solutions of the auxiliary LODEs (3), which depend on different choices of λ_1, μ_1, λ_2 and μ_2, a series of complexiton solutions of eq. (6) can be constructed immediately.

(I) Setting $\begin{cases}\Delta_1 > 0,\\ \Delta_2 > 0,\end{cases}$ the complexiton solutions consisting of hyperbolic functions can be derived as

$$u_1 = \frac{F_1\sqrt{\Delta_1}}{2}\left(\frac{M_1\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}{M_1\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}\right) + \frac{F_2\sqrt{\Delta_2}}{2}\left(\frac{N_1\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}{N_1\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}\right), \quad (12)$$

where F_1, F_2, M_1, M_2, N_1 and N_2 are arbitrary constants, and ξ_1 and ξ_2 are determined using eqs (9) and (10).

If $M_2 \neq 0$, $N_2 \neq 0$, $M_2^2 > M_1^2$, $N_2^2 > N_1^2$, the double solitary-like wave solution can be obtained as

$$u_1 = \frac{F_1\sqrt{\Delta_1}}{2}\tanh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1 + \xi_{10}\right) + \frac{F_2\sqrt{\Delta_2}}{2}\tanh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2 + \xi_{20}\right), \quad (13)$$

with $\xi_{10} = \tanh^{-1}(M_1/M_2)$ and $\xi_{20} = \tanh^{-1}(N_1/N_2)$.

(II) Setting $\begin{cases}\Delta_1 < 0,\\ \Delta_2 < 0,\end{cases}$ the complexiton solutions consisting of trigonometric functions can be derived as

$$u_2 = \frac{F_1\sqrt{-\Delta_1}}{2}\left(\frac{M_1\cos\left(\frac{\sqrt{-\Delta_1}}{2}\xi_1\right) - M_2\sin\left(\frac{\sqrt{-\Delta_1}}{2}\xi_1\right)}{M_1\sin\left(\frac{\sqrt{-\Delta_1}}{2}\xi_1\right) + M_2\cos\left(\frac{\sqrt{-\Delta_1}}{2}\xi_1\right)}\right) + \frac{F_2\sqrt{-\Delta_2}}{2}\left(\frac{N_1\cos\left(\frac{\sqrt{-\Delta_2}}{2}\xi_2\right) - N_2\sin\left(\frac{\sqrt{-\Delta_2}}{2}\xi_2\right)}{N_1\sin\left(\frac{\sqrt{-\Delta_2}}{2}\xi_2\right) + N_2\cos\left(\frac{\sqrt{-\Delta_2}}{2}\xi_2\right)}\right), \quad (14)$$

where F_1, F_2, M_1, M_2, N_1 and N_2 are arbitrary constants, and ξ_1 and ξ_2 are determined using eqs (9) and (10).

(III) Setting $\begin{cases}\Delta_1 > 0,\\ \Delta_2 = 0,\end{cases}$ the complexiton solutions consisting of hyperbolic functions and rational functions can be derived as

$$u_3 = \frac{F_1\sqrt{\Delta_1}}{2}\left(\frac{M_1\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}{M_1\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}\right) + \frac{F_2N_2}{N_1 + N_2\xi_2}, \quad (15)$$

with

$$\xi_1 = F_1x - \frac{\alpha F_1F_1^2\Delta_1 t}{4} + F_3, \quad \xi_2 = F_2x - \frac{3\alpha F_2F_1^2\Delta_1 t}{4} + F_4, \quad (16)$$

where F_1, F_2, F_3, F_4, M_1, M_2, N_1 and N_2 are arbitrary constants.

(IV) Setting $\begin{cases}\Delta_1 < 0,\\ \Delta_2 = 0,\end{cases}$ the complexiton solutions consisting of trigonometric functions and rational functions can be derived as

$$u_4 = \frac{F_1\sqrt{-\Delta_1}}{2}\left(\frac{M_1\cos\left(\frac{\sqrt{-\Delta_1}}{2}\xi_1\right) - M_2\sin\left(\frac{\sqrt{-\Delta_1}}{2}\xi_1\right)}{M_1\sin\left(\frac{\sqrt{-\Delta_1}}{2}\xi_1\right) + M_2\cos\left(\frac{\sqrt{-\Delta_1}}{2}\xi_1\right)}\right) + \frac{F_2N_2}{N_1 + N_2\xi_2}, \tag{17}$$

where $F_1, F_2, F_3, F_4, M_1, M_2, N_1$ and N_2 are arbitrary constants, and ξ_1 and ξ_2 are determined using (16).

When setting $\begin{cases}\Delta_1 > 0\\ \Delta_2 < 0\end{cases}$, $\begin{cases}\Delta_1 < 0\\ \Delta_2 > 0\end{cases}$, $\begin{cases}\Delta_1 = 0\\ \Delta_2 > 0\end{cases}$, $\begin{cases}\Delta_1 = 0\\ \Delta_2 < 0\end{cases}$, some other complexiton solutions of eq. (6) can be derived similarly. We omit them here for convenience.

In fact, there are 2^3 groups of complexiton solutions for eq. (6) in all. Similarly, 2^3 groups of complexiton solutions for every equation in the following can be derived. For brevity, we only list one group of corresponding solutions for every equation.

3.2 *The sixth-order Ramani equation*

In this subsection we consider the sixth-order nonlinear Ramani equation, or the KdV6 equation [37–39], given by

$$u_{xxxxxx} + 15u_xu_{xxxx} + 15u_{xx}u_{xxx} + 45u_x^2u_{xx} - 5(u_{xxxt} + 3u_xu_{xt} + 3u_tu_{xx}) - 5u_{tt} = 0. \tag{18}$$

Balancing the highest-order partial derivative term and the highest-order nonlinear term of eq. (18), we suppose that the solution of eq. (18) is of the form

$$u = A_1\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + A_0 + a_1\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \tag{19}$$

where $\xi_1, \xi_2, A_1, A_0, a_1$ are functions of $\{x, t\}$ to be determined, and $G_1(\xi_1), G_2(\xi_2)$ satisfy eq. (3).

By using the same procedure as in §3.1, the general solutions of sixth-order nonlinear Ramani equation (18) can be derived as follows:

$$u = 2F_1\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + A_0 + 2F_2\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \tag{20}$$

with

$$A_0 = -\frac{3}{25}\left(F_1^4\Delta_1^2 + 3F_1^2F_2^2\Delta_1^2\Delta_2^2 + F_2^4\Delta_2^2\right)t + F_5, \tag{21}$$

$$\xi_1 = F_1x + \frac{1}{5}F_1\left(3F_2^2\Delta_2 + 2F_1^2\Delta_1\right)t + F_3, \tag{22}$$

$$\xi_2 = F_2x + \frac{1}{5}F_2\left(3F_1^2\Delta_1 + 2F_2^2\Delta_2\right)t + F_4, \tag{23}$$

where F_1, F_2, F_3, F_4 and F_5 are arbitrary constants.

Setting $\begin{cases}\Delta_1 > 0\\ \Delta_2 > 0\end{cases}$, the complexiton solutions consisting of hyperbolic functions can be derived as

$$\begin{aligned} u_1 = {} & A_0 - (\lambda_1 F_1 + \lambda_2 F_2) \\ & + F_1\sqrt{\Delta_1}\left(\frac{M_1\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}{M_1\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}\right) \\ & + F_2\sqrt{\Delta_2}\left(\frac{N_1\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}{N_1\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}\right), \end{aligned} \tag{24}$$

where F_1, F_2, M_1, M_2, N_1 and N_2 are arbitrary constants, and A_0, ξ_1 and ξ_2 are determined using (21)–(23).

If $M_2 \neq 0$, $N_2 \neq 0$, $M_2^2 > M_1^2$, $N_2^2 > N_1^2$, the double solitary-like wave solution can be obtained as

$$\begin{aligned} u_1 = {} & A_0 - (\lambda_1 F_1 + \lambda_2 F_2) + F_1\sqrt{\Delta_1}\tanh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1 + \xi_{10}\right) \\ & + F_2\sqrt{\Delta_2}\tanh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2 + \xi_{20}\right), \end{aligned} \tag{25}$$

with $\xi_{10} = \tanh^{-1}(M_1/M_2)$ and $\xi_{20} = \tanh^{-1}(N_1/N_2)$.

3.3 *The generalized shallow water wave equation*

In this subsection we study the generalized shallow water wave (GSWW) equation [12, 40],

$$u_{xxxt} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xt} - \gamma u_{xx} = 0, \tag{26}$$

where α, β and γ are arbitrary, nonzero constants.

Balancing the highest-order partial derivative term and the highest-order nonlinear term of eq. (26), we suppose that the solution of eq. (26) is of the form

$$u = A_1\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + A_0 + a_1\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \tag{27}$$

where $\xi_1, \xi_2, A_1, A_0, a_1$ are functions of $\{x, t\}$ to be determined, and $G_1(\xi_1), G_2(\xi_2)$ satisfy eq. (3).

By using the same procedure as in §3.1, the general solutions of GSWW equation (26) can be derived as follows:

$$u = \frac{6F_1}{\alpha}\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + A_0 + \frac{6\delta\sqrt{\Delta_2(2 - F_1^2\Delta_1)}}{\alpha\Delta_2}\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \tag{28}$$

with

$$\gamma = k\alpha, \quad \beta = \alpha, \quad A_0 = \frac{1 - F_1^2\Delta_1}{F_1\alpha} p_0(t) + kt + F_3, \tag{29}$$

$$\xi_1 = F_1 x + p_0(t), \quad \xi_2 = \frac{\delta\sqrt{\Delta_2(2 - F_1^2\Delta_1)}[F_1 x + p_0(t)]}{\Delta_2 F_1} + F_2, \tag{30}$$

where F_1, F_2, F_3 and k are arbitrary constants, $\delta = \pm 1$ and $p_0(t)$ is an arbitrary function of t.

Setting $\begin{cases}\Delta_1 > 0\\ \Delta_2 > 0\end{cases}$, the complexiton solutions consisting of hyperbolic functions can be derived as

$$\begin{aligned} u_1 = {} & A_0 - \frac{3F_1\lambda_1}{\alpha} - \frac{3\delta\lambda_2\sqrt{\Delta_2(2 - F_1^2\Delta_1)}}{\alpha\Delta_2} \\ & + \frac{3F_1\sqrt{\Delta_1}}{\alpha}\left(\frac{M_1\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}{M_1\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}\right) \\ & + \frac{3\delta\sqrt{2 - F_1^2\Delta_1}}{\alpha}\left(\frac{N_1\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}{N_1\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}\right), \end{aligned} \tag{31}$$

where F_1, M_1, M_2, N_1 and N_2 are arbitrary constants, $\delta = \pm 1$, and A_0, ξ_1 and ξ_2 are determined using eqs (29) and (30).

If $M_2 \neq 0$, $N_2 \neq 0$, $M_2^2 > M_1^2$, $N_2^2 > N_1^2$, the double solitary-like wave solution can be obtained as

$$\begin{aligned} u_1 = {} & A_0 - \frac{3F_1\lambda_1}{\alpha} - \frac{3\delta\lambda_2\sqrt{\Delta_2(2 - F_1^2\Delta_1)}}{\alpha\Delta_2} \\ & + \frac{3F_1\sqrt{\Delta_1}}{\alpha}\tanh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1 + \xi_{10}\right) \\ & + \frac{3\delta\sqrt{2 - F_1^2\Delta_1}}{\alpha}\tanh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2 + \xi_{20}\right), \end{aligned} \tag{32}$$

with $\xi_{10} = \tanh^{-1}(M_1/M_2)$ and $\xi_{20} = \tanh^{-1}(N_1/N_2)$.

3.4 *The Caudrey–Dodd–Gibbon–Sawada–Kotera equation*

In this subsection, we study Caudrey–Dodd–Gibbon–Sawada–Kotera (CDGSK) equation [41,42]

$$u_t + u_{xxxxx} + 5(uu_{xxx} + u_x u_{xx} + u^2 u_x) = 0. \tag{33}$$

Balancing the highest-order partial derivative term and the highest-order nonlinear term of eq. (33), we suppose that the solution of eq. (33) is of the form

$$u = A_2\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)^2 + A_1\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + A_0 + a_2\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right)^2 + a_1\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right) + a_0\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \tag{34}$$

where $\xi_1, \xi_2, A_2, A_1, A_0, a_2, a_1, a_0$ are functions of $\{x, t\}$ to be determined, and $G_1(\xi_1)$, $G_2(\xi_2)$ satisfy eq. (3).

By using the same procedure as in §3.1, the general solutions of CDGSK equation (33) can be derived as follows:

$$u = -6F_1^2\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)^2 - 6\lambda_1 F_1^2\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + A_0 - 6F_2^2\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right)^2 - 6\lambda_2 F_2^2\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \tag{35}$$

with

$$A_0 = F_1^2\Delta_1 + F_2^2\Delta_2 - \frac{3}{2}\left(F_1^2\lambda_1^2 + F_2^2\lambda_2^2\right), \tag{36}$$

$$\xi_1 = F_1 x + \frac{1}{4}F_1\left(F_1^4\Delta_1^2 - 5F_2^2\Delta_2^2\right)t + F_3, \tag{37}$$

$$\xi_2 = F_2 x + \frac{1}{4}F_2\left(F_2^4\Delta_2^2 - 5F_1^2\Delta_1^2\right)t + F_4, \tag{38}$$

where F_1, F_2, F_3 and F_4 are arbitrary constants.

Setting $\begin{cases}\Delta_1 > 0\\ \Delta_2 > 0\end{cases}$, the complexiton solutions consisting of hyperbolic functions can be derived as

$$u_1 = F_1^2\Delta_1 + F_2^2\Delta_2 - \frac{3\Delta_1 F_1^2}{2}\left(\frac{M_1\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}{M_1\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}\right)^2 - \frac{3\Delta_2 F_2^2}{2}\left(\frac{N_1\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}{N_1\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}\right)^2, \tag{39}$$

where F_1, F_2, M_1, M_2, N_1 and N_2 are arbitrary constants, and ξ_1 and ξ_2 are determined using eqs (36)–(38).

If $M_2 \neq 0$, $N_2 \neq 0$, $M_2^2 > M_1^2$, $N_2^2 > N_1^2$, the double solitary-like wave solution can be obtained as

$$u_1 = -\frac{F_1^2\Delta_1 + F_2^2\Delta_2}{2} + \frac{3F_1^2\Delta_1}{2}\operatorname{sech}^2\left(\frac{\sqrt{\Delta_1}}{2}\xi_1 + \xi_{10}\right) + \frac{3F_2^2\Delta_2}{2}\operatorname{sech}^2\left(\frac{\sqrt{\Delta_2}}{2}\xi_2 + \xi_{20}\right), \tag{40}$$

with $\xi_{10} = \tanh^{-1}(M_1/M_2)$ and $\xi_{20} = \tanh^{-1}(N_1/N_2)$.

3.5 *The sixth-order Boussinesq equation*

In this subsection, we consider sixth-order Boussinesq (SB) equation [17,43]

$$u_{tt} - u_{xx} - \left(15uu_{4x} + 30u_x u_{3x} + 15u_{xx}^2 + 45u^2 u_{xx} + 90uu_x^2 + u_{6x}\right) = 0. \tag{41}$$

Balancing the highest-order partial derivative term and the highest-order nonlinear term of eq. (41), we suppose that the solution of eq. (41) is of the form

$$u = A_2\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)^2 + A_1\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + A_0 + a_2\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right)^2 + a_1\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right) + a_0\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \tag{42}$$

where $\xi_1, \xi_2, A_2, A_1, A_0, a_2, a_1, a_0$ are functions of $\{x, t\}$ to be determined, and $G_1(\xi_1)$, $G_2(\xi_2)$ satisfy eq. (3).

By using the same procedure as in §3.1, the general solutions of SB equation (41) can be derived as follows:

$$u = -2F_1^2\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)^2 - 2F_1^2\lambda_1\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + A_0 - 2F_2^2\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right)^2 - 2F_2^2\lambda_2\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \tag{43}$$

with

$$A_0 = \frac{1}{3}\left(F_1^2\Delta_1 + F_2^2\Delta_2\right) - \frac{1}{2}\left(F_1^2\lambda_1^2 + F_2^2\lambda_2^2\right), \tag{44}$$

$$\xi_1 = F_1 x + \frac{\delta_1 F_1}{2}\sqrt{4 + 5F_2^4\Delta_2^2 - F_1^4\Delta_1^2}\,t + F_3, \tag{45}$$

$$\xi_2 = F_2 x + \frac{\delta_2 F_2}{2}\sqrt{4 + 5F_1^4\Delta_1^2 - F_2^4\Delta_2^2}\,t + F_4, \tag{46}$$

where F_1, F_2, F_3 and F_4 are arbitrary constants, $\delta_1 = \pm 1$ and $\delta_2 = \pm 1$.

(I) Setting $\begin{cases}\Delta_1 > 0\\ \Delta_2 > 0\end{cases}$, the complexiton solutions consisting of hyperbolic functions can be derived as

$$u_1 = \frac{F_1^2\Delta_1 + F_2^2\Delta_2}{3} - \frac{\Delta_1 F_1^2}{2}\left(\frac{M_1\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}{M_1\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}\right)^2 - \frac{\Delta_2 F_2^2}{2}\left(\frac{N_1\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}{N_1\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}\right)^2, \tag{47}$$

where F_1, F_2, M_1, M_2, N_1 and N_2 are arbitrary constants, and ξ_1 and ξ_2 are determined using eqs (44)–(46).

If $M_2 \neq 0$, $N_2 \neq 0$, $M_2^2 > M_1^2$, $N_2^2 > N_1^2$, the double solitary-like wave solution can be obtained as

$$u_1 = -\frac{F_1^2\Delta_1 + F_2^2\Delta_2}{6} + \frac{F_1^2\Delta_1}{2}\operatorname{sech}^2\left(\frac{\sqrt{\Delta_1}}{2}\xi_1 + \xi_{10}\right) + \frac{F_2^2\Delta_2}{2}\operatorname{sech}^2\left(\frac{\sqrt{\Delta_2}}{2}\xi_2 + \xi_{20}\right), \tag{48}$$

with $\xi_{10} = \tanh^{-1}(M_1/M_2)$ and $\xi_{20} = \tanh^{-1}(N_1/N_2)$.

3.6 *The Hirota–Satsuma equations*

In this subsection, we consider the Hirota–Satsuma(HS) equations [9,44,45],

$$\begin{cases} u_t + 6\alpha u u_x - 2\gamma v v_x + \alpha u_{xxx} = 0, \\ v_t + 3\beta u v_x + \beta v_{xxx} = 0, \end{cases} \tag{49}$$

where α, β and γ are arbitrary, nonzero constants.

Balancing the highest-order partial derivative term and the highest-order nonlinear term of eq. (49), we suppose that the solutions of eq. (49) is of the form

$$\begin{cases} u = A_2\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)^2 + A_1\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + A_0 + a_2\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right)^2 \\ \qquad + a_1\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right) + a_0\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \\ v = B_2\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)^2 + B_1\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + B_0 + b_2\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right)^2 \\ \qquad + b_1\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right) + b_0\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \end{cases} \tag{50}$$

where $\xi_1, \xi_2, A_2, A_1, A_0, a_2, a_1, a_0, B_2, B_1, B_0, b_2, b_1, b_0$ are functions of $\{x, t\}$ to be determined, and $G_1(\xi_1)$, $G_2(\xi_2)$ satisfy eq. (3).

By using the same procedure as in §3.1, the general solutions of HS equation (49) can be derived as follows:

$$\begin{cases} u = -2F_1^2\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)^2 - 2\lambda_1 F_1^2\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + A_0 \\ \qquad - 2F_2^2\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right)^2 - 2\lambda_2 F_2^2\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \\ v = \frac{F_3\lambda_2}{2}\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right) + B_0 + \frac{F_3\lambda_1}{2}\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right) \\ \qquad + F_3\left(\frac{G_1'(\xi_1)}{G_1(\xi_1)}\right)\left(\frac{G_2'(\xi_2)}{G_2(\xi_2)}\right), \end{cases} \tag{51}$$

with

$$\alpha = -\frac{\beta}{2}, \quad \gamma = -\frac{12\beta F_1^2 F_2^2}{F_3^2}, \quad B_0 = \frac{F_3\lambda_1\lambda_2}{4}, \tag{52}$$

$$A_0 = -\frac{F_1^2\Delta_1 + F_2^2\Delta_2}{4} - \frac{F_1^2\lambda_1^2 + F_2^2\lambda_2^2}{2}, \tag{53}$$

$$\xi_1 = F_1 x + \frac{1}{4}\beta F_1 \left(F_1^2\Delta_1 - 3F_2^2\Delta_2\right) t + F_4, \tag{54}$$

$$\xi_2 = F_2 x + \frac{1}{4}\beta F_2 \left(F_2^2\Delta_2 - 3F_1^2\Delta_1\right) t + F_5, \tag{55}$$

where F_1, F_2, F_3, F_4 and F_5 are arbitrary constants.

Setting $\begin{cases}\Delta_1 > 0\\ \Delta_2 > 0\end{cases}$, the complexiton solutions consisting of hyperbolic functions can be derived as

$$\begin{cases} u_1 = -\dfrac{F_1^2\Delta_1 + F_2^2\Delta_2}{4} - \dfrac{\Delta_1 F_1^2}{2}\left(\dfrac{M_1\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}{M_1\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}\right)^2 \\ \qquad -\dfrac{\Delta_2 F_2^2}{2}\left(\dfrac{N_1\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}{N_1\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}\right)^2, \\ v_1 = \dfrac{F_3\sqrt{\Delta_1\Delta_2}}{4}\left(\dfrac{M_1\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}{M_1\sinh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right) + M_2\cosh\left(\frac{\sqrt{\Delta_1}}{2}\xi_1\right)}\right) \\ \qquad \times\left(\dfrac{N_1\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}{N_1\sinh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right) + N_2\cosh\left(\frac{\sqrt{\Delta_2}}{2}\xi_2\right)}\right), \end{cases} \tag{56}$$

where $F_1, F_2, F_3, M_1, M_2, N_1$ and N_2 are arbitrary constants, and ξ_1 and ξ_2 are determined using eqs (54) and (55).

If $M_2 \neq 0$, $N_2 \neq 0$, $M_2^2 > M_1^2$, $N_2^2 > N_1^2$, the double solitary-like wave solution can be obtained as

$$\begin{cases} u_1 = -\dfrac{3F_1^2\Delta_1 + 3F_2^2\Delta_2}{4} + \dfrac{F_1^2\Delta_1}{2}\operatorname{sech}^2\left(\dfrac{\sqrt{\Delta_1}}{2}\xi_1 + \xi_{10}\right) \\ \qquad +\dfrac{F_2^2\Delta_2}{2}\operatorname{sech}^2\left(\dfrac{\sqrt{\Delta_2}}{2}\xi_2 + \xi_{20}\right), \\ v_1 = \dfrac{F_3\sqrt{\Delta_1\Delta_2}}{4}\tanh\left(\dfrac{\sqrt{\Delta_1}}{2}\xi_1 + \xi_{10}\right)\tanh\left(\dfrac{\sqrt{\Delta_2}}{2}\xi_2 + \xi_{20}\right), \end{cases} \tag{57}$$

with $\xi_{10} = \tanh^{-1}(M_1/M_2)$ and $\xi_{20} = \tanh^{-1}(N_1/N_2)$.

4. Summary and discussions

In summary, an extended multiple (G'/G)-expansion method has been first proposed and then applied to the Sharma–Tasso–Olver equation, the sixth-order Ramani equation, the generalized shallow water wave equation, the Caudrey–Dodd–Gibbon–Sawada–Kotera equation, the sixth-order Boussinesq equation and the Hirota–Satsuma equations. With the aid of symbolic computation, a rich variety of complexiton solutions consisting of hyperbolic functions, trigonometric functions, rational functions and their mixture are obtained. When the parameters are taken as special values, the known double solitary-like wave solution are derived from the double hyperbolic function solution. In addition, this method can be used to obtain various complexiton solutions, especially multiple solitary-like wave solutions, for some high-dimensional and variable coefficients' NLEEs in mathematical physics.

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